Practice Test #7

$$g(u,v) = (u, v, v^2 - u^2)$$

$$0 \le u \le 2$$

$$0 \le v \le 2$$

a) $$\iint_S \sqrt{1 + 4z + 8x^2}\, dS$$

$$g_u = (1, 0, -2u)$$

$$g_v = (0, 1, 2v)$$

$$N = (2u, -2v, 1)$$

$$|N| = \sqrt{4u^2 + 4v^2 + 1}$$

$$\sqrt{1 + 4z + 8x^2} = \sqrt{1 + 4(v^2 - u^2) + 8u^2}$$

$$= \sqrt{1 + 4v^2 + 4u^2}$$

$x = u$

$y = u$

$z = v^2 - u^2$

(a) $$\iint_S \sqrt{1 + 4z + 8x^2}\, \underbrace{dS}_{\|N\|\, du\, dv}$$

$$= \int_0^2 \int_0^2 \sqrt{1 + 4v^2 + 4u^2}\, \sqrt{1 + 4u^2 + 4v^2}\, du\, dv$$

$$= \int_0^2 \int_0^2 (1 + 4v^2 + 4u^2)\, du\, dv = \dots \text{ HW}$$

N

(b) $\iint F \cdot dS = \int_0^2 \int_0^2 (2u, v^2-u^2, v) \cdot (2u, -2v, 1)\, du\, dv$

$F = (2x, z, y)$

$g = (u, v, v^2-u^2)$

$F(g(u,v)) = (2u, v^2-u^2, v)$

$N = (2u, -2v, 1)$

$$= \int_0^2 \int_0^2 (4u^2 - 2v^3 + 2u^2 v + v)\, du\, dv$$

$$= \int_0^2 \frac{4u^3}{3} - 2uv^3 + \frac{2u^3}{3}v + uv \Big|_{u=0}^{u=2} dv$$

$$= \int_0^2 \left(\frac{32}{3} - 4v^3 + \frac{16}{3}v + 2v\right) dv$$

$$= \frac{32}{3}v - v^4 + \frac{8}{3}v^2 + v^2 \Big|_{v=0}^{v=2}$$

$$= \frac{64}{3} - 16 + \frac{32}{3} + 4$$

$$= 32 - 16 + 4 = 20.$$

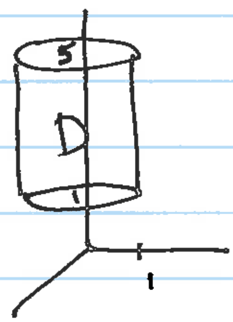
Practice

#11

Verify Gauss' Thm

for $D = \{ (x,y,z) \mid x^2+y^2 \le 1, \; 1 \le z \le 5 \}$

$F = (x^2, y, z)$



$$\iiint_D \operatorname{div} F \cdot dV = \iint_{\partial D} F \cdot dS$$

$$\operatorname{div} F = 2x + 1 + 1 = 2x + 2$$

(i)
$$\iiint_D \operatorname{div} F \cdot dV = \int_0^{2\pi} \int_0^1 \int_1^5 (2r\cos\theta + 2)\, r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (2r^2\cos\theta + 2r)\, z \Big|_1^5 \, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 8r^2\cos\theta + 8r \; dr\, d\theta$$

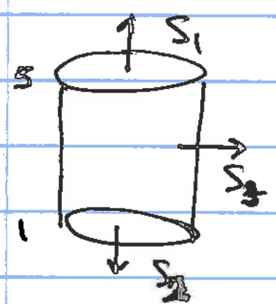
$$= \int_0^1 \int_0^{2\pi} 8r^2\cos\theta + 8r \; d\theta\, dr$$

$$= \int_0^1 8r^2\sin\theta + 8r\theta \Big|_0^{2\pi} \, dr$$

$$= \int_0^1 0 + 16\pi r \, dr$$

$$= 8\pi r^2 \Big|_0^1 = 8\pi$$

(ii)



$F = (x^2, y, z)$

$s^2+t^2 \leq 1$

S_1: $X(s,t) = (s, t, 5)$ $\quad F(X(s,t)) = (s^2, t, 5)$

$X_s = (1,0,0)$

$X_t = (0,1,0)$

$N = (0,0,1)$

$$\iint_{S_1} (s^2, t, 5) \cdot (0,0,1) \, ds\,dt = 5(\text{area } (s^2+t^2 \leq 1)) = 5\pi$$

S_2: $X_2(s,t) = (s, t, 1)$ $\quad F(X_2(s,t)) = (s^2, t, 1)$

$X_{2,s} = (1,0,0)$

$X_{2,t} = (0,1,0)$

$N = (0,0,1)$ Wrong normal.

$$\iint_{S_2} F \cdot dS = - \iint_{s^2+t^2 \leq 1} (s^2, t, 1) \cdot (0,0,1) \, ds\,dt = -\pi$$

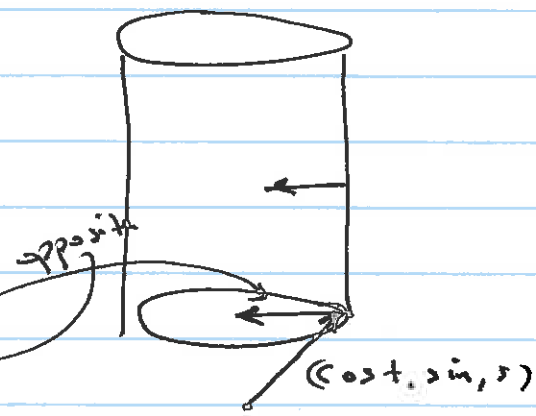
$F = (x^2, y, z)$ $\qquad 1 \le s \le 5$, $\ 0 \le t \le 2\pi$ $\qquad F(\mathbb{X}_3(s,t)) = (\cos^2 t, \sin t, s)$

$\mathbb{X}_3(s,t) = (\cos t, \sin t, s)$

$\mathbb{X}_{3,s} = (0, 0, 1)$

$\mathbb{X}_{3,t} = (-\sin t, \cos t, 0)$

$N = (-\cos t, -\sin t, 0)$



$$\iint_{S_3} F \cdot dS = -\int_1^5 \int_0^{2\pi} (\cos^2 t, \sin t, s) \cdot (-\cos t, -\sin t, 0)\, dt\, ds$$

$$= -\int_1^5 \int_0^{2\pi} (-\cos^3 t - \sin^2 t + 0)\, dt\, ds$$

$$= +4 \int_0^{2\pi} \cos^3 t + \sin^2 t \, dt$$

$$= 4\pi$$

$$\int_0^{2\pi} \sin^2 t \, dt = \int_0^{2\pi} \frac{1 - \cos 2t}{2}\, dt = \frac{t}{2} - \frac{\sin 2t}{4} \Big|_0^{2\pi} = \pi$$

$$\int_0^{2\pi} \cos^3 t \, dt = \int_0^{2\pi} \cos t (1 - \sin^2 t)\, dt = \int_0^0 = 0$$

$u = \sin t$
$du = -\cos t \, dt$

$$\iiint \mathrm{div}\, F \cdot dV = 8\pi$$

$$\iint_{S_1} F \cdot dS = 5\pi$$

$$\iint_{S_2} F \cdot dS = -\pi$$

$$\iint_{S_3} F \cdot dS = 4\pi$$

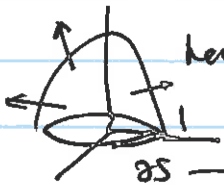
$$\iiint \mathrm{div}\, F \cdot dV \checkmark$$

$$= \left(\iint_{S_1} + \iint_{S_2} + \iint_{S_3} \right) F \cdot d\vec{S}$$

#16

$$\iint_S \nabla \times F \, dS = \oint_{\partial S} F \cdot d\vec{s}$$

$$F = (x^2, z^2, y^2)$$



hemisphere

$\partial S \longrightarrow (\cos t, \sin t, 0) = x(t) \qquad x'(t) = (-\sin t, \cos t, 0)$

$$F(x(t)) = (\cos^2 t, 0, \sin^2 t)$$

$$\int F \cdot dS$$

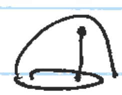
$$= \int_0^{2\pi} (\cos^2 t, 0, \sin^2 t) \cdot (-\sin t, \cos t, 0) \, dt$$

$$= \int_0^{2\pi} -\sin t \cos^2 t \, dt = \frac{\cos^3 t}{3} \Big|_0^{2\pi} = 0.$$

OR $\longrightarrow$ Calculate Directly

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2 & z^2 & y^2 \end{vmatrix} = (2y - 2z, 0, 0) = \nabla \times F$$

$$\underline{X}(s,t) = (s, t, \sqrt{1 - s^2 - t^2})$$



$s^2 + t^2 \leq 1.$

$$\underline{X}_s = \left(1, 0, \frac{-2s}{2\sqrt{1-s^2-t^2}}\right)$$

$$\underline{X}_t = \left(0, 1, \frac{-2t}{2\sqrt{1-s^2-t^2}}\right)$$

$$N = \left(\frac{s}{\sqrt{\quad}}, \frac{t}{\sqrt{\quad}}, 1\right)$$

$$(\nabla \times F)(\underline{X}(s,t)) = (2t - 2\sqrt{\quad}, 0, 0)$$

$$\iint_S \nabla \times F \cdot dS = \iint_{s^2+t^2 \le 1} \left(2t - 2\sqrt{}, 0, 0\right) \cdot \left(\frac{s}{\sqrt{}}, \frac{t}{\sqrt{}}, 1\right) ds\,dt$$

$$= \iint_{s^2+t^2 \le 1} \left(\frac{2st}{\sqrt{}} - 2s\right) ds\,dt$$

use polar coordinates

$$= \int_0^{2\pi} \int_0^1 \left(\frac{2r\cos\theta\, r\sin\theta}{\sqrt{1-r^2}} - 2r\cos\theta\right) r\,dr\,d\theta$$

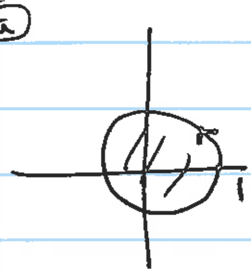
$= 0$; since

$$\int_0^{2\pi} 2\cos\theta\sin\theta\, d\theta = \int_0^{2\pi} \sin 2\theta\, d\theta = 0$$

$$\int_0^{2\pi} \cos\theta\, d\theta = 0$$

Ex 9 $\quad \displaystyle\int_{C_i} \overbrace{(x^2-y^3)}^{P}dx + \overbrace{(x^3-y^2)}^{Q}dy$

(a)



Try Green's Thm.

$$\frac{\partial Q}{\partial x} = 3x^2$$

$$\frac{\partial P}{\partial y} = -3y^2$$

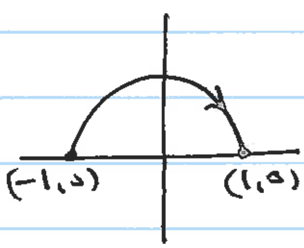
$$\int P\,dx + Q\,dy = +\iint\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \iint_{x^2+y^2\le 1} 3x^2+3y^2 \, dA$$

use polar

$$\int_0^{2\pi}\int_0^1 3r^2\cdot r\,dr\,d\theta = 2\pi\cdot\frac{3r^4}{4}\Big|_0^1$$

$$= \frac{3}{2}\pi.$$

(b)



$(-\cos t, \sin t) \qquad 0 \le t \le \pi$

$x = -\cos t \qquad dx = \sin t\,dt$

$y = \sin t \qquad dy = \cos t\,dt$

$$\int (x^2-y^3)\,dx + (x^3-y^2)\,dy = \int_0^{\pi} (\cos^2 t - \sin^3 t)\sin t\,dt + (-\cos^3 t - \sin^2 t)\cos t\,dt$$

$$\int_0^\pi \cos^2 t \sin t - \sin^4 t - \cos^4 t - \sin^2 t \cos t \, dt$$

$$\int_0^\pi \cos^2 t \sin t \, dt = \int_1^{-1} -du \cdot u^2 = \frac{u^3}{3}\Big|_{-1}^{+1} = \frac{2}{3}.$$

$u = \cos t$

$du = -\sin t \, dt$

$$\int_0^\pi \sin^2 t \cos t \, dt = \int_0^0 v^2 \, dv = 0.$$

$v = \sin t$

$dv = \cos t \, dt$

$$\int_0^\pi \sin^4 t + \cos^4 t \, dt = ?$$

$$1 = (\sin^2 t + \cos^2 t)^2 = \sin^4 t + \cos^4 t + 2\cos^2 t \sin^2 t$$

$$\sin^4 t + \cos^4 t = 1 - 2\cos^2 t \sin^2 t = 1 - \frac{1}{2}\sin^2 2t.$$

$\sin 2t = 2\sin t \cos t$

$\sin^2 2t = 4\sin^2 t \cos^2 t$

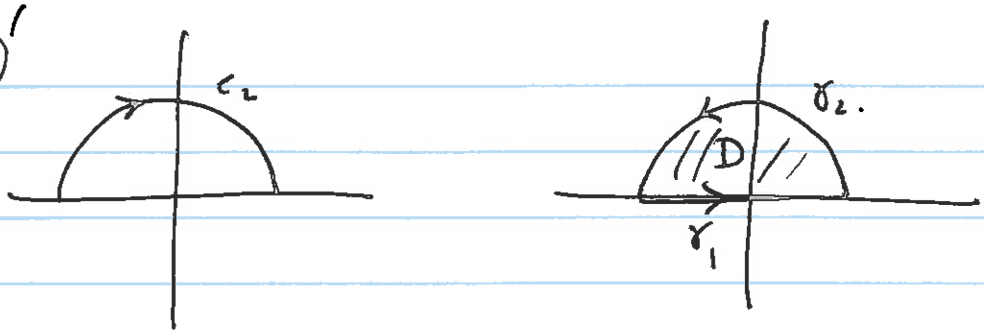
$$\int_0^\pi \left(1 - \frac{1}{2}\sin^2 2t\right) dt = \int_0^\pi 1 - \frac{1}{2}\,\frac{1-\cos 4t}{2} \, dt$$

$$= \int_0^\pi \frac{3}{4} - \frac{\cos 4t}{4} \, dt = \frac{3}{4}\pi.$$

(we need: $\int -\sin^4 t - \cos^4 t \, dt$

Ans: (B) $= -\frac{3}{4}\pi + \frac{2}{3} + 0.$

(b)'



$$\int_{\gamma_1} (x^2-y^3)\,dx + (x^3-y^2)\,dy = \int_{-1}^{1} t^2\,dt = \frac{2}{3}$$

$\gamma_1(t) = (t,0)$ $\quad x=t \quad dx = dt$
$\quad y=0 \quad dy=0$

Want

$$-\int_{C_2} = \int_{\gamma_2} (x^2-y^3)\,dx + (x^3-y^2)\,dy$$

$$\Big(\underbrace{\int_{\gamma_1}}_{\frac{2}{3}} + \int_{\gamma_2}\Big) P\cdot dx + Q\,dy = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$$= \iint_{\substack{x^2+y^2\le 1 \\ y\ge 0}} 3x^2+3y^2\,dA$$

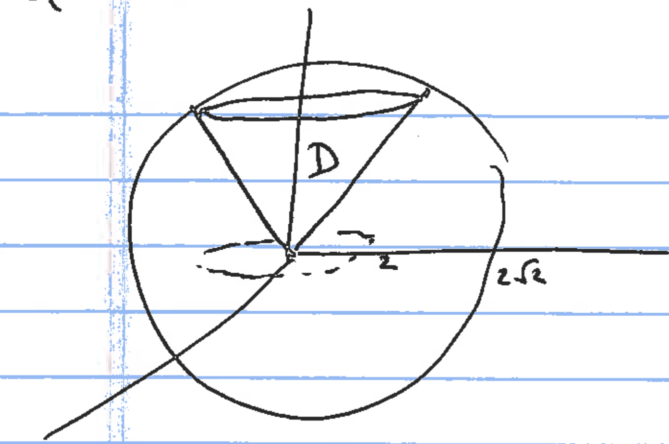
$$= \int_0^{\pi} d\theta \int_0^1 r\,dr\; 3r^2 = \frac{3}{4}\pi$$

$$\int_{C_2} P\,dx + Q\,dy = -\Big(\frac{3}{4}\pi - \frac{2}{3}\Big) = -\frac{3\pi}{4} + \frac{2}{3}$$

C_2 goes opposite of γ_2 direction.

Practice test

Q #21



$x^2+y^2+z^2=8$

$\sqrt{x^2+y^2}=z$

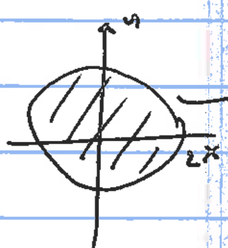
$x^2+y^2+x^2+y^2=8$

$2(x^2+y^2)=8$

$x^2+y^2=4$

$r=2.$

Rect ⓐ $\displaystyle\iint_B x^2+y^2\,dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} (x^2+y^2)\,dz\,dy\,dx$



Cylin. ⓑ $\displaystyle\iint_B x^2+y^2\,dV = \int_0^{2\pi}\int_0^{2}\int_r^{\sqrt{8-r^2}} r^2 \quad dz\,r\,dr\,d\theta$

Sphr. ⓒ $\displaystyle\iint_B x^2+y^2\,dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\sqrt{2}} \rho^2\sin^2\varphi\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

$x^2+y^2=\rho^2\sin^2\varphi$

$x=\rho\sin\varphi\cos\theta$
$y=\rho\sin\varphi\sin\theta$
$z=\rho\cos\varphi$

easiest to calculate.