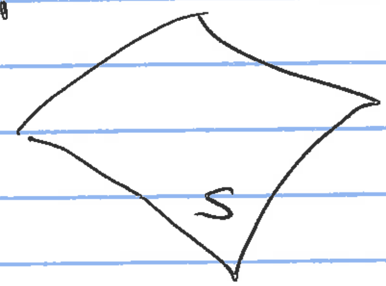
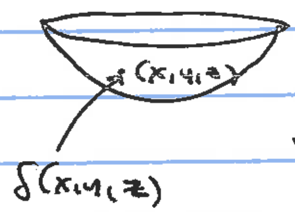
7.2 Surface Integrals

Types / Purpose

- 

$\text{Area } S = \iint_S 1 \, dS = \iint_D \|N\| \, ds \, dt$

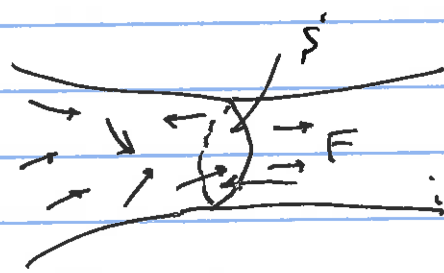
for a given parametrization

- 

metallic bowl with variable density $\delta(x,y,z)$

$$\text{Mass} = \iint_S \delta(x,y,z) \, dS$$

- Flux

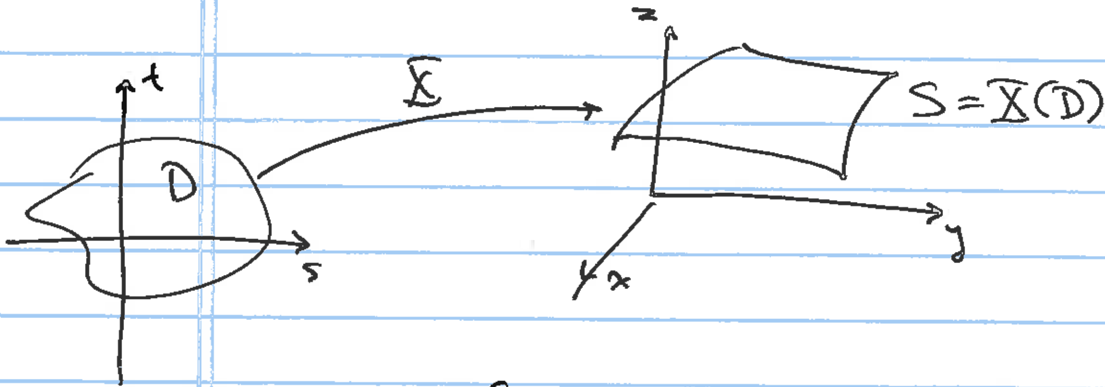


Fluid flow, varying velocity in direction & magnitude

Total amount passing thru a surface with a preferred side.

$$\iint_S F \cdot d\vec{S}$$

<u>Defn</u> Let $\bar{X}(s,t) : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a smooth parametrized^{s,t} oriented surface



① If $f : S \longrightarrow \mathbb{R}$ real valued function, one defines

$$\iint_{\bar{X}} f\,dS = \iint_{D} f(\bar{X}(s,t)) \, \Big\| \underbrace{\frac{\partial \bar{X}}{\partial s} \times \frac{\partial \bar{X}}{\partial t}}_{N} \Big\| \, ds\,dt$$

$f \equiv 1$ corresponds to area of S

② If $\vec{F} : S \longrightarrow \mathbb{R}^3$ is a vector field defined along S, one defines

$$\iint_{\bar{X}} \vec{F} \cdot d\vec{S} = \iint_{D} \underbrace{\vec{F}(\bar{X}(s,t)) \cdot \frac{\partial \bar{X}}{\partial s} \times \frac{\partial \bar{X}}{\partial t}}_{\text{real valued}} \, ds\,dt$$

<u>ORIENTATION</u> : Curves: Choice of direction of the curve

Surfaces: Choice of a side

For a parametrized surface
$\bar{X}(s,t)$

$\frac{\partial X}{\partial s} \times \frac{\partial X}{\partial t} = N$ normal. (

$\vec{n} = \frac{\vec{N}}{\|N\|}$ unit normal

$dS = \|N\| \, ds\,dt$

$d\vec{S} = \vec{N} \, ds\,dt = \vec{n} \cdot dS = \frac{\vec{N}}{\|\vec{N}\|} \|\vec{N}\| \, ds\,dt$

Ex. 1

$$\Sigma(s,t) = (\underbrace{s\cos t}_{x}, \underbrace{s\sin t}_{y}, \underbrace{s^2}_{z}) \qquad \begin{array}{l} 0 \le s \le 2 \\ 0 \le t \le 2\pi \end{array}$$

Calculate

i) Area

ii) $\iint_{\Sigma} x^2 \, dS$

iii) $\iint_{\Sigma} (xz, yz, xy) \cdot d\vec{S}$

$$\Sigma_s = (\cos t, \sin t, 2s)$$

$$\Sigma_t = (-s\sin t, s\cos t, 0)$$

$$\Sigma_s \times \Sigma_t = \vec{N} = (-2s^2\cos t, -2s^2\sin t, s)$$

$$\|\vec{N}\| = \sqrt{4s^4\cos^2 t + 4s^4\sin^2 t + s^2} = \sqrt{4s^4 + s^2}$$

$$= s\sqrt{4s^2+1}$$

(i) $$\text{Area} = \iint_{\Sigma} 1 \, dS = \iint_{D} \|N\| \, ds\,dt$$

$$= \int_0^{2\pi} \int_0^2 s\sqrt{4s^2+1} \, ds\,dt = \left(\int_0^{2\pi} dt\right)\left(\int_1^{17} u^{\frac{1}{2}} \cdot \frac{1}{8} \, du\right)$$

$u = 4s^2+1$

$du = 8s\,ds$

$$= 2\pi \cdot u^{\frac{3}{2}} \cdot \frac{2}{3} \cdot \frac{1}{8} \Big|_1^{17} = \frac{\pi}{6}\left(17\sqrt{17} - 1\right)$$

(ii) $\iint x^2\, dS$

$$X(s,t) = (\underbrace{s\cos t}_{x}, \underbrace{s\sin t}_{y}, \underbrace{s^2}_{z})$$

$x^2 = (s\cos t)^2$

$dS = \|N\|\, ds\, dt = s\sqrt{4s^2+1}\, ds\, dt$

$$= \underset{dt}{\int_0^{2\pi}} \underset{ds}{\int_0^{2}} \underbrace{s^2 \cos^2 t}_{x^2} \cdot s\sqrt{4s^2+1}\, ds\, dt$$

$$= \left(\int_0^{2\pi} \cos^2 t\, dt\right)\left(\int_0^2 s^2\sqrt{4s^2+1} \cdot s\, ds\right)$$

$u = 4s^2+1 \rightarrow s^2 = \frac{u-1}{4}$

$du = 8s\, ds$

$$= \left(\int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt\right)\left(\int_1^{17} \frac{u-1}{4} \cdot \sqrt{u} \cdot \frac{1}{8}\, du\right)$$

$$= \pi \cdot \frac{1}{32}\int_1^{17} \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du$$

$$= \frac{\pi}{32}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\Big|_1^{17}\right)$$

$$= \frac{\pi}{32}\left(\frac{2}{5}17^2\sqrt{17} - \frac{2}{5} - \frac{2}{3}17\sqrt{17} + \frac{2}{3}\right).$$

iii $\iint_S (xz, yz, xy) \cdot d\vec{S}$

$$\iint F(\Phi(s,t)) \cdot \vec{N}\, ds\, dt$$

$F = (xz, yz, xy)$

$\Phi(s,t) = (\underbrace{s\cos t}_{x}, \underbrace{s\sin t}_{y}, \underbrace{s^2}_{z})$

$$= \int_0^{2\pi}\int_0^2 (s^3\cos t,\ s^3\sin t,\ s^2\cos t\sin t) \cdot \underbrace{(-2s^2\cos t, -2s^2\sin t, s)}_{N} \cdot ds\, dt$$

$$= \int_0^{2\pi}\int_0^2 \left(\underbrace{-2s^5\cos^2 t - 2s^5\sin^2 t}_{-2s^5} + s^3\cos t\sin t\right) ds\, dt$$

$$= \int_0^{2\pi}\int_0^2 -2s^5 + s^3\cos t\sin t\ ds\, dt$$

$$= \int_0^{2\pi} -\frac{s^6}{3} + \frac{s^4}{4}\cos t\sin t\ \Big|_0^2\ dt$$

$$= \int_0^{2\pi} -\frac{64}{3} + \underbrace{4\cos t\sin t}_{\frac{1}{2}\sin 2t}\ dt = -\frac{64}{3}\cdot 2\pi + 0$$

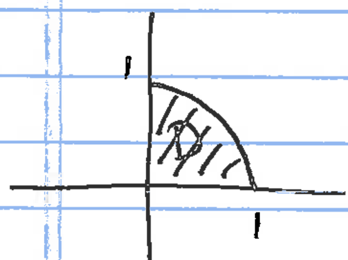
$\int_0^{2\pi} \sin 2t\, dt = 0.$

$$= \frac{-128\pi}{3}.$$

7.2 Exc #2 p488

$$\bar{X}(s,t) = (s+t,\ s-t,\ st)$$

$$D = \{(s,t) \mid s^2+t^2 \le 1,\ s \geq 0,\ t \geq 0\}$$



b) Want $\iint_{X} \vec{F} \cdot d\vec{S}$ where $F(x,y,z) = (x,y,z)$

$i \quad j \quad k$

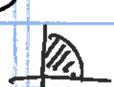
$$\frac{\partial X}{\partial s} = (1,\ 1,\ t)$$

$$\frac{\partial X}{\partial t} = (1,\ -1,\ s)$$

$$N = (s+t,\ t-s,\ -2)$$

$$\iint_{X} F \cdot d\vec{S} = \iint_{D} F(X(s,t)) \cdot \vec{N}(s,t)\, dt\, ds$$

$$= \int\int (s+t,\ s-t,\ st) \cdot (s+t,\ t-s,\ -2)\, ds\, dt$$

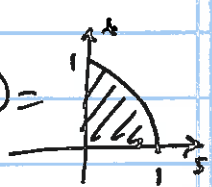


$$= \iint_{D} \left((s+t)^2 + (s-t)(t-s) - 2st\right) ds\, dt$$

$$= \iint_D \cancel{s^2} + \cancel{t^2} + \cancel{2st} - \cancel{s^2} - \cancel{t^2} + 2st - \cancel{2st} \quad ds\,dt$$

$$= \iint_D 2st \quad ds\,dt$$

D =



Use polar coordinates

$s = r\cos\theta$

$t = r\sin\theta$

$$= \int_0^{\pi/2} \int_0^1 2\underbrace{r\cos\theta}_{s}\,\underbrace{r\sin\theta}_{t} \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^1 2r^3 \cos\theta \sin\theta \; dr\; d\theta$$

$$= \left(\int_0^{\pi/2} 2\cos\theta\sin\theta\, d\theta\right)\left(\int_0^1 r^3\, dr\right)$$

$$= \int_0^{\pi/2} \sin 2\theta\, d\theta \cdot \frac{1}{4}$$

$$= \frac{-\cos 2\theta}{2}\Bigg|_0^{\pi/2} \cdot \frac{1}{4}$$

$$= \underbrace{\frac{-\overbrace{\cos\pi}^{-1} + \overbrace{\cos 0}^{1}}{2}}_{1} \cdot \frac{1}{4} = \frac{1}{4} \quad (\text{not } 1/8)$$