Dec 1

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6.3

$n=3$, To find Potential functions for Conservative v.f. & FTLI.

Ex $F = (y, 2x-z, z^2)$

Is F conservative? NO

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & 2x-z & z^2 \end{vmatrix}$$

$$= (\underbrace{0-(-1)}_{+1}, \; * \;, \; * \;) \neq (0,0,0)$$

F conservative $\Longrightarrow$ $\operatorname{curl} F = 0$

F is not " $\Longleftarrow$ $\operatorname{curl} F \neq \vec{0}$

Ex $F = (2xy+3+z, \; x^2+ze^{yz}-4, \; ye^{yz}+x)$

Is F conservative? If so, find a potential function

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2xy+3+z & x^2+ze^{yz}-4 & ye^{yz}+x \end{vmatrix}$$

$$= \left((e^{yz} + yze^{yz} + 0) - (0 + e^{yz} + yze^{yz} - 0), \; 1-1, \; 2x-2x \right) = (0,0,0)$$

Domain $= \mathbb{R}^3$; convex. Thm⊛ $\Longrightarrow$ $\exists f$ $\nabla f = F$

How do I find f?

$$F = (2xy + 3 + z,\ x^2 + ze^{yz} - 4,\ ye^{yz} + x)$$
$$\nabla f = (\ f_x\ ,\ f_y\ ,\ f_z\ )$$

$$f_x = 2xy + 3 + z \xrightarrow{\int dx} f = x^2y + 3x + xz + c_1(y,z)$$
$$f_y = x^2 + ze^{yz} - 4 \longrightarrow f = x^2y + e^{yz} - 4y + c_2(x,z)$$
$$f_z = ye^{yz} + x \longrightarrow f = e^{yz} + xz + c_3(x,y)$$

$$f = x^2y + 3x + xz + e^{yz} - 4y \ (+ \text{ any real } \#)$$

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p449 Exc #28 6.3

Find $\displaystyle\int_C (2y - 3z)\,dx + (2x+z)\,dy + (y - 3x)\,dz$

$C: \begin{cases} (0,0,0) \to (0,1,1) & \text{line segment} \\ (0,1,1) \to (1,2,3) & \text{" \quad "} \end{cases}$

Want to apply FTLI!

Is $F = (2y - 3z,\ 2x + z,\ y - 3x)$ conservative?

Is curl $F = 0$?

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2y - 3z & 2x + z & y - 3x \end{vmatrix} = (1 - 1, -3+3, 2-2)$$
$$= (0,0,0) \checkmark$$

Is domain $= \mathbb{R}^2$ convex? Yes.

$\exists f \quad \nabla f = F$

$$\left.\begin{aligned} f_x &= 2y - 3z \\ f_y &= 2x + z \\ f_z &= y - 3x \end{aligned}\right\} \quad f = ?$$

$$f_x = 2y - 3z \xrightarrow{\int dx} f = 2xy - 3xz + c_1(y,z)$$
$$f_y = 2x + z \xrightarrow{\int dy} f = 2xy + yz + c_2(x,z)$$
$$f_z = y - 3x \xrightarrow{\int dz} f = yz - 3xz + c_3(x,y)$$

$$f = 2xy - 3zx + yz + \underset{\substack{\uparrow \\ \mathbb{R}.}}{C}$$

$$\int_C (2y-3z)\,dx + (2x+z)\,dy + (y-3x)\,dz$$

$$= \int \nabla f \cdot d\vec{s} \overset{FTLI}{=} f(\text{end pt}) - f(\text{initial pt})$$

$$= 2xy - 3xz + yz + C \Big|_{(0,0,0)}^{(1,2,3)}$$

$$= (4 - 9 + 6 + \cancel{C}) - (0 + 0 + 0 + \cancel{C})$$

$$= 1$$

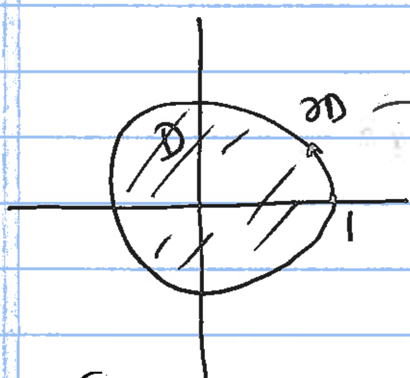
D

$$F(x,y) = \Big(\underbrace{\frac{-y}{x^2+y^2}}_{P} , \underbrace{\frac{x}{x^2+y^2}}_{Q} \Big) \text{ on } \mathbb{R}^2 - \{(0,0)\}$$

$$\left.\begin{aligned} \frac{\partial Q}{\partial x} &= \frac{1\cdot(x^2+y^2) - 2x\cdot x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \\ \frac{\partial P}{\partial y} &= \frac{-1\cdot(x^2+y^2) - 2y\cdot(-y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}. \end{aligned}\right\} \text{if } (x,y)\neq(0,0)$$

$$\left(\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \implies \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0. \right) \text{ if } (x,y)\neq(0,0)$$

Does Green's Thm apply to F over the unit disc?



$\partial D \longrightarrow (\cos t, \sin t) \qquad 0 \le t \le 2\pi$

$x = \cos t \qquad dx = -\sin t\, dt$

$y = \sin t \qquad dy = \cos t\, dt$

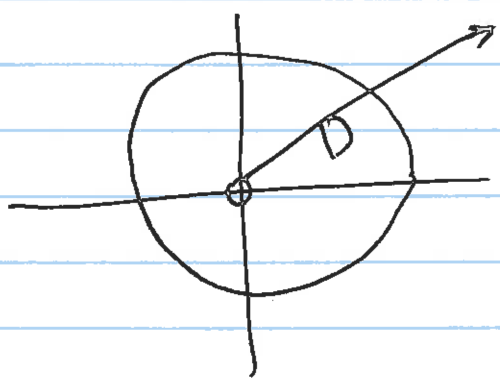
$$\int_{\partial D} P\,dx + Q\,dy = \int_{\partial D} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$= \int_0^{2\pi} \frac{-\sin t\cdot(-\sin t)\,dt}{1} + \frac{\cos t\cdot\cos t\,dt}{1}$$

$$= \int_0^{2\pi} 1\,dt = 2\pi$$

$$\underbrace{\int P\,dx + Q\,dy}_{2\pi} \neq \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{0} dA$$

① Green's Thm doesn't apply, since P and Q are not defined on all of Disc D



P, Q aren't defined at $(0,0)$

② Domain $P, Q = \mathbb{R}^2 - \{(0,0)\}$ not convex

③ Even though $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$

$$\int \frac{-y\,dx + x\,dy}{x^2+y^2} = 2\pi$$

$$\Rightarrow \quad F = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) \text{ is NOT conservative.}$$

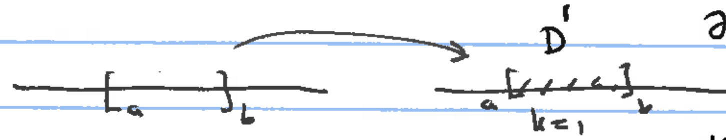
If F were conservative; then $f(\text{end pt}) - f(\text{initial pt}) = 0$

same on a closed curve.

OVERVIEW of Classical Fundamental Theorems:

dimension of ambient space $= n = 1$ $\mathbb{R}^1$ $\partial D = \{a, b\}$.

$k = 1$ // dimension of curve/surface/solid

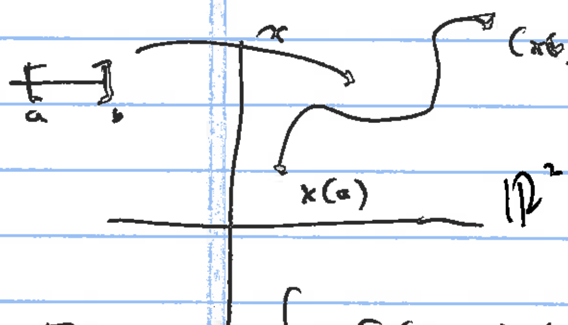


$\mathbb{R}^1$ $n=1$

FTCalculus $$\underbrace{\int_a^b f'(x)\,dx}_{\text{interior integral}} = \underbrace{f(b) - f(a)}_{\text{boundary}}$$

$n=2$: $\mathbb{R}^2$

$k=1$



$\mathbb{R}^2$

FTLI $n=2$

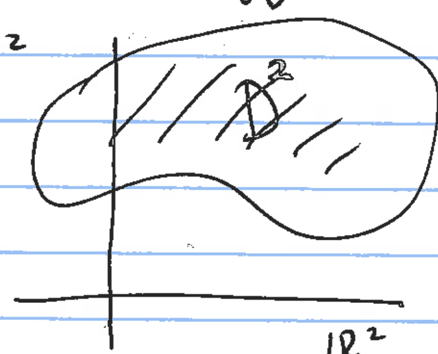
$$\underbrace{\int_{\vec{x}} \nabla f(\vec{x}(t))\,d\vec{x}}_{\text{over the curve}} = \underbrace{f(x(b)) - f(x(a))}_{\text{end pts of the curve}}$$

6.3

$n=2$

$k=2$

$\partial D'$



$\mathbb{R}^2$

Green's Thm:

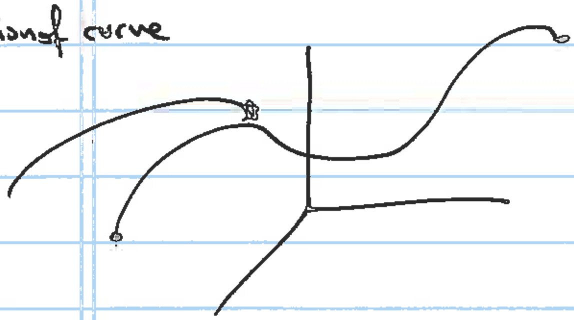
$$\underbrace{\int_{\partial D} P\,dx + Q\,dy}_{\text{1 dim'l boundary integral}} = \underbrace{\iint_D Q_x - P_y\,d}_{\text{2 dim'l interior integral}}$$

6.2

$n = 3$ $\quad \mathbb{R}^3$

dimension of curve
$k = 1$

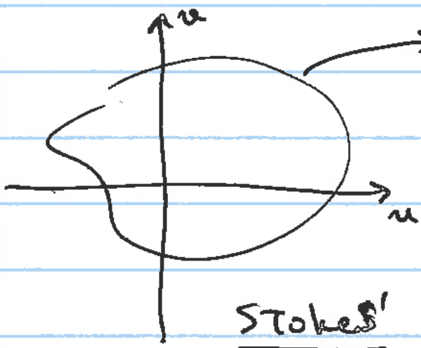
x
$[a,b]$

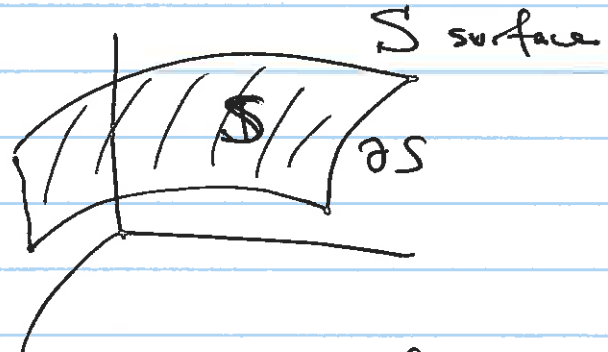


$$\int_{x(t)} \nabla f \cdot d\vec{s} = f(x(t)) \Big|_{t=a}^{t=b}$$

FTLI 6.3

$k = 2$
dim of surface

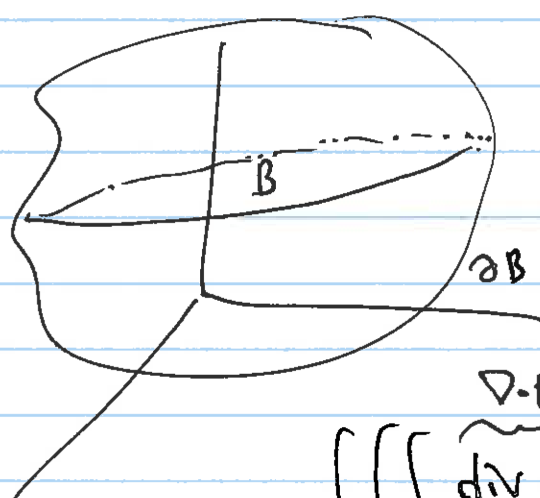




Stokes' 7.3

$$\iint_S \underbrace{\text{curl}\, F}_{\nabla \times F} \cdot d\vec{S} = \oint_{\partial S} F \cdot d\vec{s}$$

$k = 3$
dimension of solid



Gauss' Thm
Divergence Thm
7.3

$$\underbrace{\iiint_B \overbrace{\text{div}\, F}^{\nabla \cdot F}\, dV}_{\text{interior integral}} = \underbrace{\iint_{\partial B} F \cdot d\vec{S}}_{\text{boundary}}$$