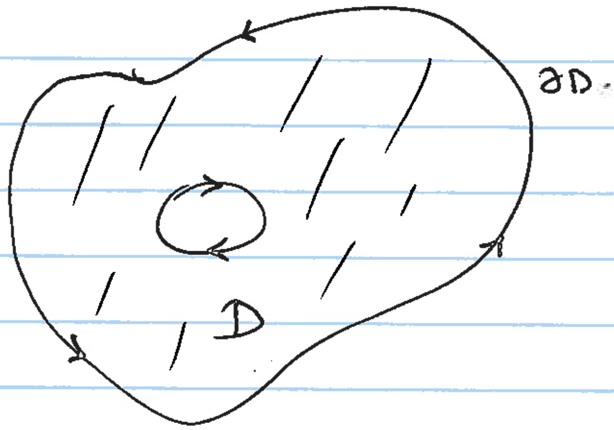
Nov 30

①

6.2 To Conclude.



Recall Green's Thm:

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$$

$$\text{Area}(D) = \int_{\partial D} -y\,dx = \int_{\partial D} +x\,dy = \int_{\partial D} \frac{1}{2}(-y\,dx + x\,dy) = \iint_D 1\,dA$$

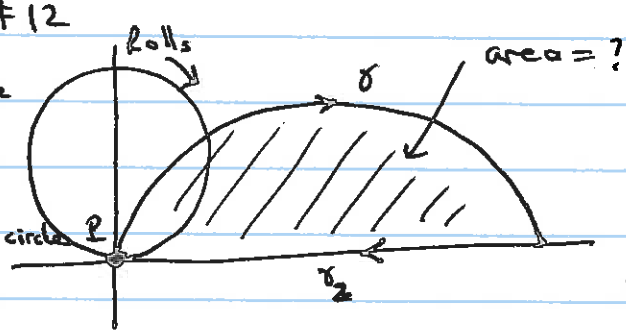
$P = -y \quad P_y = -1.$

$Q = 0 \quad Q_x = 0$

$Q_x - P_y = 1$

p 437 Ex #12

As the ball/circle rolls along x-axis; we observe a fixed pt P of the circle & how it moves.



cycloid

$x = a(t - \sin t) \quad 0 \le t \le 2\pi.$

$y = a(1 - \cos t)$

$dx = a(1 - \cos t)$

$dy = a \sin t$

Which one? $\displaystyle\int_\gamma x\,dy$ OR $\displaystyle\int_\gamma -y\,dx$ easier

$$= \int_0^{2\pi} \underbrace{-a(1-\cos t)}_{-y} \cdot \underbrace{a(1-\cos t)}_{dx}\,dt$$

$$= a^2 \int_0^{2\pi} -(1-\cos t)^2 \, dt$$

$$= -a^2 \int_0^{2\pi} (1 - 2\cos t + \underbrace{\cos^2 t}_{\frac{1+\cos 2t}{2}}) \, dt$$

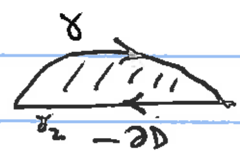
$$= -a^2 \left(t - 2\sin t + \frac{t}{2} + \frac{\sin 2t}{4} \right) \Big|_0^{2\pi}$$

$$= -a^2 \left(\frac{3}{2} \cdot 2\pi + 0 \right) = -3\pi a^2.$$

$$\int_{\gamma_2} -y \, dx = 0$$

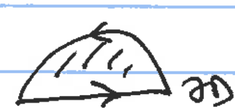
$y = 0$ along γ_2

$$\int_{-\partial D} -y \, dx = -3\pi a^2$$



negative orientation

We need positive orientation of ∂D.



$$-3\pi a^2 = \int_{-\partial D = \gamma \cup \gamma_2} -y\,dx = -\int_{\partial D} -y\,dx = -\text{area}(D)$$

$$\text{area}(D) = 3\pi a^2$$

6.3 **Def** A vector field $F : U^{open} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is called conservative (or gradient field) if $\exists$ a function $f : U^{open} \subseteq \mathbb{R}^n \to \mathbb{R}$ s.t.

$$\nabla f = F.$$

We call F conservative
We call f potential function.

Ex ① $\nabla(x^3y) = (3x^2y, x^3)$ → conservative

potential function

② $F = (x, x)$ is not conservative

Suppose $\exists f$ $\nabla f = (f_x, f_y) = (x, x)$

$$f_x = x \longrightarrow (f_x)_y = f_{xy} = 0$$
$$f_y = x \qquad (f_y)_x = f_{yx} = 1$$

contradiction

p137 Thm 4.3 Thm: If f is twice diff'ble then $f_{xy} = f_{yx}$

③ $$f = \frac{1}{\|\vec{x}\|} = (x^2+y^2+z^2)^{-\frac{1}{2}}$$

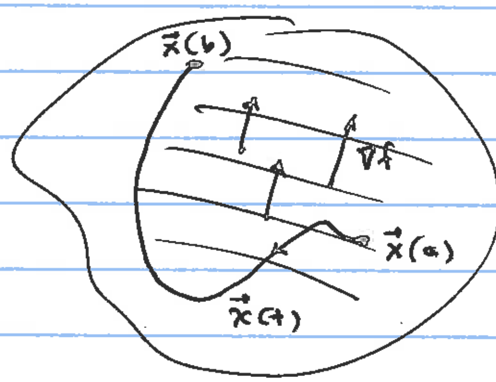
$$\nabla f = \left(-\tfrac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}} \cdot 2x, \ -\tfrac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}} \cdot 2y, \ -\tfrac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}} \cdot 2z\right)$$

$$= (-x, -y, -z) \cdot (x^2+y^2+z^2)^{-\frac{3}{2}} = -\frac{\vec{x}}{\|\vec{x}\|^3} \left(= -\frac{1}{r^2}\right)$$

magnitude.

(FTLI) FUNDAMENTAL THM of LINE INTEGRALS

Let $f: U^{\text{open}} \subseteq \mathbb{R}^n \to \mathbb{R}$ be continuously diff'ble.
Let $\vec{x}: [a,b] \to U$ be a piecewise diff'ble curve

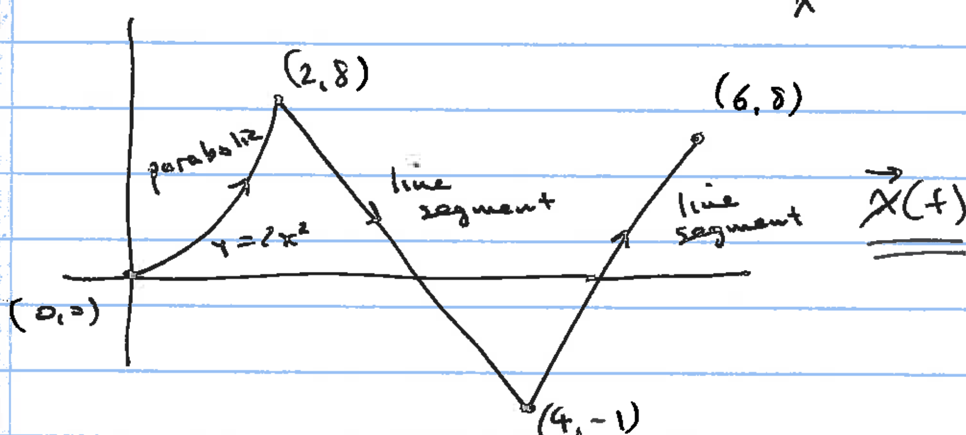


Then:

$$\int_{\vec{x}(t)} \vec{\nabla} f \cdot d\vec{s} = f(\vec{x}(b)) - f(\vec{x}(a)).$$

Ex. $$\int_{\vec{x}} 3x^2y\,dx + x^3\,dy = \int_{\vec{x}} \nabla(x^3y)\,d\vec{s} = x^3y\Big|_{(0,0)}^{(6,8)}$$

$$= 6^3 \cdot 8 = 1728$$



Q · When vector fields are conservative?

(*) **Theorem:** Let $F(\underbrace{x_1, x_2, \dots x_n}_{n \text{ variables}}) = (\underbrace{F_1, F_2, \dots F_n}_{n \text{ components}})$

$F : U^{\text{open}} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be continuously diff'ble.

Then

(1) If F is conservative, that is $F = \nabla f$,

then
$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \quad \forall i, j$$

AND

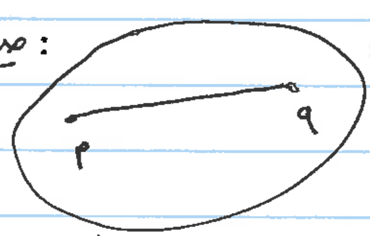
(2) $\left.\begin{array}{l} \text{If } \dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i} \ \forall i, j \\ \underline{\underline{\text{and}}} \\ U \text{ is a convex set} \end{array}\right\} \Longrightarrow$ F is conservative $F = \nabla f$ on U.

What is a convex set?

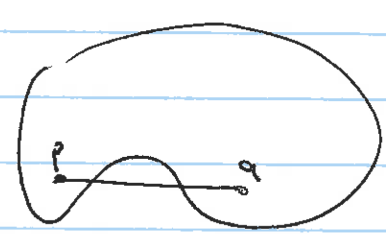
<u>Defn</u> U is convex if $\forall\, p, q \in U$,

the line segment $\overline{pq} \subseteq U$.

<u>Convex</u>:



$\forall\, pq$.



For some p, q $\overline{pq} \not\subseteq U$

not Convex.

Obs:

$n=2$ $\quad F=(P,Q)$ $\qquad F(\underset{x_1}{x},\underset{x_2}{y}) = (\underset{F_1}{P(x,y)}, \underset{F_2}{Q(x,y)})$

$$\underline{\underline{\forall ij}} \quad \frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \iff \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$n=3$ $\quad F = (\underbrace{F_1, F_2, F_3}_{\text{components.}})$ $\qquad F(\underbrace{x,y,z}_{\text{variables}})$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \ \forall ij \iff \text{curl } F = 0.$$

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p448 6.3 $\underline{\underline{\text{Ex}}}$ #4 $\quad F = 2x\sin y \, i + x^2\cos y \, j$

Is it conservative?

What is f s.t. $\nabla f = F$?

$$(\underbrace{2x\sin y}_{P=F_1}, \underbrace{x^2\cos y}_{F_2=Q})$$

$$2x\cos y = \frac{\partial}{\partial y} 2x\sin y \overset{?}{=} \frac{\partial}{\partial x} x^2\cos y = 2x\cos y \qquad \text{Yes}, =\checkmark$$

Domain $= \mathbb{R}^2$ convex. ✓

(*) $\underline{\text{Thm:}} \Rightarrow \exists f \quad \nabla f = (2x\sin y, x^2\cos y) = (f_x, f_y)$

$$f_x = 2x\sin y \xrightarrow{\int dx} f = x^2\sin y + c_1(y)$$

$$f_y = x^2\cos y \xrightarrow{\int dy} f = x^2\sin y + c_2(x)$$

$f = x^2\sin y$