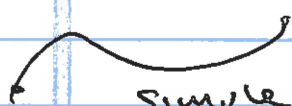
6.2 GREEN'S THEOREM

Defn A curve $\vec{x} : [a,b] \to \mathbb{R}^n$ is called

- simple if $\vec{x}$ is a 1-1 function;
- closed if $\vec{x}(a) = \vec{x}(b)$
- simple-closed if
  - it is closed and
  - x is 1-1 on $[a,b)$

$x(a) = x(b)$

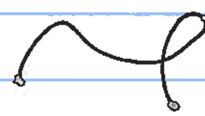


simple

closed

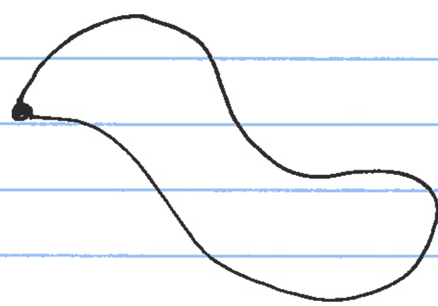
$x(a) = x(b)$

not simple

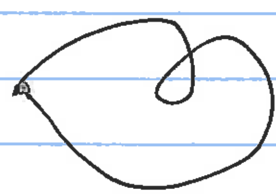


not closed.

$x(a)=x(b)$

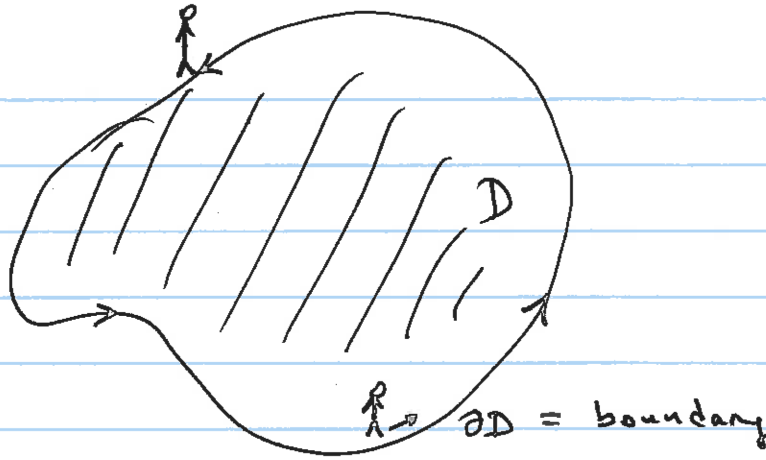


simple closed curve



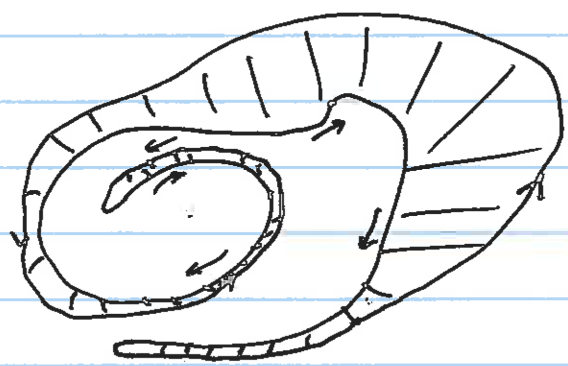
not a simple closed curve

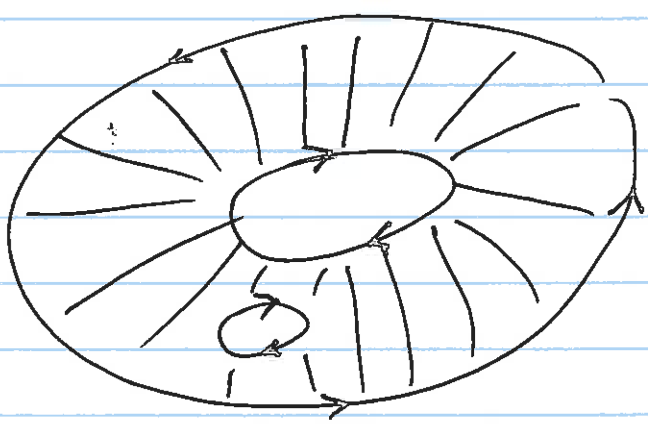
How to positively orient boundary



$D \subseteq \mathbb{R}^2$

If one walks along ∂D, then D stays on the left.





THEOREM (GREEN)

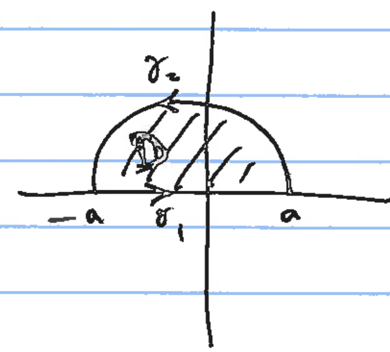
Let D be a closed and bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple closed curves which are piecewise diff'ble, and positively oriented.

Let $P, Q : D \to \mathbb{R}$ be defined on all of D and continuously diff'ble. Then

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

p 436 Exc # 4

$$F = 2y\vec{i} + x\vec{j}$$
$$F = (P, Q)$$
$$P = 2y$$
$$Q = x$$



$x^2 + y^2 \le a^2$
$y \ge 0$

Want $\displaystyle\int_{\partial D} P\,dx + Q\,dy$

$\gamma_1(t) = (t, 0) \quad -a \le t \le a.$

$\gamma_2(t) = (a\cos t, a\sin t) \quad 0 \le t \le \pi$

$\gamma_1(t)$: $\quad x = t \quad dx = dt$

$\quad y = 0 \quad dy = 0 \cdot dt$

$$\int_{\gamma_1} P\,dx + Q\,dy = \int_{\gamma_1} 2y\,dx + x\,dy = \int_{-a}^{a} 0 \cdot dt + t \cdot 0\,dt = 0$$

$$\int_{\gamma_2} P\,dx + Q\,dy = \int_{\gamma_2} 2y\,dx + x\,dy$$

$$x = a\cos t \qquad dx = -a\sin t\,dt$$
$$y = a\sin t \qquad dy = a\cos t\,dt$$

$$= \int_0^{\pi} 2\underbrace{(a\sin t)}_{y}\cdot\underbrace{(-a\sin t\,dt)}_{dx} + \underbrace{(a\cos t)}_{x}\underbrace{(a\cos t\,dt)}_{dy}$$

$$= a^2 \int_0^{\pi} -2\sin^2 t + \cos^2 t\,dt$$

$$= a^2 \int_0^{\pi} \left(-2\cdot\frac{1-\cos 2t}{2} + \frac{1+\cos 2t}{2}\right)dt$$

$$= a^2 \int_0^{\pi} \left(-1 + \cos 2t + \frac{1}{2} + \frac{\cos 2t}{2}\right)$$

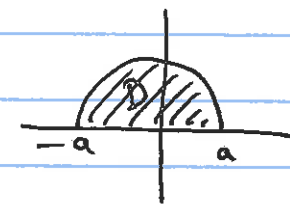
$$= a^2 \int_0^{\pi} \left(-\frac{1}{2} + \frac{3}{2}\cos 2t\right)dt$$

$$= a^2 \left(-\frac{1}{2}t + \frac{3}{4}\sin 2t\right)\Big|_0^{\pi}$$

$$= a^2\left(-\frac{\pi}{2} + 0\right) = -\frac{\pi a^2}{2}$$

Green's
$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$P = 2y \qquad \frac{\partial P}{\partial y} = 2$$
$$Q = x \qquad \frac{\partial Q}{\partial x} = 1$$



$$\iint_D (1-2)\,dA = \iint_D -1\,dA = -\text{area } D$$
$$= -\frac{\pi a^2}{2}$$

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recall
6.1
$$\int_{\vec{x}} \vec{F}\cdot d\vec{s} = \int_a^b F(\vec{x}(t))\cdot \vec{x}\,'(t)\,dt$$

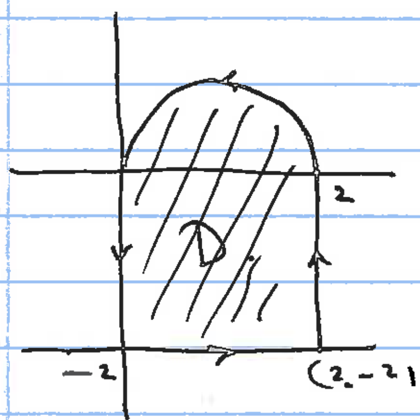
$$F = (P, Q)$$

$$= \int_a^b \big(P(x(t)), Q(x(t))\big)\cdot\big(\underbrace{x'(t)}_{\frac{dx}{dt}}, \underbrace{y'(t)}_{\frac{dy}{dt}}\big)\,dx$$

$$= \int_a^b P\,\frac{dx}{dt}\,dt + Q\,\frac{dy}{dt}\,dt$$

$$= \int_a^b P\,dx + Q\,dy$$

Exc #8 p 437



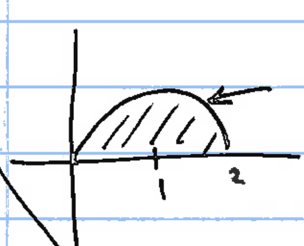
$$\int_{\partial D} F\, d\vec{s}$$

$$F = (\underbrace{3xy}_{P}, \underbrace{2x^2}_{Q})$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4x - 3x = x$$

Green's Thm:
$$\int_{\partial D} F \cdot ds = \int_{\partial D} P\,dx + Q\,dy = \iint_D x\, dA.$$

$$= \int_{-2}^{0}\int_{0}^{2} x \; dx\,dy + \int_0^2 \int_0^{\sqrt{1-(x-1)^2}} x \cdot dy\,dx$$



$(x-1)^2 + y^2 = 1.$

$y = \sqrt{1-(x-1)^2}$

$$= 4 + \int_0^2 x\sqrt{1-(x-1)^2}\,dx = 4 + \int_{-1}^{1} (u+1)\sqrt{1-u^2}$$

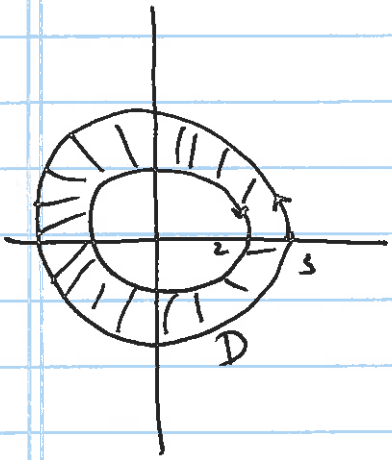
$u = x-1$

$du = dx$

$$= 4 + \int_{-1}^{1} \underbrace{u\sqrt{1-u^2}}_{odd}\,du + \int_{-1}^{1}\sqrt{1-u^2}\,du = 4 + 0 + \frac{\pi}{2}.$$

Exc #6 p 437.

$$\int F \cdot ds = \int (\underbrace{x^2y + x}_{P},\ \underbrace{y^3 - xy^2}_{Q}) \cdot \vec{x}'(t)\, dt$$



$$\frac{\partial Q}{\partial x} = -y^2$$

$$\frac{\partial P}{\partial y} = x^2.$$

Green's Thm $\displaystyle\int P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$$= \iint_D (-y^2 - x^2)\, dA$$

Use polar Coordinates

Jacobian

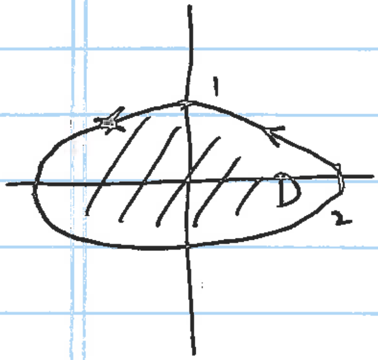
$$= \int_0^{2\pi}\int_2^3 -r^2 \cdot r \cdot dr\, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_2^3 -r^3\, dr\right)$$

$$= 2\pi \cdot \frac{-r^4}{4}\bigg|_2^3 = -\frac{\pi}{2}(81 - 16) = -\frac{65\pi}{2}.$$

p 437 #10

$$I = \int (\underbrace{4y-3x}_{P},\ \underbrace{x-4y}_{Q}) \cdot \vec{x}'(t)\,dt$$

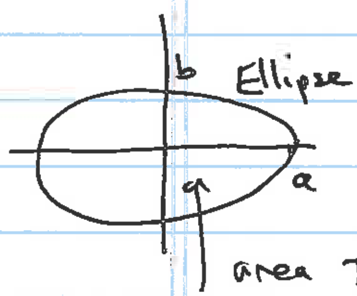


$$x^2 + 4y^2 = 4$$

$$\frac{x^2}{4} + y^2 = 1$$

$$I = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (1-4)\, dA$$

$$= \iint_D -3\, dA = -3 \cdot \underbrace{\text{area}(D)}_{2\pi} = -6\pi.$$



area πab