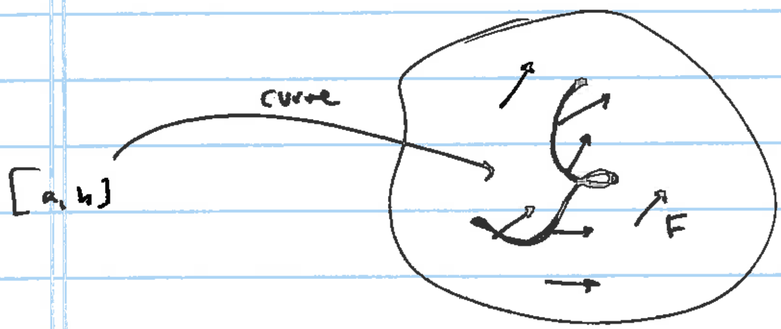
(6.1) Continue

Part II. (Vector) Line integrals

Def^n: Let $\vec{x} : [a,b] \longrightarrow X \subseteq \mathbb{R}^n$ be a piecewise diff'ble curve

$$F : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

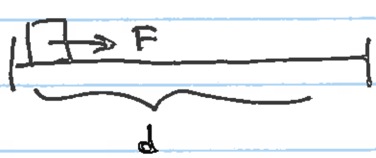


$F : X \longrightarrow \mathbb{R}^n$ vector field

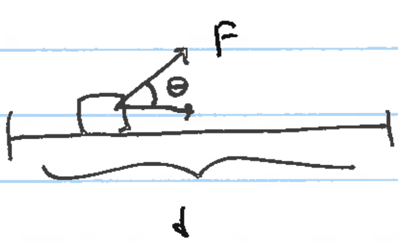
The vector line integral $\displaystyle\int_{\vec{x}} \vec{F} \cdot d\vec{s} := \int_a^b \vec{F}(\vec{x}(t)) \cdot \underbrace{x'(t)\,dt}_{d\vec{s}}$. (A)

$$\left(\text{compare} \quad \int_{\vec{x}} f\,ds := \int_a^b f(\vec{x}(t))\,\underbrace{\|x'(t)\|\,dt}_{ds} \right)$$

Motivation from Physics

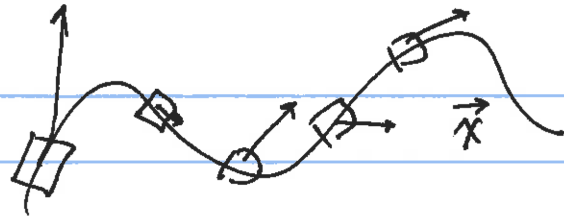


$W = Fd$ work, force, distance



$W = F\cos\theta \cdot d$

②



$\vec{x} : [a,b] \to \mathbb{R}^3$ path

If F changes direction, magnitude, path is curved

then Work $= W = \int_a^b F \cdot d\vec{s} := \int_a^b F(\vec{x}(t)) \cdot x'(t)\,dt$

(B) Another Notation:

$$\int \vec{F} \cdot d\vec{s} = \int_a^b F(\vec{x}(t)) \cdot \underbrace{x'(t)\,dt}_{d\vec{s}} = \int F \cdot \underbrace{T\,ds}_{d\vec{s}}$$

(B)

$T = \frac{x'(t)}{|x'(t)|}$ unit tangent vector

Examples:

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$\vec{F} = xi + yj + zk$

$\vec{x}(t) = (2t+1, t, 3t-1) \qquad 0 \le t \le 1$

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_0^1 (2t+1, t, 3t-1) \cdot (2,1,3)\, dt$$

$x'(t) = (2,1,3)$

$F(x(t)) = (2t+1, t, 3t-1)$

$$= \int_0^1 (4t + 2 + t + 9t - 3)\,dt = \int_0^1 14t - 1\,dt = 6.$$

Exc #12 $\quad F = x\,i + xy\,j + xyz\,k \qquad 0 \le t \le 2\pi.$

$$\vec{x}(t) = (\underbrace{3\cos t}_{x}, \underbrace{3\sin t}_{y}, \underbrace{5t}_{z})$$

$$\vec{x}\,'(t) = (-3\sin t,\ 3\cos t,\ 5)$$

$$F(\vec{x}(t)) = (3\cos t,\ 9\cos t\sin t,\ 5t\cdot 9\cos t\sin t)$$

$$\int_0^{2\pi} \underbrace{(3\cos t,\ 9\cos t\sin t,\ 45t\cos t\sin t)}_{F(\vec{x}(t))} \cdot \underbrace{(-3\sin t,\ 3\cos t,\ 5)}_{\vec{x}\,'(t)}\,dt$$

$$= \int_0^{2\pi} -9\cos t\sin t + 27\cos^2 t\sin t + 225\,t\cos t\sin t\ dt$$

$u = \cos t$

$$= \underbrace{\frac{9\cos^2 t}{2} + \frac{-27\cos^3 t}{3}\Bigg|_0^{2\pi}}_{0} + 225\int_0^{2\pi} t\cos t\sin t\ dt$$

$$= 225\cdot\left(-\frac{\pi}{2}\right); \quad \text{since}$$

$$\int_0^{2\pi} t\cos t\sin t = \frac{-t}{4}\cos 2t\Big|_0^{2\pi} - \int_0^{2\pi} -\frac{1}{4}\cos 2t\,dt$$

By Parts

$t = u \qquad \cos t\sin t = dv = \frac{1}{2}\sin 2t$

$dt = du \qquad v = \frac{-1}{4}\cos 2t$

$$= -\frac{\pi}{2} + \underbrace{\int_0^{2\pi} \frac{1}{4}\cos 2t\,dt}_{0} = -\frac{\pi}{2}.$$

(c) Another way to write Line integrals

$$\int_{\vec{x}} P\,dx + Q\,dy + R\,dz = \int_{\vec{x}} \underbrace{(P,Q,R)}_{F}\cdot\underbrace{\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)}_{\vec{x}'(t)}dt$$

$$= \int_{\vec{x}} F\cdot x'(t)\,dt$$

$$= \int_{\vec{x}} F\cdot d\vec{s}$$

Ex.

$$\int_{\vec{x}} y^2\,dx + xz\,dy - z^2\,dz$$

$\vec{x}(t) = (\underbrace{2t^2}_{x}, \underbrace{t-1}_{y}, \underbrace{3t}_{z})$ $\qquad 0 \le t \le 2$

Method I $\qquad F = (y^2, xz, -z^2)$

$F(\vec{x}(t)) = ((t-1)^2, 6t^3, -9t^2)$

$$\int_0^2 F(\vec{x}(t))\cdot x'(t)\,dt = \int_0^2 ((t-1)^2, 6t^3, -9t^2)\cdot(4t, 1, 3)\,dt$$

Method II

$x = 2t^2 \qquad dx = 4t\,dt$

$y = t-1 \qquad dy = dt$

$z = 3t \qquad dz = 3\,dt$

plug them in

compare

$$\int_0^2 (t-1)^2\,4t\,dt + (2t^2\cdot 3t)\,dt + -(3t)^2\cdot 3\,dt$$

compare

$$\overbrace{((t-1)^2,\ 2t^2\cdot 3t,\ -(9t^2))}^{F(\vec{x}(t))}\cdot\overbrace{(4t, 1, 3)}^{x'(t)}\,dt$$

$$= \int_0^2 (t^2-2t+1)4t + 6t^3 - 27t^2 \; dt$$

$$= \int_0^2 (4t^3 - 8t^2 + 4t + 6t^3 - 27t^2)\, dt$$

$$= \int_0^2 (10t^3 - 35t^2 + 4t)\, dt$$

$$= \frac{10}{4}t^4 - \frac{35}{3}t^3 + 2t^2 \Big|_0^2$$

$$= \frac{10}{4}\cdot 16 - \frac{35}{3}\cdot 8 + 8 = 48 - \frac{280}{3}$$

$$= \frac{144-280}{3} = \frac{-136}{3}.$$