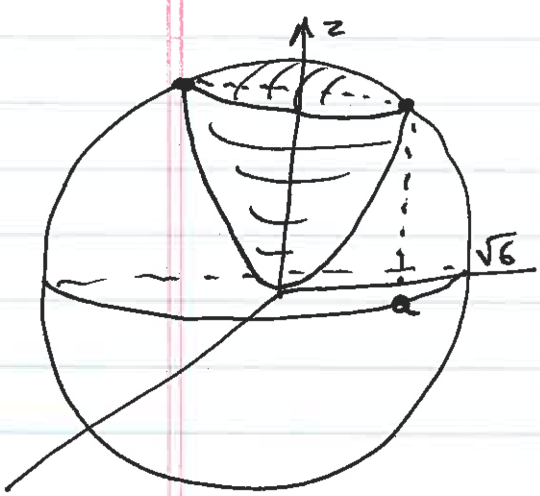
CYLINDRICAL COORDINATES CONTINUE

Ex Let W be the solid region inside the paraboloid $z = x^2+y^2$, and the sphere $x^2+y^2+z^2 = 6$. Let a substance X fill the region W, and the density of X at a point (x,y,z) is $2z$.

(i) Find the volume of W

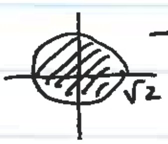
(ii) Find the mass of X.



$$z = x^2+y^2 = r^2$$
$$z = \sqrt{6-x^2-y^2} = \sqrt{6-r^2}$$

To find a

$$z = r^2 = \sqrt{6-r^2}$$
$$r^4 = 6-r^2$$
$$r^4+r^2-6 = 0$$
$$(r^2+3)(r^2-2) = 0$$
$$a = r = \sqrt{2}.$$



Top view

(i) $$V = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{\sqrt{6-r^2}} 1 \cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} rz \Big|_{z=r^2}^{z=\sqrt{6-r^2}} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \left(r(6-r^2)^{\frac{1}{2}} - r^3\right) dr\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}\cdot\frac{2}{3}(6-r^2)^{\frac{3}{2}} - \frac{r^4}{4} \Big|_0^{\sqrt{2}}$$

$$= \int_0^{2\pi} d\theta \left[\left(-\frac{1}{3} 4^{\frac{3}{2}} - \frac{(\sqrt{2})^4}{4} \right) - \left(-\frac{1}{3} 6^{\frac{3}{2}} - 0 \right) \right]$$

$$= 2\pi \cdot \left(-\frac{8}{3} - 1 + \frac{1}{3} 6\sqrt{6} \right) = \frac{2\pi}{3} \left(6\sqrt{6} - 11 \right)$$

(ii) Mass $\displaystyle\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{\sqrt{6-r^2}} 2zr \, dz \, dr \, d\theta$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} r z^2 \Big|_{z=r^2}^{z=\sqrt{6-r^2}} dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} \underbrace{r(6 - r^2 - r^4)}_{6r - r^3 - r^5} dr \, d\theta$$

$$= \int_0^{2\pi} 3r^2 - \frac{r^4}{4} - \frac{r^6}{6} \Big|_0^{\sqrt{2}} d\theta$$

$$= \int_0^{2\pi} d\theta \left(6 - 1 - \frac{8}{6} \right)$$

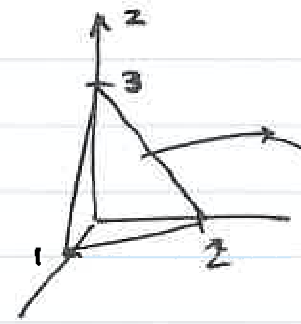
$$= 2\pi \cdot \left(5 - \frac{4}{3} \right) = \frac{22\pi}{3}.$$

Material related to MT 2:

5.4 Exc #15 p 348

$f(x,y,z) = 1 - z^2$

W tetrahedron



$6x + 3y + 2z = 6$

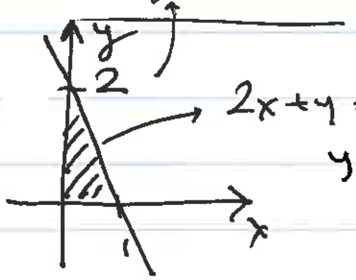
$z = \frac{6 - 3y - 6x}{2}$

Calculate $\iiint_W (1 - z^2)\, dV$

$dz\,dy\,dx$

$$\int_0^1 \int_0^{2-2x} \int_0^{3 - \frac{3}{2}y - 3x} (1 - z^2)\, dz\, dy\, dx \longrightarrow \text{Very Long}$$

View from z-axis direction $\longrightarrow$



$2x + y = 2$

$y = 2 - 2x$

Try another order of integration

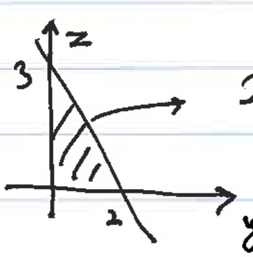
$$\int_0^3 \int_0^{2 - \frac{2}{3}z} \int_0^{1 - \frac{1}{2}y - \frac{1}{3}z} (1 - z^2)\, dx\, dy\, dz$$

$x = \frac{6 - 3y - 2z}{6}$

$x = 1 - \frac{1}{2}y - \frac{1}{3}z$

Top viewed from x-axis.

View from x-axis direction



$2z + 3y = 6$

$y = \frac{6 - 2z}{3} = 2 - \frac{2}{3}z$

$$I = \int_0^3 \int_0^{2-\frac{2}{3}z} (1-z^2)\,x \Big|_{x=0}^{x=1-\frac{1}{2}y-\frac{1}{3}z} \, dy\, dz .$$

$$= \int_0^3 \int_0^{2-\frac{2}{3}z} (1-z^2)\left(1-\frac{1}{2}y-\frac{1}{3}z\right) dy\, dz .$$

$$= \int_0^3 (1-z^2)\left(y-\frac{1}{4}y^2-\frac{1}{3}yz\right)\Big|_{y=0}^{y=2-\frac{2}{3}z} dz .$$

$$= \int_0^3 (1-z^2)\, y \left(1-\frac{1}{4}y-\frac{1}{3}z\right)\Big|_{y=0}^{y=2-\frac{2}{3}z} dz$$

$$= \int_0^3 (1-z^2)\left(2-\frac{2}{3}z\right)\left(1-\underbrace{\frac{1}{4}\left(2-\frac{2}{3}z\right)-\frac{1}{3}z}\right) dz$$

$$\underbrace{\left(1-\frac{1}{2}+\frac{1}{6}z-\frac{1}{3}z\right)}$$

$$\underbrace{\left(\frac{1}{2}-\frac{1}{6}z\right)}$$

$$1-z^2 \qquad 2\left(1-\frac{1}{3}z\right) \qquad \frac{1}{2}\left(1-\frac{1}{3}z\right)$$

$$= \int_0^3 (1-z^2)\left(1-\frac{1}{3}z\right)^2 dz .$$

$$= \int_0^3 (1-z^2)\left(1-\frac{2}{3}z+\frac{1}{9}z^2\right)dz.$$

$$= \int_0^3 1-\frac{2}{3}z+\frac{1}{9}z^2-z^2+\frac{2}{3}z^3-\frac{1}{9}z^4\,dz$$

$$= \int_0^3 1-\frac{2}{3}z-\frac{8}{9}z^2+\frac{2}{3}z^3-\frac{1}{9}z^4\,dz$$

$$= z-\frac{1}{3}z^2-\frac{8}{27}z^3+\frac{1}{6}z^4-\frac{1}{45}z^5\Big|_0^3$$

$$= 3-\frac{9}{3}-\frac{8}{\cancel{27}}\cdot\cancel{27}+\frac{1}{\underset{2}{\cancel{6}}}\cdot\overset{27}{\cancel{81}}-\frac{1}{5\cdot\cancel{9}}\cancel{9}\cdot 27$$

$$= \cancel{3}-\cancel{3}-8+\frac{27}{2}-\frac{27}{5}$$

$$= -8+27\left(\frac{1}{2}-\frac{1}{5}\right) = -8+27\cdot\frac{3}{10}$$

$$= \frac{-80+81}{10} = \frac{1}{10} \checkmark$$

MT2 Practice test #11

$$\left.\begin{aligned} x^2+y^2+z^2-w&=1 \\ x-y+z+w^2&=5 \end{aligned}\right\}$$

$$F(x,y,z,w) = (x^2+y^2+z^2-w,\ x-y+z+w^2)$$

$$DF = \begin{bmatrix} 2x & 2y & 2z & -1 \\ 1 & -1 & 1 & 2w \end{bmatrix}$$

$$DF(\underset{x}{\overset{\shortparallel}{1}},\underset{y}{\overset{\shortparallel}{1}},\underset{z}{\overset{\shortparallel}{1}},\underset{w}{\overset{\shortparallel}{2}}) = \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \begin{bmatrix} 2 & 2 & 2 & -1 \\ 1 & -1 & 1 & 4 \end{bmatrix} \end{array}$$

$(x,y) = g(z,w)$

Want (x,y) to be solved

$$\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} = -4 \neq 0$$

By Imp. F. Thm $\exists\, g$ locally solving $(x,y) = g(z,w)$ via $F(x,y,z,w) = (1,5)$

$$g'(1,1,1,2) = -\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$$

$$\text{locally: } g' = -\begin{bmatrix} 2x & 2y \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 2z & -1 \\ 1 & 2w \end{bmatrix}$$

$$= \frac{-1}{-2x-2y}\begin{bmatrix} -1 & -2y \\ -1 & 2x \end{bmatrix}\begin{bmatrix} 2z & 1 \\ 1 & 2w \end{bmatrix} = \frac{1}{2x+2y}\begin{bmatrix} -2z-2y & -1-4yw \\ -2z+2x & -1+4x \end{bmatrix}$$