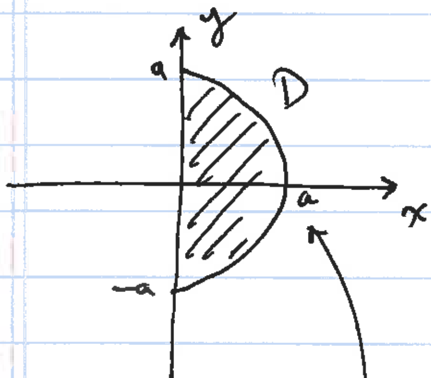
S+S Exc #16 p 372

Correct Version

Compare to the next page (done in class.)

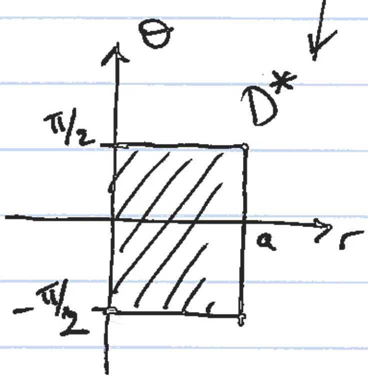
$$I = \int_{-a}^{a} \int_{0}^{\sqrt{a^2-y^2}} e^{x^2+y^2}\, dx\, dy$$



$-a \le y \le a$

$0 \le x \le \sqrt{a^2-y^2}$

$$\left.\begin{array}{l} r\cos\theta = x \\ r\sin\theta = y \end{array}\right] \longleftrightarrow \begin{array}{l} \sqrt{x^2+y^2} = r \\ \theta = \tan^{-1}\frac{y}{x} + k\pi \end{array}$$



$0 \le r \le a$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$I = \int_0^a \int_{-\pi/2}^{\pi/2} e^{r^2} r\, d\theta\, dr$$

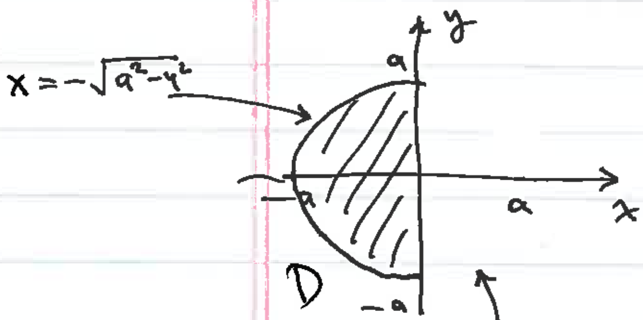
$$= \left(\int_0^a e^{r^2} r\, dr\right) \underbrace{\left(\int_{-\pi/2}^{\pi/2} 1\, d\theta\right)}_{\pi} = \left(\int_0^{a^2} \frac{1}{2} e^u\, du\right) \cdot \pi$$

$$= \frac{\pi}{2}\left(e^{a^2} - 1\right).$$

Corrected (1b)

Example done in class, look at the * bounds

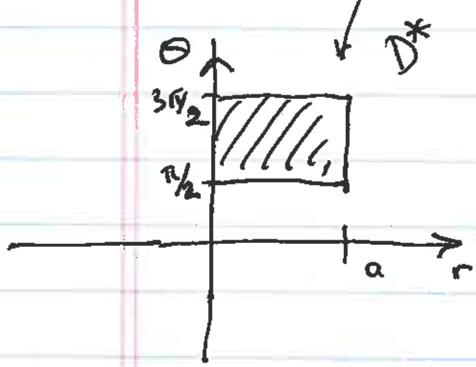
$$I = \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{0} e^{x^2+y^2}\, dx\, dy$$



$-a \le y \le a$

$-\sqrt{a^2-y^2} \le x \le 0$

$\left.\begin{array}{l} r\cos\theta = x \\ r\sin\theta = y \end{array}\right] \longleftrightarrow \begin{array}{l} \sqrt{x^2+y^2} = r \\ \tan\theta = \frac{y}{x} \end{array}$



$0 \le r \le a$

$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

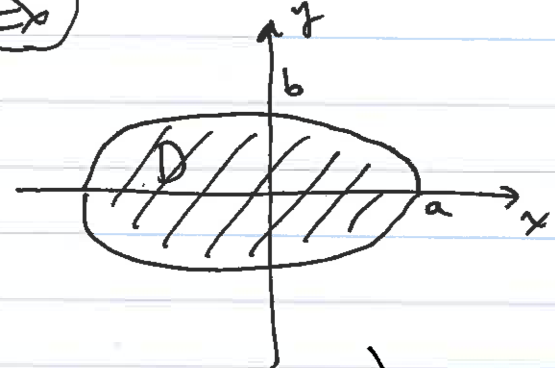
$$I = \int_0^a \int_{\pi/2}^{3\pi/2} e^{r^2}\, r\, d\theta\, dr$$

$$= \left(\int_0^a e^{r^2} r\,dr\right)\underbrace{\left(\int_{\pi/2}^{3\pi/2} 1 \cdot d\theta\right)}_{\pi} = \left(\int_0^{a^2} \tfrac{1}{2} e^u\, du\right)\cdot \pi$$

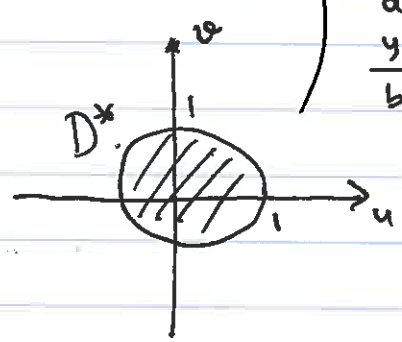
$u = r^2$
$du = 2r\,dr$

$$= \frac{\pi}{2}\left(e^{a^2} - 1\right)$$

Ex



Area enclosed by the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b > 0$$
$$(\text{Area} = \pi ab)$$



$$\frac{x}{a} = u, \quad \frac{y}{b} = v \qquad \text{OR} \qquad x = au, \quad y = bv$$

$$\text{Area enclosed by ellipse } D = \iint_D 1\, dx\,dy = \iint_{D^*} 1 \cdot \underbrace{ab\, du\,dv} = ab \underbrace{\iint_{D^*} 1\, du\,dv}_{\pi = \text{area of the unit disc/circle.}} = \pi ab$$

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab.$$

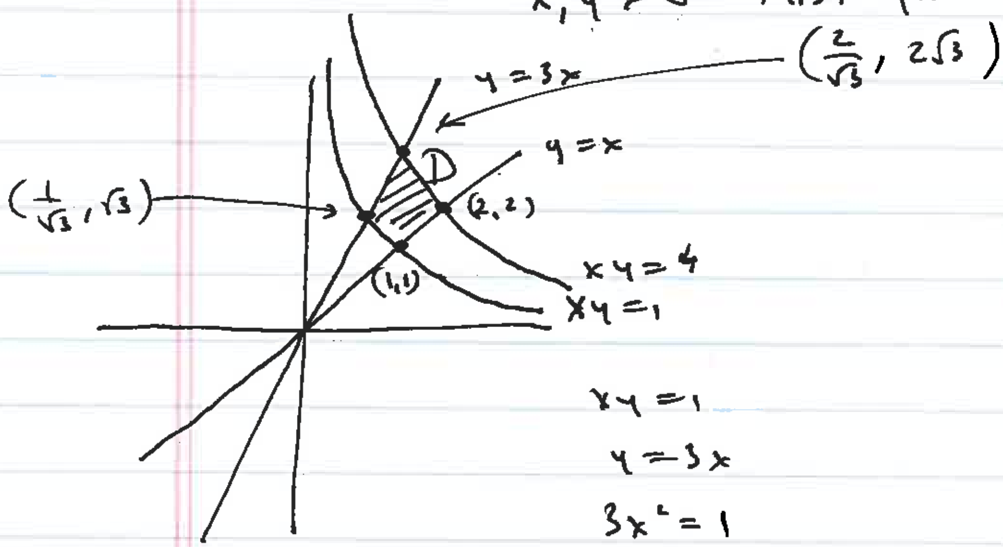
Ex

Calculate $\displaystyle\iint_D y^2 e^{y^2}\,dy\,dx$

where D is the region bounded by

$$\left.\begin{aligned} y &= x \\ y &= 3x \\ xy &= 4 \\ xy &= 1 \end{aligned}\right\}$$

$x, y \geq 0$ First quadrant



$xy = 1$

$y = 3x$

$3x^2 = 1$

$x = \frac{1}{\sqrt{3}} \quad y = \sqrt{3}$

$xy = 4$

$y = 3x$

$3x^2 = 4$

$x^2 = \frac{4}{3}$

$x = \frac{2}{\sqrt{3}} \quad y = 2\sqrt{3}$

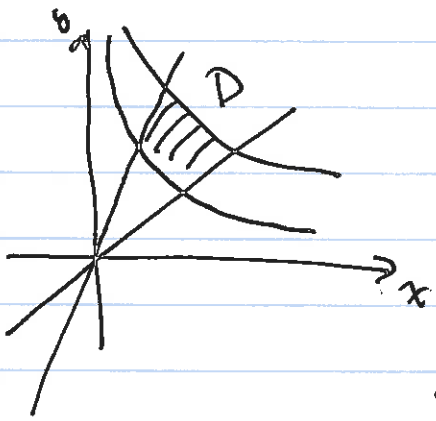
$1 \leq xy \leq 4 \longrightarrow xy = u$

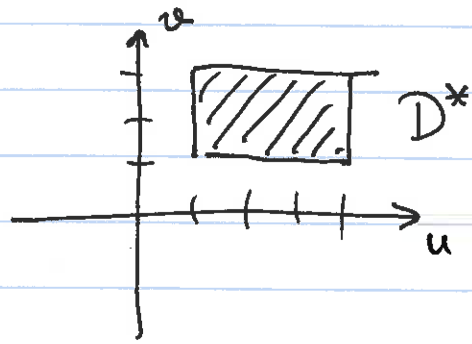
$1 \leq \frac{y}{x} \leq 3 \longrightarrow \frac{y}{x} = v$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{y}{x} + \frac{y}{x} = \frac{2y}{x}.$$

$$du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$$

$$\boxed{du\,dv = \frac{2y}{x}\,dx\,dy}$$





$u = xy$ $\qquad$ $1 \le u \le 4$

$v = \frac{y}{x}$ $\qquad$ $1 \le v \le 3$

$$\iint_D y^2 e^{y^2}\,dy\,dx = \int_1^4 \int_1^3 uv \cdot e^{uv}\ ?\ dv\,du$$

$$\frac{\partial(x,y)}{\partial(u,v)}$$

$$y^2 = u \cdot v =$$

$$du\,dv = \frac{2y}{x}\,dx\,dy$$

$$\frac{1}{2v}\,du\,dv = \frac{x}{2y}\,du\,dv = dx\,dy$$

$$v = \frac{y}{x} \qquad \frac{1}{v} = \frac{x}{y}$$

$$I = \int_1^4 \int_1^3 u\cancel{v} \cdot e^{uv} \frac{1}{2\cancel{v}} \, dv \, du$$

$$= \frac{1}{2} \int_1^4 \int_1^3 \underbrace{u e^{uv} \, dv} \, du$$

$$= \frac{1}{2} \int_1^4 \left(e^{uv} \Big|_{v=1}^{v=3} \right) du$$

$$= \frac{1}{2} \int_1^4 \left(e^{3u} - e^{u} \right) du$$

$$= \frac{1}{2} \left(\frac{1}{3} e^{3u} - e^{u} \right) \Big|_{u=1}^{u=4}$$

$$= \frac{1}{2} \left(\frac{1}{3} e^{12} - e^{4} - \frac{1}{3} e^{3} + e \right)$$