Nov 3

5.5

2 variable substitutions in Double Integrals.

Def Let $T : U^{open} \subseteq \mathbb{R}^n \longrightarrow V^{open} \subseteq \mathbb{R}^n$ be

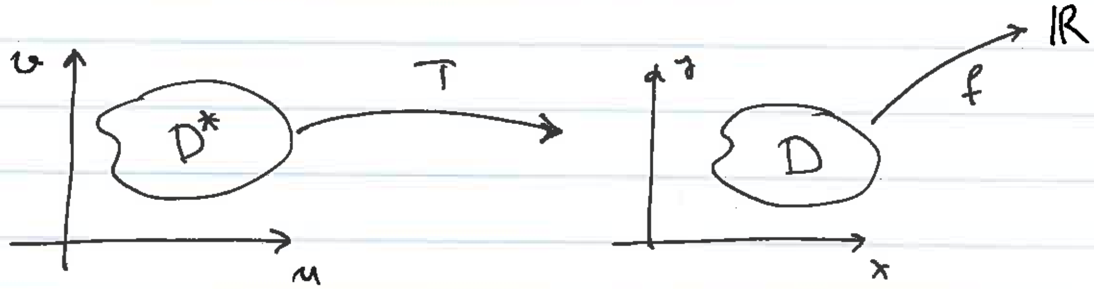
- continuously diff'ble,
- 1–1 and onto V, and
- $\det(DT) \neq 0$.

Then T is called a change of coordinates.

Def $T(x_1, x_2 .. x_n) = (y_1, y_2 \cdots y_n)$

$$\frac{\partial(y_1, y_2, y_3 \cdots, y_n)}{\partial(x_1, x_2, \cdots x_n)} = \det DT.$$

Thm: (Jacobi) $n=2$ Case



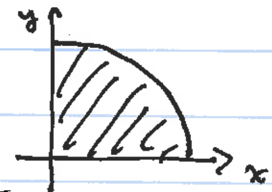
Let $T : D^* \longrightarrow D$ be a change of coordinates
Let $f : D \longrightarrow \mathbb{R}$ be continuous

$$\iint_D f(x,y)\, \underbrace{dA}_{dx\,dy} = \iint_{D^* = T^{-1}(D)} f(T(u,v)) \underbrace{|\det DT|}_{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|} \underbrace{dA}_{(du\,dv)}.$$

Obs: $dx\,dy = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

Ex 1

$$\int_0^2 \int_0^{\sqrt{4-y^2}} \sqrt{x^2+y^2}\, dx\, dy$$



$0 \le y \le 2$

$0 \le x \le \sqrt{4-y^2}$

$T \downarrow$

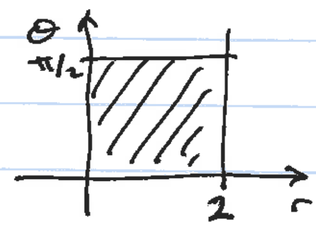
$0 \le r \le 2$

$0 \le \theta \le \frac{\pi}{2}$

$$T \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$\left|\dfrac{\partial(x,y)}{\partial(r,\theta)}\right|$

$$\int_0^2 \int_0^{\pi/2} r \cdot r \cdot d\theta\, dr$$



$$\sqrt{x^2+y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r$$

$$\det DT = r = \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$dx\,dy = \underbrace{\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|}_{r} dr\,d\theta = r\,dr\,d\theta$$

$$\int_0^2 \int_0^{\pi/2} r^2\, d\theta\, dr. = \left(\int_0^2 r^2\,dr\right)\left(\int_0^{\pi/2} d\theta\right)$$

$$= \left.\frac{r^3}{3}\right|_0^2 \cdot \frac{\pi}{2} = \frac{8}{3}\cdot\frac{\pi}{2} = \frac{4\pi}{3}$$

Prop

$$\int_a^b \int_c^d f(x)\cdot g(y)\,dy\,dx$$

$$= \left(\int_a^b f(x)\,dx\right)\left(\int_c^d g(y)\,dy\right)$$

CAUTION
- All bounds must be constants, AND
- $\underbrace{f(x)}_{\text{no } y}\cdot\underbrace{g(y)}_{\text{no } x}$ can be separated into two factors each of which is a function of only one variable or constant

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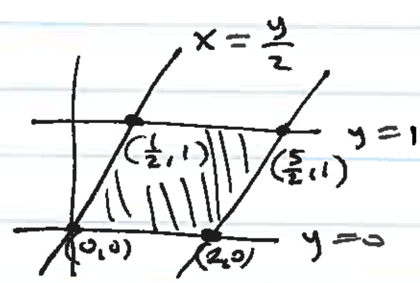
5.5 Exc # 8 p371

HW (a)

$$\int_0^1 \int_{y/2}^{(y/2)+2} (2x-y)\,dx\,dy$$

$0 \le y \le 1$

$y/2 \le x \le (y/2)+2$



$x = \left(\frac{y}{2}\right) + 2$

$2x - 4 = y$

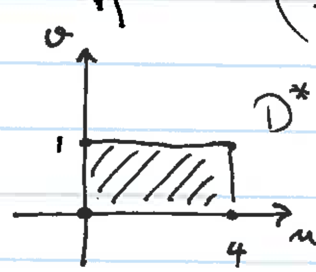
$T \quad T^{-1}$

(b) $\left.\begin{aligned} u &= 2x-y \\ v &= y \end{aligned}\right\}$ substitution

(T linear $\Rightarrow$ T takes lines to lines.)

Linear $T(x,y) = (2x-y,\ y) = (u,v)$

$T(0,0) = (0,0)$
$T(2,0) = (4,0)$
$T(\frac{5}{2},1) = (4,1)$
$T(\frac{1}{2},1) = (0,1)$



(c) $$\int_0^1 \int_0^4 u \cdot \frac{1}{2} \, du \, dv$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1 \Big/ \left|\frac{\partial(u,v)}{\partial(x,y)}\right|$$

$$\left.\begin{array}{l} u = 2x - y \\ v = y \end{array}\right\} \qquad \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2$$

$$= \int_0^1 \int_0^4 u \cdot \frac{1}{2} \, du \, dv$$

$$= \left(\int_0^4 u \, du\right)\left(\int_0^1 \frac{1}{2} \, dv\right) = \frac{u^2}{2}\Big|_0^4 \cdot \frac{1}{2} = 4.$$

<u>Obs</u>

If T is invertible, with T^{-1}; both diff'ble, then

$$(DT)^{-1} = D(T^{-1}) \quad \text{By Chain Rule.}$$

$$(\det DT) \cdot (\det D(T^{-1})) = 1$$

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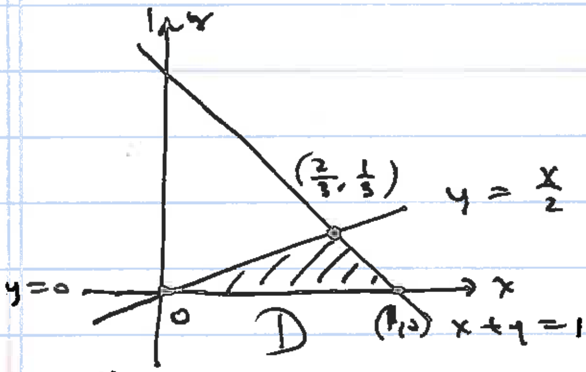
5.5 Exc #10 $\displaystyle\iint_D \sqrt{\frac{x+y}{x-2y}}\, dA$

$\left\{\begin{array}{l} y = \frac{x}{2} \\ y = 0 \\ x+y=1 \end{array}\right.$ D is bounded by these lines



$x + y = 1$

$x + \frac{x}{2} = 1$

$\frac{3}{2}x = 1$

$x = \frac{2}{3}$

$u = x + y$

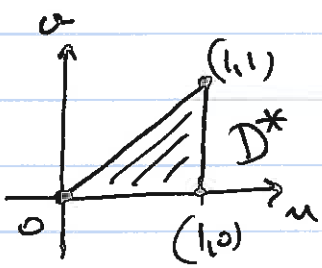
$v = x - 2y$

Linear T takes lines to lines

$T(0,0) = (0,0)$

$T(1,0) = (1,1)$

$T\left(\frac{2}{3}, \frac{1}{3}\right) = (1,0)$



$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -3$$

Type I $\left\{\begin{array}{l} 0 \le u \le 1 \\ 0 \le v \le u \end{array}\right.$

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy = du\,dv$$

$$3\, dx\,dy = du\,dv$$

$$dx\,dy = \frac{1}{3}\, du\,dv$$

$$\iint_D \sqrt{\frac{x+y}{x-2y}}\, dA \;(dx\,dy) = \int_0^1 \int_0^u \sqrt{\frac{u}{v}}\ \frac{1}{3}\ \underbrace{dv\,du}_{\text{Type I region, } D^*}$$

$$= \int_0^1 \int_0^u \frac{1}{3} u^{\frac{1}{2}} v^{-\frac{1}{2}}\, dv\,du.$$

$$= \int_0^1 \frac{1}{3} u^{\frac{1}{2}} v^{\frac{1}{2}} \cdot \frac{1}{1/2} \Bigg|_{v=0}^{v=u} du$$

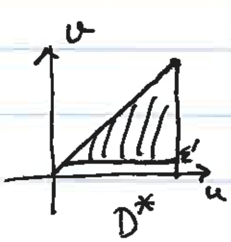
$$= \int_0^1 \frac{2}{3}\ u^{1/2}\, u^{1/2}\, du = \int_0^1 \frac{2}{3} u\, du = \frac{1}{3} u^2 \Big|_0^1$$

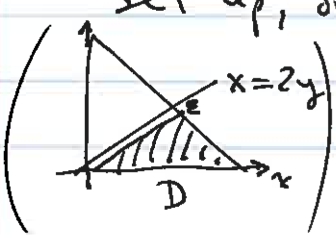
$$= \frac{1}{3}.$$

Caution: ① $\int_0^a x^{-\frac{1}{2}}\, dx$ is an improper integral:

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^a x^{-\frac{1}{2}} dx = \lim_{\varepsilon \to 0^+} 2x^{\frac{1}{2}} \Big|_\varepsilon^a = \lim_{\varepsilon \to 0^+} 2(\sqrt{a} - \sqrt{\varepsilon}) = 2\sqrt{a}$$

② Actually, the original integral $\iint \sqrt{\frac{x+y}{x-2y}}\, dA$ is improper, since $\frac{1}{\sqrt{x-2y}}$ is unbounded & undefined along the line $x = 2y$. In a formal set up, one needs to remove an ε-ribbon along $x=2y$ (





). Find the integral and let $\varepsilon \to 0$.