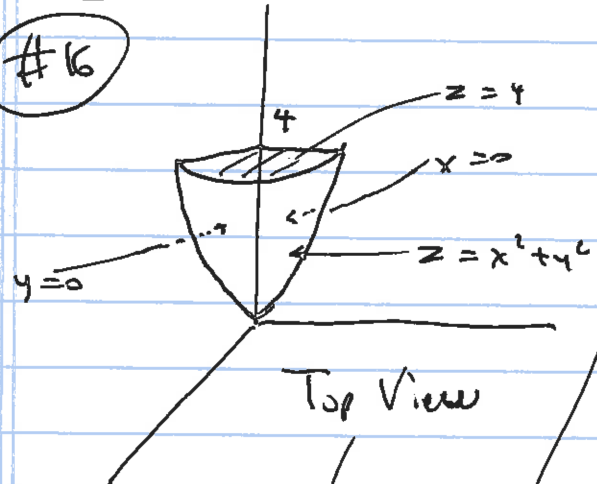
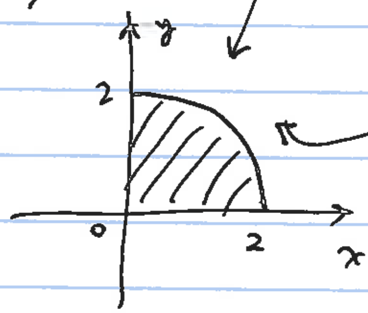
5.4 Examples

p348 #16



Top View



WANT:

$$\left\{\begin{array}{l} a \le x \le b \\ \sigma(x) \le y \le \delta(x) \end{array}\right.$$

$$\varphi(x,y) \le z \le \psi(x,y)$$

$$0 \le x \le 2$$

$$0 \le y \le \sqrt{4-x^2}$$

$$x^2+y^2 \le z \le 4$$

$$I = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} 3x \, dz \, dy \, dx$$

$$= \int_0^2 \int_0^{\sqrt{4-x^2}} 3xz \Big|_{z=x^2+y^2}^{z=4} dy \, dx$$

$$= \int_0^2 \int_0^{\sqrt{4-x^2}} 12x - 3x^3 - 3xy^2 \, dy \, dx.$$

$$= \int_0^2 12xy - 3x^3y - xy^3 \Big|_{y=0}^{y=\sqrt{4-x^2}} dx$$

$$= \int_0^2 \left(12x\sqrt{4-x^2} - 3x^3\sqrt{4-x^2} - x\left(\sqrt{4-x^2}\right)^3 - 0 \right) dx$$

$$= \int_0^2 \sqrt{4-x^2}\left[12x - 3x^3 - x(4-x^2) \right] dx$$

$$= \int_0^2 \sqrt{4-x^2}\left[12x - 3x^3 - 4x + x^3 \right] dx$$

$$= \int_0^2 \sqrt{(4-x^2)}\left[8x - 2x^3 \right] dx$$

$u = 4 - x^2$

$du = -2x\,dx$

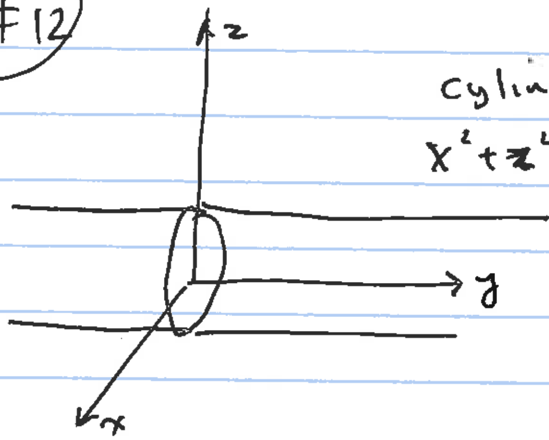
$$= \int_0^2 \sqrt{4-x^2} \cdot 2x\,dx \left[-x^2 + 4 \right]$$

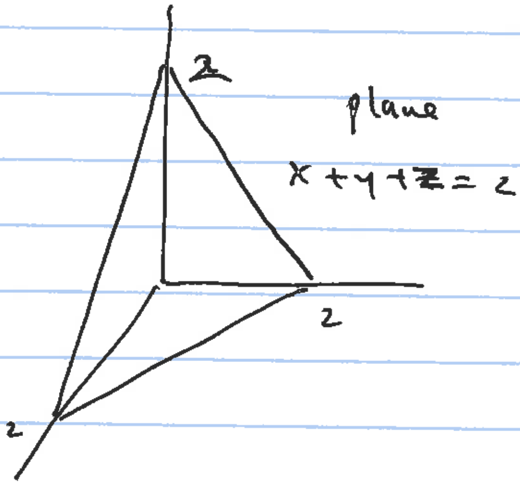
$$= \int_4^0 u^{1/2}(-du) \cdot u$$

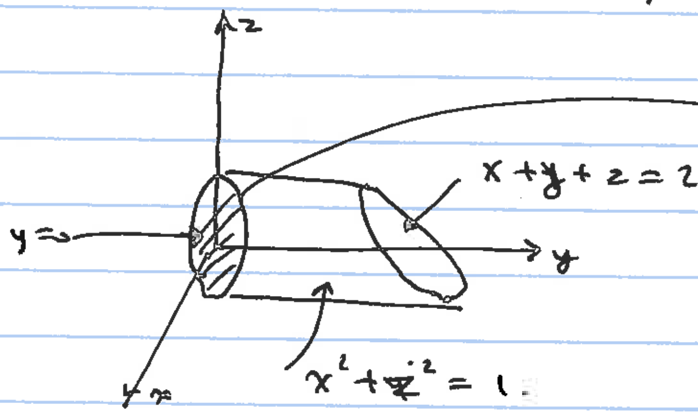
$$= \int_0^4 u^{3/2}\,du = u^{5/2} \cdot \frac{2}{5} \Big|_0^4$$

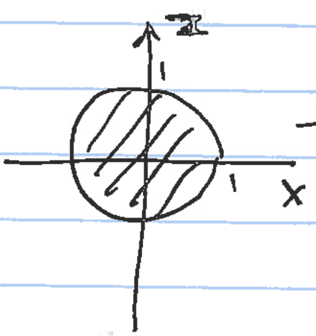
$$= \left(4^{5/2} - 0^{5/2}\right) \cdot \frac{2}{5} = \frac{64}{5}.$$

p 348 #12









$$\begin{cases} -1 \le x \le 1 \\ -\sqrt{1-x^2} \le z \le \sqrt{1-x^2} \\ 0 \le y \le 2-x-z \end{cases}$$

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{2-x-z} y \, dy \, dz \, dx$$

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{2} y^2 \Big|_{0}^{2-x-z}$$

5.4 ④

p348 #12

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{2-x-z} y \, dy \, dz \, dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \tfrac{1}{2}(2-x-z)^2 \, dz \, dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \tfrac{1}{2}\left(4 + x^2 + z^2 - 4x - 4z + 2zx\right) dz \, dx$$

$$= \int_{-1}^{1} \tfrac{1}{2}\left((4+x^2-4x)z + \frac{z^3}{3} \underbrace{- \frac{4z^2}{2} + \frac{2z^2x}{2}}_{(\sqrt{1-x^2})^2 - (-\sqrt{1-x^2})^2 = 0} \right\rvert_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}$$

$$= \int_{-1}^{1} \tfrac{1}{2}\left[(4+x^2-4x)2\sqrt{1-x^2} + \frac{(1-x^2)\sqrt{1-x^2}}{3} \cdot 2 \right] dx$$

$$= \int_{-1}^{1} \sqrt{1-x^2}\left[4 + x^2 - 4x + \frac{1}{3} - \frac{x^2}{3} \right] dx.$$

$$= \int_{-1}^{1} \sqrt{1-x^2}\left[\frac{13}{3} - 4x + \frac{2}{3}x^2 \right] dx = \boxed{\frac{13}{3}\cdot\frac{\pi}{2} + \frac{2}{3}\cdot\frac{\pi}{8} = \pi\frac{9}{4}}$$

- $\int_{-1}^{1} x\sqrt{1-x^2}\,dx = 0$ / odd

- $\int_{-1}^{1} \sqrt{1-x^2} =$ area  $= \pi/2.$

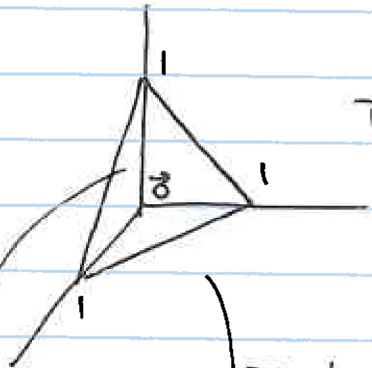
- $\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx = \int_{-\pi/2}^{\pi/2} \sin^2\theta\cos^2\theta \, d\theta = \int_{-\pi/2}^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^2 d\theta = \int_{-\pi/2}^{\pi/2} \frac{1 - \cos 4\theta}{8} = \frac{\pi}{8}$

  $x = \sin\theta$

  $dx = \cos\theta \, d\theta$

  $\sqrt{1-x^2} = \cos\theta$

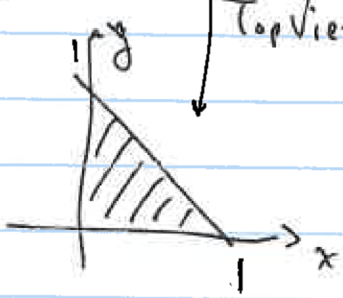
Ex $\quad \text{Volume}(D) = \iiint_D 1 \, dV$



Tetrahedron in the first octant

$\text{Volume} = \frac{1}{6}$

Top View



$$\begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1-x. \end{cases}$$

Top is $x+y+z=1$ plane

$$0 \le z \le 1-x-y$$

$$V = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1 \; dz\,dy\,dx.$$

$$= \int_0^1 \int_0^{1-x} z \Big|_{z=0}^{z=1-x-y} dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)-0 \;\; dy\;dx$$

$$= \int_0^1 (1-x)y - \frac{y^2}{2} \Big|_{y=0}^{y=1-x} dx$$

$$\int_0^1 \left((1-x)^2 - \frac{(1-x)^2}{2} - 0 \right) dx$$

$$= \int_0^1 \frac{(1-x)^2}{2} dx = \int_1^0 -\frac{u^2}{2} du$$

$u = 1-x$

$du = -dx$

$$= \int_0^1 \frac{u^2 du}{2} = \frac{u^3}{6}\Big|_0^1 = \frac{1}{6}$$