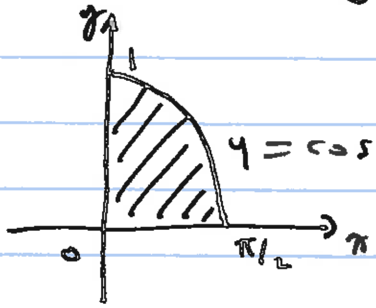
5.3 Exc # 8 p 337

$$A = \int_0^{\pi/2} \int_0^{\cos x} \sin x \, dy \, dx$$

$$\text{I} \begin{cases} 0 \le x \le \pi/2 \\ 0 \le y \le \cos x \end{cases}$$



Changing the order of integration

$$\text{II} \begin{cases} 0 \le y \le 1 \\ 0 \le x \le \cos^{-1} y \end{cases}$$

$$B = \int_0^1 \int_0^{\cos^{-1} y} \sin x \, dx \, dy$$

Evaluate: $$A = \int_0^{\pi/2} \int_0^{\cos x} \sin x \, dy \, dx = \int_0^{\pi/2} y \sin x \Big|_{y=0}^{y=\cos x} dx$$

$$= \int_0^{\pi/2} (\cos x \sin x - 0) \, dx = \int_1^0 u \cdot (-du)$$

$$u = \cos x$$
$$du = -\sin x \, dx$$

$$= \int_0^1 u \, du = \frac{u^2}{2} \Big|_0^1 = \frac{1}{2}.$$

(B) $$\int_0^1 \int_0^{\cos^{-1} y} \sin x \, dx \, dy$$

$$= \int_0^1 \left(-\cos x \Big|_{x=0}^{x=\cos^{-1} y} \right) dy$$

$$= \int_0^1 \left(-\cos(\cos^{-1} y)) + \cos 0 \right) dy$$

$$= \int_0^1 (-y+1)\, dy = -\frac{y^2}{2} + y \Big|_0^1$$

$$= \left(-\frac{1}{2} + 1 - 0\right) = \frac{1}{2}$$

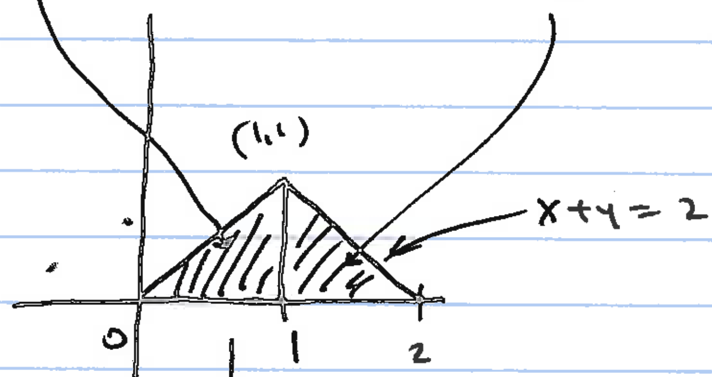
5.3 p 337

Exc # 12

$$\int_0^1 \int_0^x \sin x \, dy\, dx + \int_1^2 \int_0^{2-x} \sin x \, dy\, dx.$$

$0 \le x \le 1$ $\quad$ $1 \le x \le 2$

$0 \le y \le x$ $\quad$ $0 \le y \le 2-x$



combine 2 regions.

$0 \le y \le 1$

$y \le x \le 2-y$

$$\int_0^1 \int_y^{2-y} \sin x \, dx\, dy$$

$$= \int_0^1 -\cos x \Big|_{x=y}^{x=2-y} dy = \int_0^1 -\cos(2-y) + \cos y \, dy$$

$$= \sin(2-y) + \sin y \Big|_0^1$$

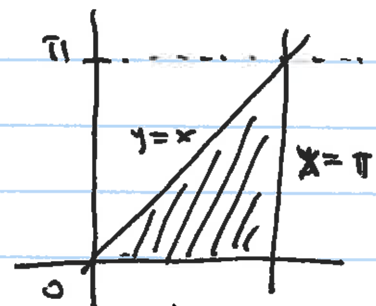
$$= (\sin 1 + \sin 1) - (\sin 2 + \cancel{\sin 0}^{\,0}) = 2\sin 1 - \sin 2$$

5.3

p387 #16 $\displaystyle\int_0^{\pi}\int_y^{\pi} \frac{\sin x}{x}\,dx\,dy$ Evaluate

($\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$, continuously extendable to $x=0$.
$\Longrightarrow$ integral exists.)

Type II $\begin{cases} 0 \le y \le \pi \\ y \le x \le \pi \end{cases}$



Type I $\begin{cases} 0 \le x \le \pi \\ 0 \le y \le x \end{cases}$

$$\int_0^{\pi}\int_0^{x} \frac{\sin x}{x}\,dy\,dx = \int_0^{\pi} \frac{\sin x}{x}\cdot y \Big|_{y=0}^{y=x} dx$$

$$= \int_0^{\pi}\left(\frac{\sin x}{\cancel{x}}\cancel{x} - 0\right) dx$$

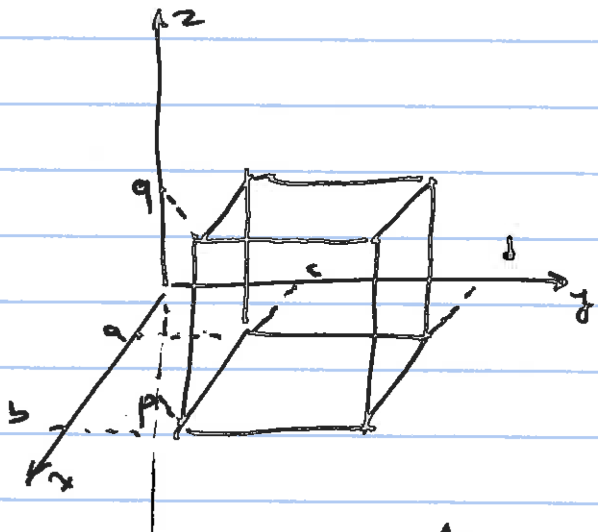
$$= \int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi} = -\cos\pi + \cos 0$$

$$= 2$$

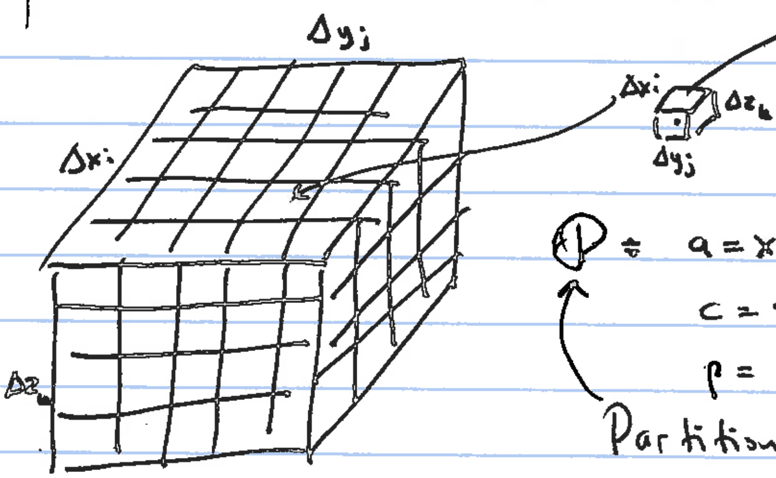
5.4 Main idea of how to extend double to triple integrals

Let $f(x,y,z) = \underbrace{[a,b]\times[c,d]\times[p,q]}_{\text{Box } B} \longrightarrow \mathbb{R}$

$$\| \quad \left\{ (x,y,z) \;\middle|\; \begin{array}{l} a \le x \le b \\ c \le y \le d \\ p \le z \le q \end{array} \right\}$$



Slice parallel to coordinate planes



$f(c_{ijk})$

$c_{ijk} \in$ small box.

$$\mathcal{P} \div \begin{array}{l} a = x_0 < x_1 \dots < x_n = b \\ c = y_0 < y_1 \dots < y_m = d \\ p = z_0 < z_1 \dots < z_\ell = q \end{array}$$

Partition

Riemann Sum

$$\sum_{i,j,k} \underbrace{f(c_{ijk})}_{\substack{\text{approximate}\\ \text{density}\\ \text{in small box}}} \cdot \underbrace{\Delta x_i \Delta y_j \Delta z_k}_{\substack{\text{volume}\\ \text{of small box}}} = R(f, \mathcal{P}, \{c_{ijk}\})$$

Approximation of the mass with density function f

as $\|\mathcal{P}\| \to 0$

$$\iiint_B f \, dV \quad (\text{if possible})$$

Fubini's Thm

Let $f: \underbrace{[a,b]\times[c,d]\times[p,q]}_{B} \longrightarrow \mathbb{R}$ be bounded

Let S be the set of discontinuities of f.

If
- S has 0 volume, and
- All lines parallel to axes x, y, z intersect S at finitely many pts

then $\iiint_B f\,dV$ exists, and equals to

$$\int_a^b \int_c^d \int_p^q f(x,y,z)\,dz\,dy\,dx$$

(as well as the other 5 orders.)

(There are 6 different orders)

5.4 Example #2 p347.

$$\iiint_{[0,1]\times[0,2]\times[0,3]} (x^2+y^2+z^2)\,dV$$

Fubini's Thm.

$$= \int_0^3 \int_0^2 \int_0^1 (x^2+y^2+z^2)\,dx\,dy\,dz$$

$$= \int_0^3 \int_0^2 \left(\frac{x^3}{3} + x(y^2+z^2) \Big|_{x=0}^{x=1} \right) dy\,dz.$$

$$= \int_0^3 \int_0^2 \left(\frac{1}{3} + (y^2 + z^2) - 0 \right) dy\, dz$$

$$= \int_0^3 \frac{1}{3} y + \frac{y^3}{3} + z^2 y \Big|_{y=0}^{y=2} dz$$

$$= \int_0^3 \left(\frac{2}{3} + \frac{8}{3} + 2z^2 - 0 \right) dz$$

$$= \frac{10}{3} z + \frac{2}{3} z^3 \Big|_{z=0}^{z=3}$$

$$= 10 + 18 - 0 = 28$$