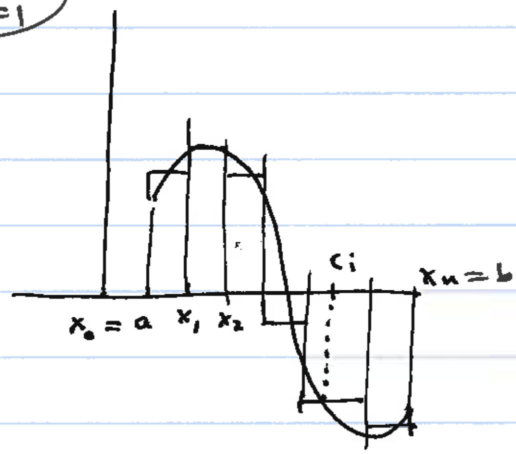
Review of Calc I, Riemann Integrals

$n=1$



Partition P : $a = x_0 < x_1 \cdots < x_n = b$.

$c_i \in [x_{i-1}, x_i]$

$\Delta x_i = x_i - x_{i-1}$

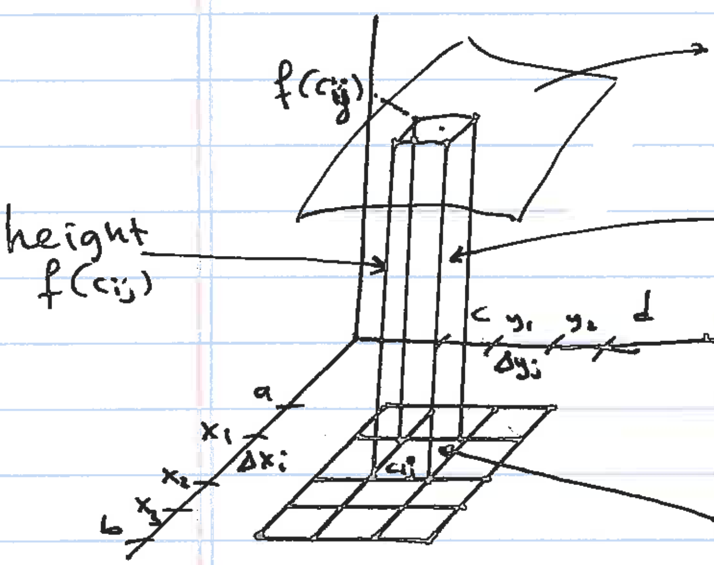
$\|P\| = \max \Delta_i$

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = R(f, P, \{c_i\}) \xrightarrow[\|P\| \to 0]{\text{as}} \int_a^b f(x)\,dx$$

5.2 $n=2$

Caution $(R \neq \mathbb{R})$

Defn let $f : \underbrace{[a,b] \times [c,d]}_{R(\text{rectangle})} \longrightarrow \mathbb{R}$ be bounded.



$z = f(x,y)$

volume $= f(c_{ij}) \cdot \Delta x_i \Delta y_j$

Let $\left.\begin{matrix} a = x_0 < x_1 \cdots < x_k = b \\ c = y_0 < y_1 \cdots < y_\ell = d \end{matrix}\right\} P$

Area $= \Delta A_{ij} = \Delta x_i \Delta y_j = |x_i - x_{i-1}||y_j - y_{j-1}|$

$\|P\| = \max(\Delta x_i, \Delta y_j)$

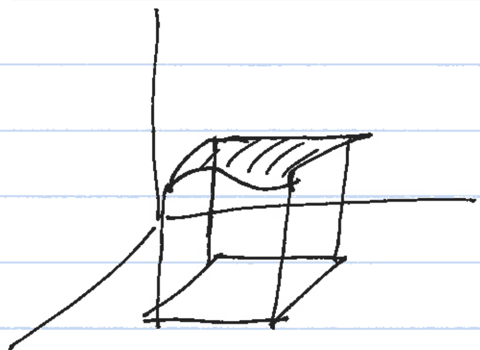
Let: $c_{ij} \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$.

If $R(f, P, \{c_{ij}\}) = \sum_{\substack{i=1 \\ j=1}}^{k,\ell} f(c_{ij})\Delta x_i \Delta_j \xrightarrow[\|P\| \to 0]{\text{limit as}} L \in \mathbb{R}$;

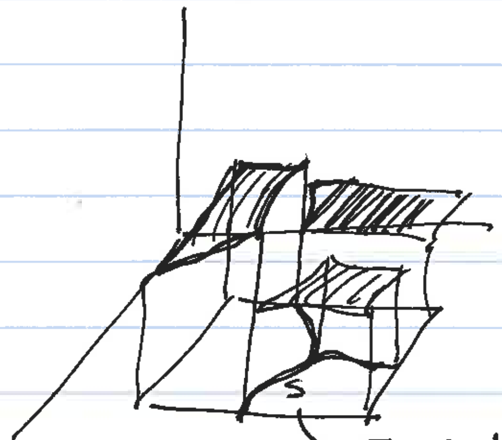
then f is called integrable and $L = \iint_R f\,dA$.

5.2

Thm: $f : [a,b] \times [c,d] \to \mathbb{R}$ is integrable if f is continuous (OR the set S of discontinuities of f has 0 area)



$S = \phi$



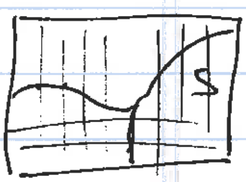
The set of discontinuities

FUBINI'S THEOREM

Let (i) $f : [a,b] \times [c,d] \to \mathbb{R}$ be bounded

(ii) $S = \{(x,y) \mid f \text{ is not continuous}\}$, the set of discontinuities have 0 area.

(iii) S intersect lines parallel to x or y-axes at finitely many points.



Then (i) f is integrable, and

(ii)
$$\iint_R f\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

~ Exc # 11 p 314.

Calculate the double integral of

$$f(x,y) = (16 - x^2 - y^2)$$

over the rectangle $\underbrace{[-2,2]}_{x} \times \underbrace{[1,3]}_{y}$.

Fubini's Thm $\Rightarrow$

$$\iint_{[-2,2]\times[1,3]} (16 - x^2 - y^2)\, dA$$

$$= \int_1^3 \int_{-2}^{2} (16 - x^2 - y^2)\, dx\, dy$$

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Let's Calculate

$$\int_1^3 \int_{-2}^{2} (16 - x^2 - y^2)\, dx\, dy$$

$$= \int_1^3 \left(16x - \frac{x^3}{3} - y^2 x \Big|_{x=-2}^{x=2} \right) dy$$

$$= \int_1^3 \left(32 - \frac{8}{3} - 2y^2\right) - \left(-32 - \left(\frac{-8}{3}\right) + 2y^2\right) dy$$

$$= \int_1^3 \left(64 - \frac{16}{3} - 4y^2\right) dy = \int_1^3 \left(\frac{176}{3} - 4y^2\right) dy$$

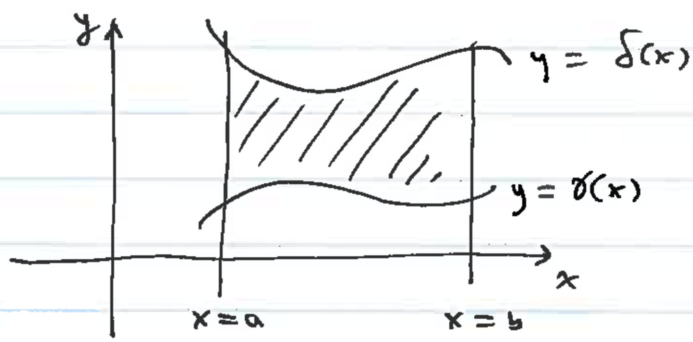
$$= \frac{176}{3}y - \frac{4y^3}{3}\Big|_1^3$$

$$= (176 - 36) - \left(\frac{176}{3} - \frac{4}{3}\right)$$

$$= 140 - \frac{172}{3} = \frac{420-172}{3} = \frac{248}{3}$$

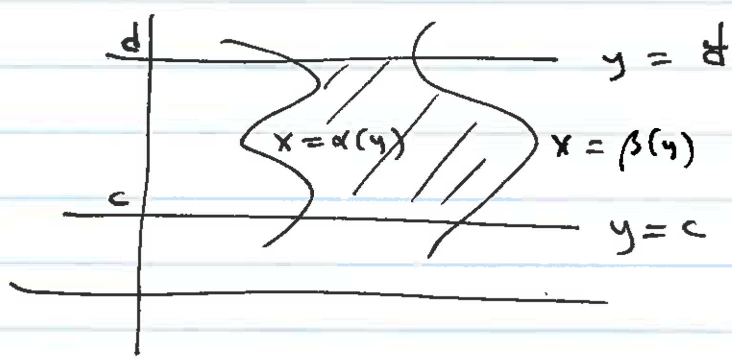
ELEMENTARY REGIONS:

Type I



$$\left\{(x,y) \;\middle|\; \begin{array}{c} a \le x \le b \\ \gamma(x) \le y \le \delta(x) \end{array}\right\}$$

Type II



$$\left\{(x,y) \;\middle|\; \begin{array}{c} c \le y \le d \\ \alpha(y) \le x \le \beta(y) \end{array}\right\}$$

Type III (a region can be written as type I and II)

<u>Theorem:</u> Let $f : D \longrightarrow \mathbb{R}$ be continuous,

and let D be of type I:

$$D = \{ (x,y) \mid a \leq x \leq b, \quad \gamma(x) \leq y \leq \delta(x) \}$$

Then $$\iint_D f\,dA = \int_a^b \int_{\gamma(x)}^{\delta(x)} f(x,y)\,dy\,dx.$$

p 332 Ex #4

a) $$\int_0^1 \underbrace{\int_0^{x^2} 3\,dy}\,dx$$

$$= \int_0^1 \left(3y \Big|_{y=0}^{y=x^2} \right) dx$$

$$= \int_0^1 \left(3x^2 - 0 \right) dx$$

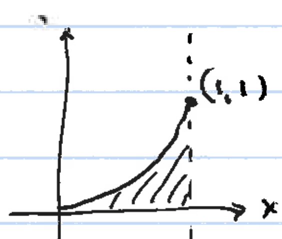
$$= x^3 \Big|_{x=0}^{x=1} = 1^3 - 0^3 = 1.$$

The answer is the volume of this solid.

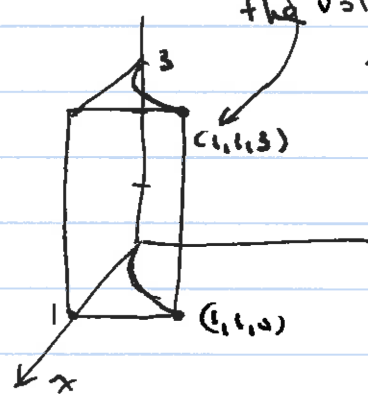
b)

$0 \leq x \leq 1$

$0 \leq y \leq x^2$



Domain of function
" " integration



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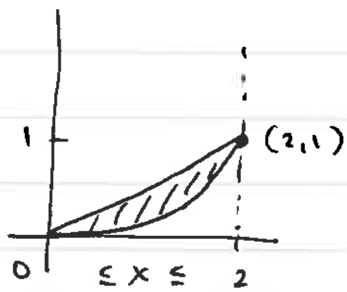
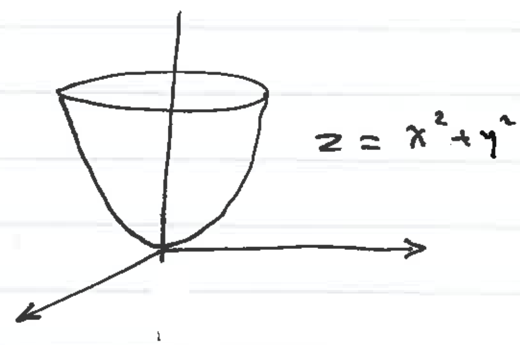
Exc #8 $\displaystyle\int_0^2 \int_{x^2/4}^{x/2} (x^2+y^2)\,dy\,dx$





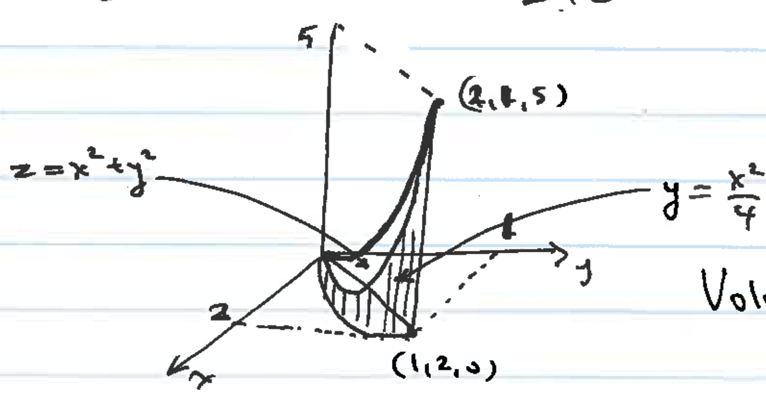
$$I = \int_0^2 \left(x^2 y + \frac{y^3}{3} \Bigg|_{y=x^2/4}^{y=x/2} \right) dx$$

$$= \int_0^2 \left(\frac{x^3}{2} + \frac{x^3}{24}\right) - \left(\frac{x^4}{4} + \frac{x^6}{192}\right) dx.$$

$$= \frac{x^4}{8} + \frac{x^4}{96} - \frac{x^5}{20} - \frac{x^7}{7\cdot 192}\Bigg|_0^2$$

$$= \frac{16}{8} + \frac{\cancel{16}^{\,1}}{\cancel{96}_{\,6}} - \frac{\cancel{32}^{\,8}}{\cancel{20}_{\,5}} - \frac{\cancel{64}\cdot 2}{7\cdot\cancel{192}_{\,3}} - 0$$

$$= 2 + \frac{1}{6} - \frac{8}{5} - \frac{2}{21} = \frac{420+35-336-20}{210} = \frac{99}{210} = \frac{33}{70}$$



Volume $= \frac{33}{70}$.