Lagrange Multipliers

Recall Linear algebra

A is an $n \times n$ matrix,

$$A \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{a System of Linear Eq}^{n}\text{'s}$$

$\det A \neq 0$ (A^{-1} exists) $\Longrightarrow$ $\underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\text{Trivial sol}^n}$ <u>only solution</u>

$\det A = 0 \Longleftarrow$ If $A\vec{x} = 0$ has non-trivial solutions

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Ex3 Max/min $f(x,y) = (x-2y)^2$ subject to $x^2+y^2=1$.

$$\nabla f = \left(2(x-2y), \; -2(x-2y)\cdot 2 \right)$$

$$\nabla g = (2x, 2y)$$

$$\nabla f = \lambda \nabla g \qquad \left\{ \begin{array}{l} 2x - 4y = 2\lambda x \\ -4x + 8y = 2\lambda y \\ \\ x^2 + y^2 = 1 \end{array} \right.$$

$$\left.\begin{aligned} 2x - 2\lambda x - 4y &= 0 \\ -4x + 8y - 2\lambda y &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} (2-2\lambda)x - 4y &= 0 \\ -4x + (8-2\lambda)y &= 0 \end{aligned}\right\}$$

want non-trivial sol^n. since $x^2+y^2=1$

$$\Longrightarrow \begin{vmatrix} 2-2\lambda & -4 \\ -4 & 8-2\lambda \end{vmatrix} = 0.$$

$$(2-2\lambda)(8-2\lambda) - 16 = 0.$$

$$\cancel{16} - 4\lambda - 16\lambda + 4\lambda^2 - \cancel{16} = 0.$$

$$4\lambda^2 - 20\lambda = 0$$

$$\lambda = 0 \text{ OR } \lambda = 5$$

Case: $\lambda = 0$

$$2x - 4y = 0$$

$(-4x + 8y = 0)$ multiple of 1st eq.

$$x^2 + y^2 = 1$$

$$2x = 4y$$

$$x = 2y$$

$$1 = x^2 + y^2 = 4y^2 + y^2 = 5y^2$$

$$y = \pm \frac{1}{\sqrt{5}} \qquad x = \pm \frac{2}{\sqrt{5}}$$

c.p.: $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$, $\left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$

Case $\lambda = 5$

$$-8x - 4y = 0$$
$$-4x - 2y = 0 \quad \text{(multiple of } 1^{\text{st}} \text{ eq}^{n}\text{)}$$
$$x^2 + y^2 = 1$$

$$8x = -4y$$
$$-2x = y$$

$$1 = x^2 + y^2 = x^2 + 4x^2 = 5x^2$$

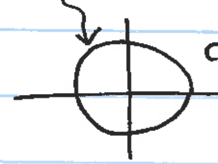
$$x = \pm \frac{1}{\sqrt{5}} \qquad y = \mp \frac{2}{\sqrt{5}}$$

c.p.: $\left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$, $\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

All c.p.	$f = (x-2y)^2$	
$\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$	0	min value
$\left(-\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$	0	
$\left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$	5	max value
$\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$	5	

$\Longleftarrow$

$x^2 + y^2 = 1$



compact

f is continuous.

Ex 4

Find Closest pt of the plane

$$x + 3y - 2z = 7$$

to the origin $(0,0,0)$.

We did this (Ex#6, 4.2)

max/min $f = x^2 + y^2 + z^2$

$$\nabla f = (2x, 2y, 2z)$$
$$\nabla g = (1, 3, -2)$$

$\nabla f = \lambda \nabla g$

$$2x = \lambda \implies x = \frac{\lambda}{2}$$
$$2y = 3\lambda \implies y = \frac{3\lambda}{2}$$
$$2z = -2\lambda \implies z = -\lambda.$$
$$x + 3y - 2z = 7.$$

plug in

$$7 = \frac{\lambda}{2} + 3 \cdot \frac{3\lambda}{2} - 2 \cdot (-\lambda)$$

$$7 = \lambda\left(\frac{1}{2} + \frac{9}{2} + 2\right)$$
$$7 = 7\lambda$$

$$\lambda = 1 \implies \left.\begin{aligned} x &= \tfrac{1}{2} \\ y &= \tfrac{3}{2} \\ z &= -1 \end{aligned}\right]$$

The closest pt, since there is only one closest pt of a plane to origin.

Caution: $x + 3y - 2z = 7$ is unbounded

is <u>not</u> compact

f has no maximum on this plane.

Multiple constraints

Ex 4 max/min $f = \frac{x^2+y^2}{2}$ subject to

$g_1 \quad x^2+y^2+z^2=1$ → sphere

$g_2 \quad x+y+z=0$ → plane

intersection is a "great" circle

$\nabla f = (x, y, 0)$

$\nabla g_1 = (2x, 2y, 2z)$

$\nabla g_2 = (1, 1, 1)$

$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$ L.M. equation

$$\left.\begin{array}{ll}
① & x = \lambda \cdot 2x + \mu \cdot 1 \\
② & y = \lambda \cdot 2y + \mu \cdot 1 \\
③ & 0 = \lambda \cdot 2z + \mu \cdot 1 \\
④ & x^2+y^2+z^2=1 \\
⑤ & x+y+z=0
\end{array}\right\} \text{ solve}$$

subtract

① − ② $\quad x-y = 2\lambda x - 2\lambda y = 2\lambda(x-y)$

$$0 = (2\lambda-1)(x-y)$$

<u>case 1</u> $\quad\quad$ <u>case 2</u>

$\lambda = \frac{1}{2}$ $\quad\quad$ $x = y$

Case 1:

① $\quad x = x + \mu$

$\mu = 0$

③ $\quad 0 = z + 0$

$z = 0$

④ $\quad x^2+y^2 = 1$

⑤ $\quad x+y = 0$

$y = -x$

$1 = x^2+y^2 = 2y^2$

$y = \pm\frac{1}{\sqrt{2}}$

$x = \mp\frac{1}{\sqrt{2}}$

$z = 0$

Case 2:

④ $\quad 2x^2+z^2 = 1$

⑤ $\quad 2x+z = 0$

$z = -2x$

$2x^2+4x^2 = 1$

$6x^2 = 1$

$y = x = \pm\frac{1}{\sqrt{6}}$

$z = \mp\frac{2}{\sqrt{6}}$

This is consistent with ①, ②, ③ if one takes:

$\lambda = 1/6$ (both cases)

$\mu = \pm\frac{\sqrt{6}}{9}$

c.p.

$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ $\quad\quad$ $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$

$\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$ $\quad\quad$ $\left(-\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$

C.p.	$\frac{x^2+y^2}{2}$	
$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$	$1/2$	max value
$\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$	$1/2$	
$\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$	$1/6$	min value
$\left(-\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$	$1/6$	

Constraint set is a circle, compact

Thm: Let $X^{\text{open}} \subseteq \mathbb{R}^n$

k equations.

$$S = \left\{ \vec{x} \in X \;\middle|\; \begin{array}{l} g_1(x) = c_1 \\ g_2(x) = c_2 \\ \vdots \\ g_k(x) = c_k \end{array} \right\}$$

$f, \underset{i=1,\dots k}{g_i} : X^{\text{open}} \subseteq \mathbb{R}^n \to \mathbb{R}$ cont. diff'ble

If $x_0 \in S$, is an extremum for $f|S$, then $\exists\, \lambda_1, \lambda_2 \dots \lambda_k \in \mathbb{R}$ s.t.

$$\nabla f(x_0) = \lambda_1 \nabla g_1(x_0) + \lambda_2 \nabla g_2(x_0) + \dots + \lambda_k \nabla g_k(x_0)$$

provided that $\{\nabla g_1(x_0), \dots \nabla g_k(x_0)\}$ linearly independent.