Oct 17, 2016

1

4.2 To finish

Exc # 32 p276.

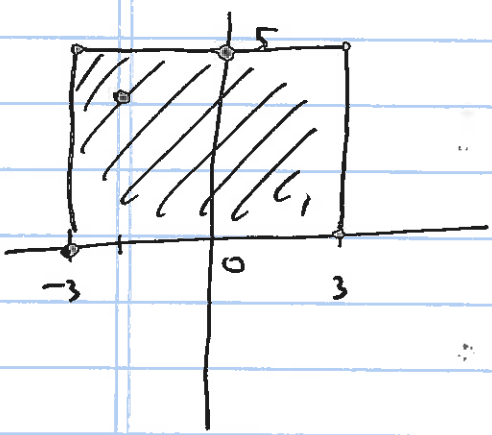
$$f(x,y) = x^2 + xy + y^2 - 6y$$

Find $\underbrace{\text{max/min values}}_{\text{they}}$ of f for $\begin{cases} -3 \le x \le 3 \\ 0 \le y \le 5. \end{cases}$

We know: they exist by Ext. Value. Thm,

+ continuity

+ compactness (closed + bounded) of domain



· Interior I^{st} D.T.

· Boundary.

$$\nabla f = (2x + y,\ x + 2y - 6)$$

$$\left.\begin{aligned} 2x + y &= 0 \\ x + 2y &= 6 \end{aligned}\right\} \qquad (-2, 4) = (x, y)$$

Boundary:

1) $y = 0$ & $-3 \le x \le 3$
2) $y = 5$ & $-3 \le x \le 3$
3) $x = -3$ & $0 \le y \le 5$
4) $x = 3$ & $0 \le y \le 5$

} Union of 4 segments

We use reduction of number of variables on the boundary ②

1) $f(x,y) = x^2 + xy + y^2 - 6y$ $(y=0)$

$f(x,0) = x^2$ $-3 \le x \le 3$

$0 = \frac{d}{dx} f(x,0) = 2x$ $(0,0)$

$x = 0$ $(-3,0)$

$(3,0)$

2) $f(x,5) = x^2 + 5x + 25 - 30$

$(y=5)$ $= x^2 + 5x - 5$ $-3 \le x \le 3.$

$\frac{d}{dx} f(x,5) = 2x + 5$ $(-\frac{5}{2}, 5)$

$(-3, 5)$

$(3, 5)$

3) $f(-3,y) = 9 - 3y + y^2 - 6y$ $0 \le y \le 5$

$= y^2 - 9y + 9$

$\frac{d}{dy} f(-3,y) = 2y - 9$ $(-3, \frac{9}{2})$

$(-3, 0)$

$(-3, 5)$

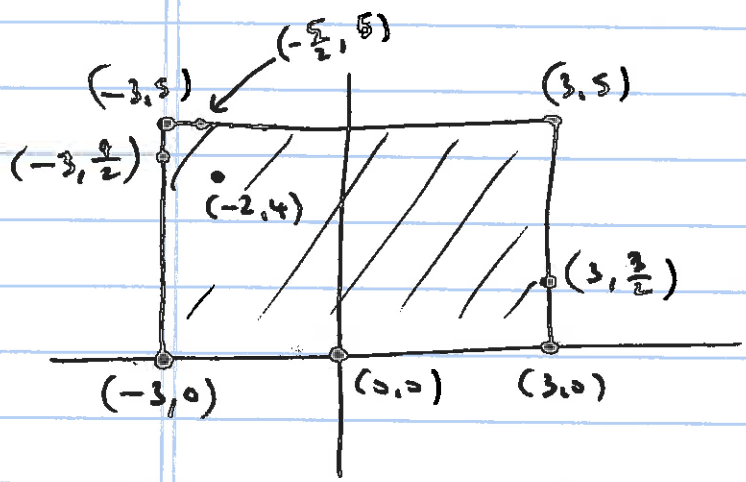
4) $f(3,y) = 9 + 3y + y^2 - 6y$ $0 \le y \le 5$

$= y^2 - 3y + 9$ $(3, 3/2)$

$\frac{d}{dy}$ " $= 2y - 3$ $(3, 0)$

$(3, 5)$

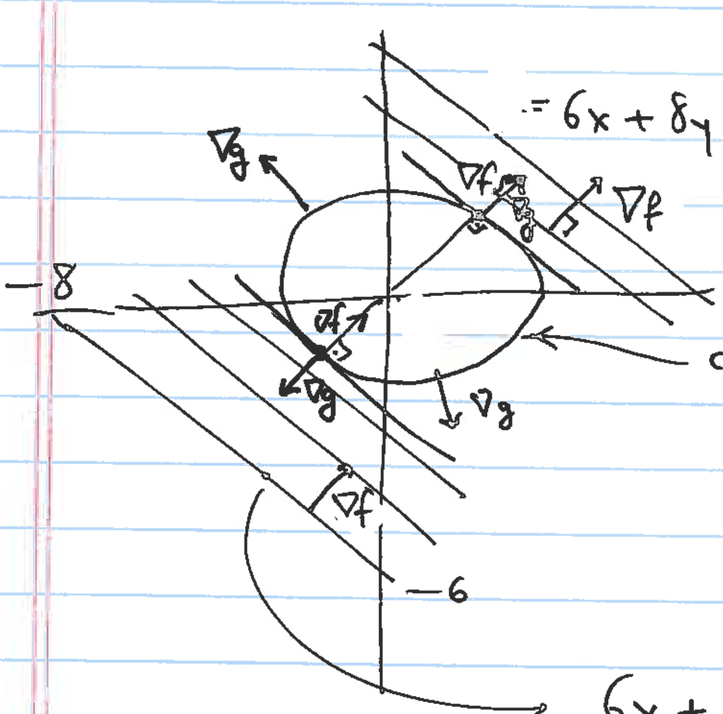
	(x,y)	$f(x,y) = x^2 + xy + y^2 - 6y$	
✓	$(-2,4)$	-12	min value attained at $(-2,4)$
✓	$(0,0)$	0	
✓	$(-3,0)$	9	
✓	$(3,0)$	9	
✓	$(-\frac{5}{2},5)$	$-11\frac{1}{4}$	
✓	$(-3,5)$	-11	
corners Duplicated	$(3,5)$		
✓	$(-3,\frac{9}{2})$	$-11\frac{1}{4}$	
Duplicated	$(-3,0)$		
Duplicated	$(-3,5)$		
✓	$(3,3/2)$	$6\frac{3}{4}$	
Dupl.	$(3,0)$		
✓	$(3,5)$	19	max value attained at $(3,5)$



End of 4.2

4.3 Lagrange Multipliers

Ex 1 Max/min $f(x,y) = 6x + 8y$ on the circle $\underbrace{x^2+y^2=25}_{g}$



One has $\nabla f \parallel \nabla g$ at the points of tangency

$$\left.\begin{aligned} \nabla f &= (6,8) \\ \nabla g &= (2x,2y) \end{aligned}\right\} \text{ want them parallel}$$

$$(6,8)=\nabla f \parallel \nabla g = (2x,2y)$$

$$(6,8) = \lambda(2x,2y)$$

$$\left.\begin{aligned} 6 &= 2\lambda x \\ 8 &= 2\lambda y \\ x^2+y^2 &= 25 \end{aligned}\right\} \text{ solve}$$

Constraint. $\longrightarrow x^2+y^2=25$

$$x = \frac{6}{2\lambda} \qquad \lambda \neq 0, (\text{otherwise } 6 \neq 2\lambda x)$$

$$y = \frac{8}{2\lambda}$$

$$25 = x^2 + y^2 = \left(\frac{6}{2\lambda}\right)^2 + \left(\frac{8}{2\lambda}\right)^2 = \frac{36+64}{4\lambda^2} = \frac{100}{4\lambda^2} = \frac{25}{\lambda^2}$$

$$25 = \frac{25}{\lambda^2} \implies \lambda^2 = 1$$

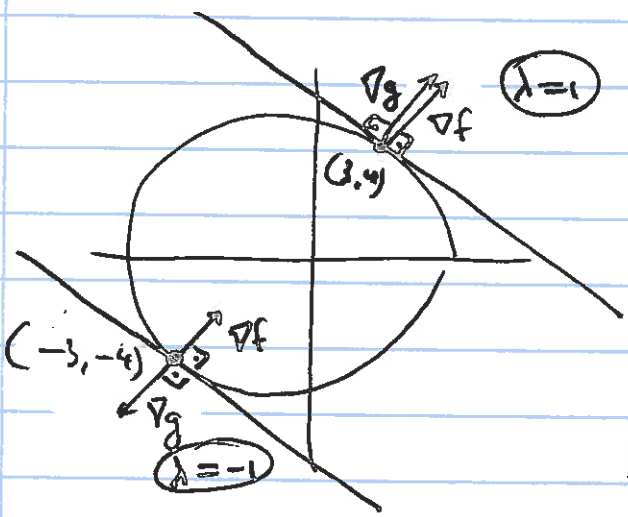
$$\lambda = \pm 1$$

$\lambda = 1$

$\left.\begin{matrix} x = 3 \\ y = 4 \end{matrix}\right]$ c.p.

$\lambda = -1$

$\left.\begin{matrix} x = -3 \\ y = -4 \end{matrix}\right]$ c.p.



$\lambda = 1$ $\nabla f(3,4) = \nabla g(3,4) = (6,8)$

$f(3,4) = 6 \cdot 3 + 8 \cdot 4 = 50$

max value

$f(-3,-4) = -50$

min value.

c.p.	$6x + 8y$
$(3,4)$	50
$(-3,-4)$	-50

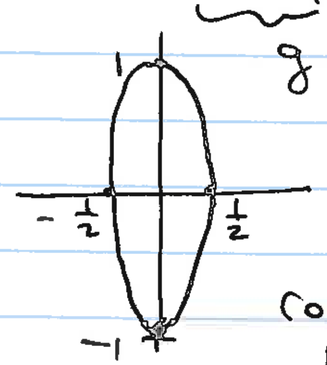
Theorem: Lagrange Multipliers

Let $f, g : \Omega^{\text{open}} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be both continuously diff'ble.

If $x_0 \in S = \{ x \in \Omega \mid g(x) = c \}$ and if x_0 is a local extremum of $\underbrace{f|S}_{f \text{ restricted to } S}$ then $\exists \lambda \in \mathbb{R}$

$$\nabla f(x_0) = \lambda \cdot \nabla g(x_0)$$

Ex 2 max/min $\underbrace{x + y^2}_{f}$ with the constraint $\underbrace{4x^2 + y^2}_{g} = 1$.



compact $\Downarrow$ $\exists$ max/min of f on this ellipse

L.M.: Solve

$$\nabla f = \lambda \nabla g$$

$$\nabla f = (1, 2y)$$

$$\nabla g = (8x, 2y)$$

$$\left.\begin{array}{ll} ① & 1 = \lambda \cdot 8x \\ ② & 2y = \lambda \cdot 2y \\ ③ & 4x^2 + y^2 = 1 \end{array}\right\}$$

need to include the constraint(s)

Start with

② $2y = \lambda 2y \implies 0 = 2y - \lambda 2y$

$$0 = 2y(1-\lambda)$$

OR

$y = 0$	$\lambda = 1$
③ $4x^2 + 0 = 1$	① $1 = 1 \cdot 8x$
$4x^2 = 1$	$x = \frac{1}{8}$
$x^2 = \frac{1}{4}$	③ $4 \cdot \left(\frac{1}{8}\right)^2 + y^2 = 1$
$x = \pm \frac{1}{2}$	$\frac{1}{16} + y^2 = 1$
	$y^2 = \frac{15}{16}$
	$y = \pm \frac{\sqrt{15}}{4}$

c.p.	$x + y^2$	
$\left(\frac{1}{2}, 0\right)$	$\frac{1}{2}$	
$\left(-\frac{1}{2}, 0\right)$	$-\frac{1}{2}$	← min value attained at $\left(-\frac{1}{2}, 0\right)$
$\left(\frac{1}{8}, \frac{\sqrt{15}}{4}\right)$	$\frac{1}{8} + \frac{15}{16} = \frac{17}{16}$	
$\left(\frac{1}{8}, -\frac{\sqrt{15}}{4}\right)$	$\frac{17}{16}$	← max value attained at $\left(\frac{1}{8}, \pm\frac{\sqrt{15}}{4}\right)$