Oct 13, 2016 ①

II^{nd} Derivative test

$n \geqslant 3$

Ex $f(x,y,z) = 2x^2 + 3y^2 + 4z^2 + 2xy + 2xz - 4x - 2y - 2z$

Find all c.p. / classify

$f_x = 4x + 2y - 4 + 2z$

$f_y = 6y + 2x - 2$

$f_z = 8z + 2x - 2$

c.p. Solve $\nabla f = 0$.

$$\left.\begin{aligned} 4x + 2y + 2z &= 4 \\ 2x + 6y \quad &= 2 \\ 2x \quad + 8z &= 2 \end{aligned}\right\} \text{ solve}$$

$x = 1$

$y = 0$

$z = 0$

One c.p. $(1,0,0)$

$$H_f = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 6 & 0 \\ 2 & 0 & 8 \end{bmatrix}$$

$$\Delta_1 = 4$$

$$\Delta_2 = \begin{vmatrix} 4 & 2 \\ 2 & 6 \end{vmatrix} = 20$$

$$\Delta_3 = \begin{vmatrix} 4 & 2 & 2 \\ 2 & 6 & 0 \\ 2 & 0 & 8 \end{vmatrix} = (24 \cdot 8 + 0 + 0) - (24 + 0 + 32)$$

$$= \underbrace{192 - 56}_{136} > 0$$

$\Delta_1, \Delta_2, \Delta_3 > 0 \implies$ + - definite

$(1, 0, 0)$ local min.

<u>Thm:</u> Let A be a $n \times n$ symmetric matrix, $\det A \neq 0$.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & & & a_{nn} \end{bmatrix}$

Let $\Delta_1 = a_{11}$

$$\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_k = \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ a_{21} & & a_{2k} \\ \vdots & & \\ a_{k1} & & a_{kk} \end{vmatrix}$$

det of top left $k \times k$ submatrix

$\Delta_n = \det A$.

If all $\Delta_1, \Delta_2 \cdots \Delta_n > 0$, then A is + definite

If $\left.\begin{matrix} \Delta_1, \Delta_3, \Delta_5, \Delta_{odd} < 0 \\ \Delta_2, \Delta_4, \cdots \Delta_{even} > 0 \end{matrix}\right\}$, then A is − definite

Otherwise A is indefinite (given: $\Delta_n = \det A \neq 0$)

Ex 2 $\quad f = x^3 + xz^2 - 3x^2 + y^2 + 2z^2$

Find all c.p. / classify.

$$\left.\begin{aligned} ① \quad f_x &= 3x^2 + z^2 - 6x \\ ② \quad f_y &= 2y \\ ③ \quad f_z &= 2xz + 4z. \end{aligned}\right\} \text{ Solve } \nabla f = 0$$

② $y = 0$ ✓

③ $2xz + 4z = 0 \longrightarrow$ ③ $2z(x+2) = 0$

$3x^2 + z^2 - 6x = 0$

$z = 0$ OR $x = -2$

① $3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0$ or $x = 2$

① $12 + z^2 + 12 = 0$

no real soln.

cp: $\left\{\begin{matrix} (0, 0, 0) \\ (2, 0, 0) \end{matrix}\right.$

$$H_f = \begin{bmatrix} 6x-6 & 0 & 2z \\ 0 & 2 & 0 \\ 2z & 0 & 2x+4 \end{bmatrix}$$

$$H_f(0,0,0) = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$\Delta_1 = -6$

$\Delta_2 = -12$

$\Delta_3 = -48$

indefinite $\Rightarrow$ Saddle at $(0,0,0)$

$$H_f(2,0,0) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\left.\begin{matrix} \Delta_1 = 6 \\ \Delta_2 = 12 \\ \Delta_3 = 96 \end{matrix}\right\} + \text{-def.}$$

$\Rightarrow$ local min at $(2,0,0)$

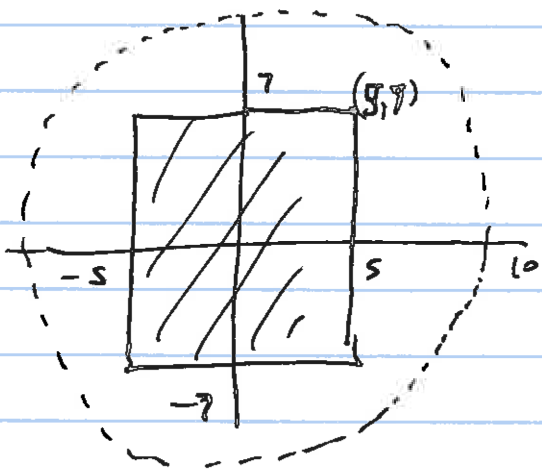
4.2 Part III Boundary, Compactness, Extreme Value Th.

Def^n: A set $X \subseteq \mathbb{R}^n$ is called bounded if $\exists R < \infty$ s.t.

$$X \subseteq \{\vec{x} \mid |\vec{x}| \le R\};$$

otherwise X is called unbounded.

Ex (a) $\{(x,y) \mid |x| \le 5,\ |y| \le 7\}$ is bounded



$R = 10$

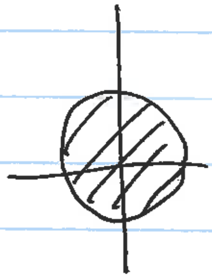
(b) x-axis in $\mathbb{R}^2$ is unbounded

Recall A set X is called closed if (Boundary of X) $\subseteq X$.

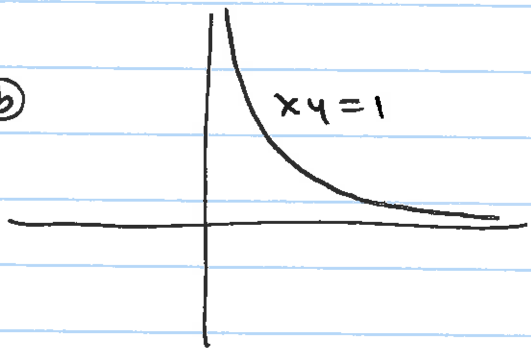
Def^n A set X in $\mathbb{R}^n$ is called compact if it closed and bounded.

Ex (a) $\{(x,y) \mid x^2+y^2 \le 1\}$ closed + bounded

compact.



(b)



$A = \{(x,y) \mid \begin{matrix} x>0 \\ xy=1 \end{matrix} \}$

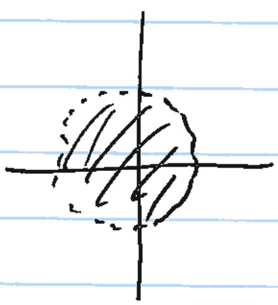
closed ($Bd\,A = A$)

but unbounded

not compact

(c) Not compact, but bounded

(need not closed)



$\{(x,y) \mid x^2+y^2 < 1\}$.

Extreme Value Thm: Let $f : \bar{X} \subseteq \mathbb{R}^n \to \mathbb{R}$ s.t.

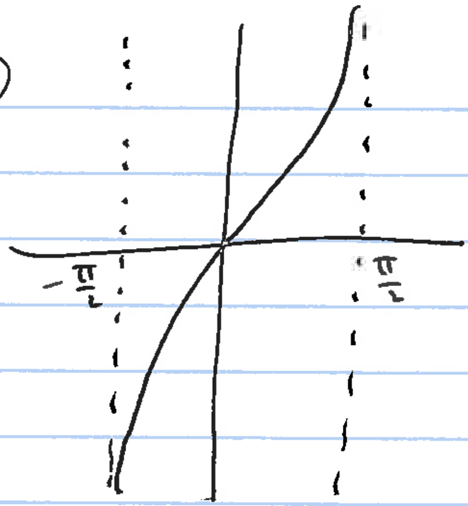
(i) f is continuous, and

(ii) $\bar{X}$ is compact, then f must attain its maximum and minimum values on $\bar{X}$.

In other words: $\exists\, a \in \bar{X}\ \exists\, b \in \bar{X}$ s.t.

$$\forall x \in \bar{X} \qquad f(a) \le f(x) \le f(b).$$

Ex



$\tan x : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$

$\underbrace{\text{unbounded}}_{\text{function}}$

Domain not compact $\left\{ \begin{array}{l} \text{Domain } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \text{bounded.} \\ \text{Domain not closed} \end{array} \right.$

Ex

$f(x) = x : \mathbb{R} \longrightarrow \mathbb{R}$ no max / no min

closed

$\left. \begin{array}{l} \text{domain } \mathbb{R} \text{ closed} \\ \text{domain } \mathbb{R} \text{ unbounded} \end{array} \right\}$ $\mathbb{R}$ is not compact.

Ex (Calculus I) $f(x) = x^3 - 3x$, $0 \le x \le 2$

find $\underbrace{\text{max/min}}_{\text{global}}$ if they exist.

$f'(x) = 3x^2 - 3$

$3x^2 - 3 = 0$

$x = \pm 1$

		$f(x) = x^3 - 3x$	
Boundary	0	0	
Boundary	2	2	max value
critical pt.	1	-2	min value
	$\cancel{-1}$		

$-1 \notin [0, 2]$.