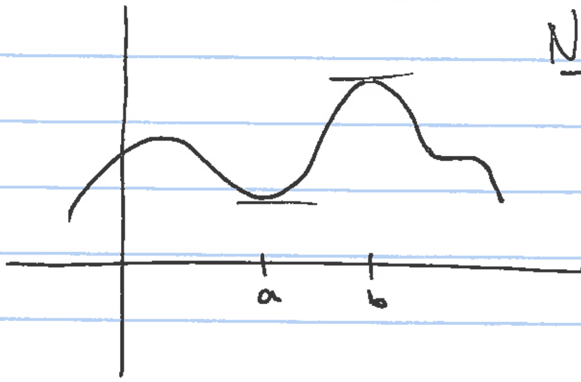
Oct 12, 2016

①

(4.2)

Second Derivative test

Recall (Calc I) $f: I \subseteq \mathbb{R}^1 \to \mathbb{R}^1$



Need: $f'(x) = 0$, $f'(x)$ DNE

Find c.p.

if f is twice diff'ble:

$f'(a) = 0$, $f''(a) > 0$ local min

$f'(b) = 0$, $f''(b) < 0$ local max

$f'(c) = 0 = f''(c)$ inconclusive

$f: X \subseteq \mathbb{R}^n \to \mathbb{R}^1$

a critical pt $\quad a \in$ interior X

$(X - \text{boundary } X) = \text{interior of } X$

f Twice diff'ble

c.p $\Rightarrow \nabla f(a) = 0$

$$P_2(\vec{x}, \vec{a}) = f(a) + \cancel{\overset{0}{\nabla f(a)}}(x-a) + \frac{1}{2}(x-a) H_f(a) (x-a)^T$$

will decide

local max/min

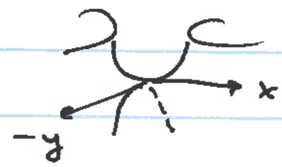
provided that $\det H_f(a) \neq 0$

Examples 2×2

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 + 3y^2$$

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -x^2 - 4y^2$$

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 - y^2$$

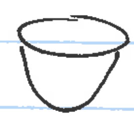


$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 - 2xy + 2y^2$$

$$= 2\left(x^2 - xy + \tfrac{1}{4}y^2 - \tfrac{1}{4}y^2\right) + 2y^2$$

$$= 2\left(x - \tfrac{1}{2}y\right)^2 - \tfrac{1}{2}y^2 + 2y^2$$

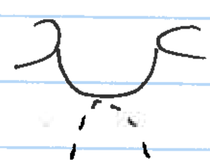
$$= 2\left(x - \tfrac{1}{2}y\right)^2 + \tfrac{3}{2}y^2$$



$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x^2 + y^2 + 4xy$$

$$= \left(x^2 + 4xy + 4y^2\right) - 3y^2$$

$$= (x + 2y)^2 - 3y^2$$



Linear Algebra

Let A be a symmetric $n \times n$ matrix.

A is called positive definite if $\forall_{v \neq 0} v \in \mathbb{R}^n$ $\quad v^T A v > 0$

A is called negative definite if $\forall_{v \neq 0} v \in \mathbb{R}^n$ $\quad v^T A v < 0$

A is called indefinite if

(i) $\det A \neq 0$

(ii) $\exists\, v, w \neq 0$ s.t.

$$v^T A v < 0 < w^T A w$$

*** A is called degenerate if $\det A = 0$.

2nd Derivative Test

Theorem: Let $f: \underline{X}^{open} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^1$ be twice continuously diff'ble. Let $\nabla f(a) = 0$, c.p. for some $a \in \underline{X}$.

Then:

If $H_f(a)$ is + definite, then f has a local minimum at a.

If $H_f(a)$ is − definite then f has a local maximum at a.

If $H_f(a)$ is indefinite ($\det H_f \neq 0$) then f has a saddle at a.

If $\det H_f(a) = 0$, then the test is inconclusive.

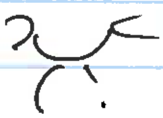
How do we find + or − definite (without finding eigenvalues)

$n=1$ $\qquad H_f = [a]$

$a > 0$	+ definite
$a < 0$	− definite
$a = 0$	degenerate

$n=2$ $\qquad f : \underset{x,y}{X^{\text{open}} \subseteq \mathbb{R}^2} \longrightarrow \mathbb{R}^1$

$$H_f(a,b) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial y\, \partial x}(a,b) \\ \frac{\partial^2 f}{\partial x \partial y}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{bmatrix} = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$



- If $AC - B^2 < 0$: indefinite
- If $AC - B^2 > 0$, $A > 0$, + definite 
  $AC - B^2 > 0$, $A < 0$ − definite 
- If $AC - B^2 = 0$ (inconclusive 2nd Derivative test)

Ex $f(x) = x^3 + 3x^2 + y^3 - 12y \; : \mathbb{R}^2 \to \mathbb{R}^1$

unbounded since
$\lim_{x\to\infty} f(x,0) = +\infty$
$\lim_{x\to-\infty} f(x,0) = -\infty$

$$\nabla f : \begin{cases} f_x = 3x^2 + 6x \\ f_y = 3y^2 - 12 \end{cases}$$

c.p. solve $\nabla f = (3x^2 + 6x,\ 3y^2 - 12) = (0,0)$

$3x^2 + 6x = 0$ and $3y^2 - 12 = 0$

$3x(x+2) = 0$ $\qquad 3(y^2 - 4) = 0$

$x = 0$ or $x = -2$ $\qquad y = \pm 2$

$(x = 0$ or $x = -2)$ and $(y = 2$ or $y = -2)$

c.p. $(0, 2)$
$(0, -2)$
$(-2, 2)$
$(-2, -2)$

$$H_f = \begin{bmatrix} 6x + 6 & 0 \\ 0 & 6y \end{bmatrix}$$

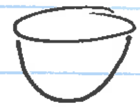
need to check at each c.p.

$$H_f(0,2) = \begin{bmatrix} 6 & 0 \\ 0 & 12 \end{bmatrix}$$

det $= 72$
$f_{xx} = 6$

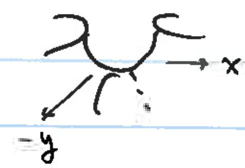
$+$ definite

local min at $(0,2)$



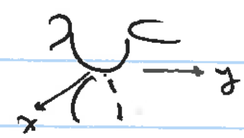
$$H_f(0,-2) = \begin{bmatrix} 6 & 0 \\ 0 & -12 \end{bmatrix}$$

det $= -72$
indefinite



$$H_f(-2,2) = \begin{bmatrix} -6 & 0 \\ 0 & 12 \end{bmatrix}$$

det $= -72$
indefinite



$$H_f(-2,-2) = \begin{bmatrix} -6 & 0 \\ 0 & -12 \end{bmatrix}$$

det $= 72 > 0$
$f_{xx} < 0$
$-$ definite

local max at $(0,0)$

p. 276

Exc #12 $f(x,y) = e^{-x}(x^2+3y^2) : \mathbb{R}^2 \to \mathbb{R}^1$

$0 \le f(x,y)$ bounded below, but not from above:

$$\lim_{y\to+\infty} f(0,y) = \lim_{y\to+\infty} 3y^2 = +\infty$$

$$f_x = -e^{-x}(x^2+3y^2) + e^{-x}\cdot 2x$$
$$= e^{-x}(-x^2-3y^2+2x)$$

$$f_y = 6y\,e^{-x}.$$

Find c.p.

Solve $\begin{cases} e^{-x}(-x^2-3y^2+2x) = 0 \\ e^{-x}\cdot 6y = 0 \end{cases}$

Solve $\begin{cases} -x^2-3y^2+2x = 0 \\ 6y = 0 \end{cases}$ since $e^{-x} > 0$

Sol^n

$y = 0$ and

$-x^2 - \cancel{3y^2}^{\,0} + 2x = 0$

$-x^2 + 2x = 0$

$x = 0$ or $x = 2$

c.p. $\begin{cases} (0, 0) \\ (2, 0) \end{cases}$

$$f_{xx} = -e^{-x}(-x^2-3y^2+2x) + e^{-x}(-2x+2)$$
$$= e^{-x}[x^2+3y^2-2x-2x+2]$$

$$H_f = \begin{bmatrix} e^{-x}[x^2+3y^2-4x+2] & -6y\,e^{-x} \\ -6y\,e^{-x} & 6\,e^{-x} \end{bmatrix}$$

$$H_f = \underset{\substack{\vee \\ 0}}{e^{-x}} \begin{bmatrix} x^2 + 3y^2 - 4x + 2 & -6y \\ -6y & 6 \end{bmatrix}.$$

$$H_f(0,0) = 1 \cdot \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$

local minimum at $(0,0)$

$$H_f(2,0) = \underset{\substack{\vee \\ 0}}{e^{-2}} \begin{bmatrix} -2 & 0 \\ 0 & 6 \end{bmatrix}$$

Saddle at $(2,0)$