Oct 7, 2016

①

4.1 Multivariable Taylor polynomials.

Let $f: \overset{\text{open}}{X} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^1$ diff'ble at least twice

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots \frac{\partial f}{\partial x_n}\right) \quad \text{gradient}$$

$$Df = \left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \frac{\partial f}{\partial x_n}\right] \quad \text{Derivative matrix}$$

Jacobian.

<u>Defn</u>

Total differential

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \dots + \frac{\partial f}{\partial x_n} dx_n$$

<u>Defn</u> Hessian of $f: \overset{\text{open}}{X} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^1$

twice diff'ble

$$Hf = D(\nabla f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \cdots & & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

<u>Remark:</u>

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^1$

$Df: \mathbb{R}^n \longrightarrow \mathbb{R}^n \; (= 1 \times n \text{ matrices})$

$Hf: \mathbb{R}^n \longrightarrow \mathbb{R}^{n^2} \; (= n \times n \text{ matrices})$

Ex 1 $\quad f(x,y) = x^2e^y + x - y$

$$Df = \begin{bmatrix} 2xe^y + 1 & x^2e^y - 1 \end{bmatrix}.$$

$$df = (2xe^y + 1)dx + (x^2e^y - 1)dy$$

$$H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2e^y & 2xe^y \\ 2xe^y & x^2e^y \end{bmatrix}$$

$$Df(3,0) = \begin{bmatrix} 7 & 8 \end{bmatrix}$$

$$df(3,0) = 7dx + 8dy$$

$$H_f(3,0) = \begin{bmatrix} 2 & 6 \\ 6 & 9 \end{bmatrix}$$

First degree Taylor Polynomial

$$P_1((x,y),(3,0)) = 12 + 7(x-3) + 8(y-0)$$

$$f(3,0) = 9 \cdot e^0 + 3 - 0$$

$$P_1((x,y),(3,0)) = 12 + \begin{bmatrix} 7 & 8 \end{bmatrix} \begin{bmatrix} x-3 \\ y-0 \end{bmatrix}$$

$$f(3,0) + Df(3,0) \cdot \begin{bmatrix} x-3 \\ y-0 \end{bmatrix}$$

2nd degree Taylor Polynomial:

Closed form

$$P_2((x,y),(3,0)) = f(3,0) + Df(3,0)\cdot\begin{bmatrix} x-3 \\ y-0 \end{bmatrix} +$$

$$\frac{1}{2}\begin{bmatrix} x-3 & y-0 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ 6 & 9 \end{bmatrix}\begin{bmatrix} x-3 \\ y-0 \end{bmatrix}$$

Simplified:

$$P_2(x,y) = 12 + 7(x-3) + 8(y-0) +$$

$$\frac{1}{2}\left(2(x-3)^2 + 12(x-3)(y-0) + 9(y-0)^2\right)$$

Open form

$$P_2((x,y)(3,0)) = f(3,0) + \underbrace{\frac{\partial f}{\partial x}(3,0)}_{7}(x-3) + \underbrace{\frac{\partial f}{\partial y}(3,0)}_{8}(xy-0) +$$

$$+ \frac{1}{2}\left[\underbrace{\frac{\partial^2 f}{\partial x^2}(3,0)}_{2}(x-3)^2 + \underbrace{\frac{\partial^2 f}{\partial y\,\partial x}(3,0)}_{6}(x-3)(y-0) + \cdots\right.$$

$$\left. + \underbrace{\frac{\partial^2 f}{\partial x\,\partial y}(3,0)}_{6}(x-3)(y-0) + \underbrace{\frac{\partial^2 f}{\partial y^2}(3,0)}_{9}(y-0)^2\right]$$

Def Let $f: \underline{X}^{\text{open}} \subseteq \mathbb{R}^n \to \mathbb{R}^1$, $a \in \underline{X}$

Let f be twice diff'ble

$$P_1((x_1 \ldots x_n),(a_1 \ldots a_n)) = f(\vec{a}) + \underset{\substack{(1\times n) \\ \text{row}}}{Df(a)}\cdot\underset{\substack{(n\times 1) \\ \text{column}}}{(\vec{x}-\vec{a})^T}$$

$$P_2((x_1 \ldots x_n),(a_1 \ldots a_n)) =$$

$$= f(\vec{a}) + Df(a)\cdot(\vec{x}-\vec{a})^T + \frac{1}{2}\underset{1\times n}{[\vec{x}-\vec{a}]}\,\underset{n\times n}{H_f(a)}\,\underset{n\times 1}{[\vec{x}-\vec{a}]^T}$$

Open form:

$$p_1(x,a) = f(a) + \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(\vec{a}) \cdot (x_k - a_k)$$

$$p_2(x,a) = p_1(x,a) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(a)\,(x_i - a_i)(x_j - a_j)$$

$$p_k(x,a) = p_{k-1}(x,a) + \frac{1}{k!} \sum_{i_1 i_2 \cdots i_k} \frac{\partial^2 f}{\partial x_{i_k} \partial x_{i_2} \partial x_{i_1}}(a) \cdot (x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2}) \cdots (x_{i_k} - a_{i_k})$$

Ex #10 $\quad f = e^{2x+y} \qquad a = (0,0)$

$f(0,0) = 1$

$f_x = 2e^{2x+y} \qquad f_x(0,0) = 2$

$f_y = e^{2x+y} \qquad f_y(0,0) = 1$

$f_{xx} = 4e^{2x+y} \qquad f_{xx}(0,0) = 4$

$f_{xy} = f_{yx} = 2e^{2x+y} \qquad f_{xy}(0,0) = f_{yx}(0,0) = 2$

$f_{yy} = e^{2x+y} \qquad f_{yy}(0,0) = 1.$

$p_1(x,y) = 1 + 2(x-0) + 1(y-0) = 1 + 2x + y$

$$p_2(x,y) = 1 + 2x + y + \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= 1 + 2x + y + \frac{1}{2}\left[4x^2 + 2xy + 2yx + y^2\right]$$

$$= 1 + 2x + y + 2x^2 + 2xy + \frac{1}{2}y^2.$$

MT 1

(This is only for MT1; other exams may have different grade cuts.)

Average 69.8

Median 74.5.

A

—— 90 — 23 %

A −

—— 84

B +

—— 79

B — 38 %

—— 74

B −

—— 69

C +

—— 64

C — 18 %

—— 58

C −

—— 53

D +

—— 48

D

—— 43 — 21 %

D −

—— 38

F