Oct 6 2016

3.2

Length of curves:

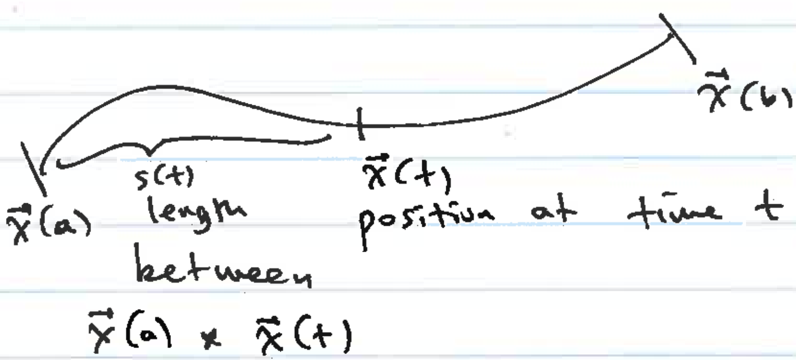
Let $\vec{x} : [a, b] \longrightarrow \mathbb{R}^n$ be a C^1 (piecewise C^1) parametrized curve.

We define the length of $\vec{x}$:

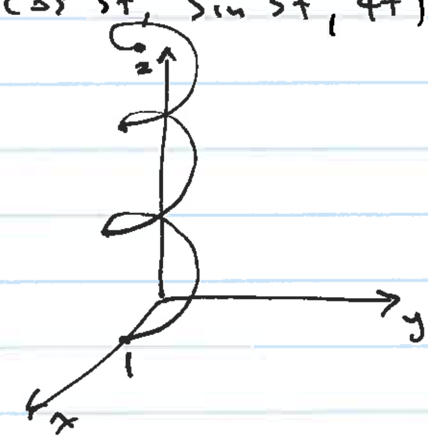
$$L(\vec{x}) = \int_a^b \|\vec{x}\,'(t)\|\, dt$$

Arclength parameter

$$s(t) = \int_a^t \|x'(u)\|\, du$$



Ex $\vec{x}(t) = (\cos 3t,\ \sin 3t,\ 4t)$ $\quad 0 \le t \le 2\pi$



velocity $X'(t) = (-3\sin 3t,\ 3\cos 3t,\ 4)$

speed $\|x'(t)\| = \sqrt{9\sin^2 3t + 9\cos^2 3t + 16} = 5$

$s(t) = \int_0^t 5\,du = 5t$

$\underbrace{s}_{} = 5t$

arclength parameter

$0 \le t \le 2\pi$

Length $\int_0^{2\pi} 5\,du = 10\pi$ length of $\vec{x}(t)$

$0 \le t \le 2\pi$.

p219 Exc #7 $\gamma(t) = \left(\ln t,\ \frac{t^2}{2},\ \sqrt{2}\,t\right)$ $1 \le t \le 4$

$\gamma'(t) = \left(\frac{1}{t},\ t,\ \sqrt{2}\right)$

$\|\gamma'(t)\| = \sqrt{\frac{1}{t^2} + t^2 + 2} = \left|t + \frac{1}{t}\right| = t + \frac{1}{t}$

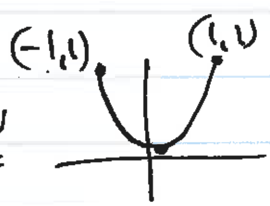
$1 \le t$

Length $= \int_1^4 \left(t + \frac{1}{t}\right) dt = \frac{t^2}{2} + \ln t \Big|_1^4$

$= \frac{16-1}{2} + \ln 4 - \ln 1$

$= \frac{15}{2} + \ln 4$

HW



$(t, t^2) = \vec{x}(t)$ $\|x'(t)\| = \sqrt{1 + 4t^2}$

$(1, 2t) = x'(t)$

$L = \int_{-1}^{1} \sqrt{1+4t^2}\,dt$ figure out!

4.1 Taylor Polynomials.

Review + Example. one Variable:

$f: \underset{\substack{\Psi \\ a}}{I} \subseteq \mathbb{R}^1 \longrightarrow \mathbb{R}^1$ diff'ble many times

Taylor polynomials at $x=a$.

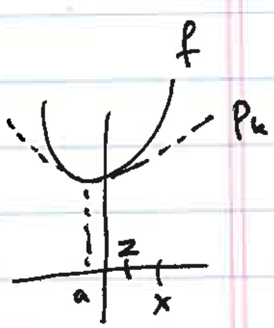
$$\begin{cases} p_1(x) = f(a) + f'(a)(x-a) \\ p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \\ p_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \end{cases}$$

$$\Big\{ P_k(x;a) = P_{k-1}(x;a) + \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

(Taylor's) Thm: (i) $\lim_{x \to a} \frac{R_k(x,a)}{(x-a)^k} = 0$ where

$$\underbrace{R_k(x,a)}_{\text{remainder}} = f(x) - \underbrace{P_k(x;a)}_{k^{\text{th}} \text{ degree T. P.}}$$

Provided that f is k-times diff'ble



(ii) $\exists\, z$ between a & x s.t.

$$R_k(x,a) = \frac{f^{(k+1)}(z)}{(k+1)!}(x-a)^{k+1}$$

if f is $k+1$ times diff'ble.

(iii) Consequently if $\|f^{(k+1)}\| \le M$, on the interval between x & a,

then $$\|R_k(x,a)\| \le \frac{M}{(k+1)!}(x-a)^{k+1}.$$

Example: $e^{0.04}$ approximate by using p_1, p_2, p_3 & estimate error, in each case.

$f(x) = e^x$, $a=0$ $\quad p_1 = f(0) + \frac{f'(0)}{1!}(x-0)$

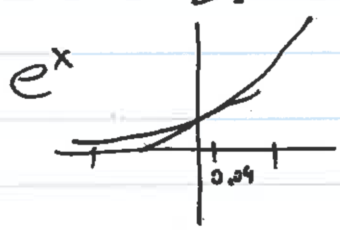
$$\left.\begin{array}{l} f(0)=1 \\ f'(0)=1 \end{array}\right\} \quad p_1 = 1 + x$$

$e^{0.04} \approx 1 + 0.04 = 1.04$. What is the error?

$$|R_1| \le \frac{M}{2!}(x-0)^2$$

$M \ge \|e^x\|$
$\|$
3

Can take $M = 3$ (generous, but not too hard to find.)



$x = 0.04$

$$|R_1| \le \frac{3}{2}(0.04)^2 = \frac{3}{2}\,0.0016 = 0.0024$$

$$e^{0.04} \approx 1.04 \pm 0.0024$$

$1.040 \pm 0.003 \leftarrow$ 1st order approx.

$$P_2(x) = 1 + x + \frac{x^2}{2}$$

$$P_2(0.04) = 1 + 0.04 + \frac{(0.04)^2}{2}$$

$$= 1.0408$$

$$e^{0.04} \approx 1.0408 \pm \underbrace{\qquad\qquad}_{\text{margin of error?}}$$

$$|R_2| \le \frac{M}{3!}(x-0)^3$$

$$M = 3$$

$$|R_2| \le \frac{3}{6}(0.04)^3 = 0.000032$$

$$e^{0.04} \approx \underbrace{1.040800 \pm 0.000032}_{2^{\text{nd}}\text{ order approx.}}$$

$$P_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

3rd order approximation

$$P_3(0.04) = 1.040810\overline{6} \pm 0.00000032$$

$$|R_3| \le \frac{3}{4!}(0.04)^4 = 0.00000032$$