2.6 IMPLICIT FUNCTION THM

Example 0 $\quad F(x,y) = c$

- if y can be solved as a function of x, $y = y(x)$ locally

$$\frac{d}{dx} F(x, y(x)) = F_x \cdot \underbrace{\frac{dx}{dx}}_{1} + F_y \frac{dy}{dx} = 0 = \frac{dc}{dx}$$

$$F_x + F_y y' = 0$$

- if $F_y \neq 0$ $\quad y' = -\dfrac{F_x}{F_y}$

Ex 1

$$\left.\begin{aligned} 2x + y + 3z &= 1 \\ 5x + 3y - z &= 2 \end{aligned}\right\} (*)$$

Solve for x & y in terms of z, & calculate $\dfrac{\partial x}{\partial z}, \dfrac{\partial y}{\partial z}$

$$2x + y = 1 - 3z$$
$$5x + 3y = 2 + z$$

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + z \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + z \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right)$$

②

$$\begin{bmatrix} \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial z} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

Compare $\begin{bmatrix} -3 \\ 1 \end{bmatrix} = -\begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

(*) $$\left.\begin{aligned} 2x + y + 3z &= 1 \\ 5x + 3y - z &= 2 \end{aligned}\right\}$$

Compare: inverse

$$F(x,y,z) = (2x + y + 3z,\ 5x + 3y - z) = (1,2)$$

$$F' = \begin{array}{c} \begin{array}{cccc} x & y & & z \end{array} \\ \left[\begin{array}{cc:c} 2 & 1 & 3 \\ 5 & 3 & -1 \end{array}\right] \end{array}$$

F_{xy} $\qquad$ F_z

want to solve x & y in terms of $\longrightarrow z$

Ex 2

$$2x^2 + yz + z^3 = 4$$
$$xyz + z^2 - x^3y^2 = 1$$

Can you solve. x & y in terms of z? Probably not.
(Does there exist a solution locally?)

If so, find $\frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}$ at $(1,1,1)$.

$$F(x,y,z) = (2x^2 + yz + z^3,\ xyz + z^2 - x^3y^2)$$

$$F' = \begin{bmatrix} 4x & z & y + 3z^2 \\ yz - 3x^2y^2 & xz - 2x^3y & xy + 2z \end{bmatrix}$$

$$F'(1,1,1) = \begin{array}{c} \begin{bmatrix} 4 & 1 & 4 \\ -2 & -1 & 3 \end{bmatrix} \\ \begin{array}{ccc} \underbrace{x \quad\quad y}_{\text{want to solve } x,y \text{ in terms of}} & & z \end{array} \end{array}$$

Implicit function Thm tells us that:

If $\begin{vmatrix} 4 & 1 \\ -2 & -1 \end{vmatrix} = -4 + 2 = -2 \neq 0$

then x, y are locally functions of z near $(1,1,1)$ and

$$\begin{bmatrix} \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial z} \end{bmatrix}(1,1,1) = -\begin{bmatrix} 4 & 1 \\ -2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$= \frac{-1}{-2} \begin{bmatrix} -1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$= \frac{+1}{2} \begin{bmatrix} -7 \\ 20 \end{bmatrix} = \begin{bmatrix} -7/2 \\ +10 \end{bmatrix}.$$

Caution if you compare to the textbook p 170, you will see that they are duals of each other, where the roles of x & y are interchanged!

IMPLICIT FUNCTION THM:

For given n equations in $n+m$ variables

n = # equations

n = # $x_1 \cdots x_n$ to be solved.

$$\left.\begin{aligned} F_1(x_1 \cdots x_n, y_1 \cdots y_m) &= c_1 \\ F_2(x_1 \cdots x_n, y_1 \cdots y_m) &= c_2 \\ &\vdots \\ F_n(x_1, \cdots x_n, y_1 \cdots y_m) &= c_n \end{aligned}\right\}$$

Want to solve $(x_1, \cdots x_n)$ in terms of $(y_1 \cdots y_m)$ locally

$$F(x_1, \cdots x_n, y_1 \cdots y_m) = (F_1, F_2 \cdots F_n)$$

$F: \mathbb{R}^{n+m} \to \mathbb{R}^n$, where F is continuously diff'ble

Let $(\vec{a}, \vec{b}) \in \mathbb{R}^{n+m}$ s.t. $F(a, b) = (c_1, c_2 \cdots c_n)$

$$\text{Let} \quad F' = \left[\begin{array}{ccc:ccc} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} & \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & & & \vdots & & \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} & \frac{\partial F_n}{\partial y_1} & \cdots & \frac{\partial F_n}{\partial y_m} \end{array}\right]$$

Want to solve $\vec{x}$ $\qquad F_x \qquad\qquad F_y \qquad$ want the solution in terms of $\vec{y}$, locally

If $(\det F_x(a, b)) \neq 0$, then there exists a local and diff'ble solution $\vec{x} = g(\vec{y})$, that is

$$F(g(y), y) = c. \quad \text{Furthermore:}$$

$$g'(a,b) = -F_x^{-1}(a, b)\, F_y(a, b)$$

↑ solving $\vec{x}$ $\qquad$ ↑ in terms of $\vec{y}$

Ex 3

$$F \begin{cases} x_1^3 + x_2 y_1 + x_3 y_1^3 + y_1 y_2^2 = 10 \\ x_1^4 + x_3 y_2 + x_2 y_1^4 + y_1^2 y_2 = 16. \end{cases}$$

$$(x_1, x_2, x_3, y_1, y_2) = (2, 1, 0, 1, -1)$$

Does there exist $\underbrace{(y_1, y_2)}_{\text{solving}} = \underbrace{f(x_1, x_2, x_3)}_{\text{in terms of } x_1, x_2, x_3}$? (locally)

$$F' = \begin{array}{c} \begin{bmatrix} 3x_1^2 & y_1 & y_1^3 & 3x_3 y_1^2 + x_2 + y_2^2 & 2y_1 y_2 \\ 4x_1^3 & y_1^4 & y_2 & 4y_1^3 \cdot x_2 + 2y_1 y_2 & x_3 + y_1^2 \end{bmatrix} \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & \underbrace{y_1 \qquad\qquad y_2}_{\text{solving for } y_1, y_2} \end{array} \end{array}$$

$$F' = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & y_1 & y_2 \end{array} \\ \left[\begin{array}{ccc:cc} 12 & 1 & 1 & 2 & -2 \\ 32 & 1 & -1 & 2 & 1 \end{array}\right] \end{array}$$

$$\det = 6 \neq 0$$

Imp. FT $\Longrightarrow \exists\ (y_1, y_2) = f(x_1, x_2, x_3)$ locally near $(2, 1, 0)$

$$Df = -\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 12 & 1 & 1 \\ 32 & 1 & -1 \end{bmatrix}$$

$$Df = -\frac{1}{6}\begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 12 & 1 & 1 \\ 32 & 1 & -1 \end{bmatrix}$$

$$= -\frac{1}{6}\begin{bmatrix} 76 & 3 & -1 \\ 40 & 0 & -4 \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \end{bmatrix}_{\text{at } (2,1,0)}$$

$(y_1, y_2) = f(x_1, x_2, x_3)$ near $(2,1,0)$.

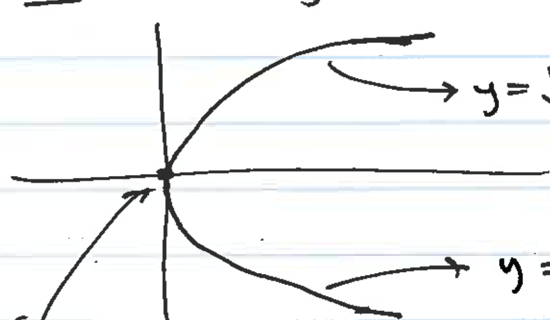
$f(2,1,0) = (1,-1)$.

$$\frac{\partial y_2}{\partial x_3}(2,1,0,1,-1) = \frac{\partial y_2}{\partial x_3}(2,1,0)$$

$$= \frac{-4}{-6} = \frac{2}{3}.$$

Remark: In Implicit Function Theorem, the existence of the solutions are always local.

Ex: $x = y^2$



$\to y = \sqrt{x}$ solves it for $y > 0$

$\to y = -\sqrt{x}$ " " for $y < 0$

at $(0,0)$ there is no function $y = f(x)$ which solves $x = y^2$ for $x \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$