26 Thm: Let $f: \mathbb{X}^{open} \subseteq \mathbb{R}^n \to \mathbb{R}^1$, $a \in \mathbb{X}$, $\nabla f(a) \neq 0$

Let $f$ be diff'ble at $a$, $a$. Then

Steepest ascend ↑ (i) $f$ increases most rapidly in the direction of $\nabla f(a)$, with a rate of $\|\nabla f(a)\|$

Steepest descend ↓ (ii) $f$ decreases most rapidly in the direction of $-\nabla f(a)$, with a rate of $-\|\nabla f(a)\|$

(iii) $\nabla f(a) \perp$ level set of $f$ thru $a$.

$$= \{ \vec{x} \in \mathbb{X} \mid f(\vec{x}) = c = f(\vec{a}) \}$$

Discussion about (iii)

If one takes a level set where the height = value of $f$ is constant.

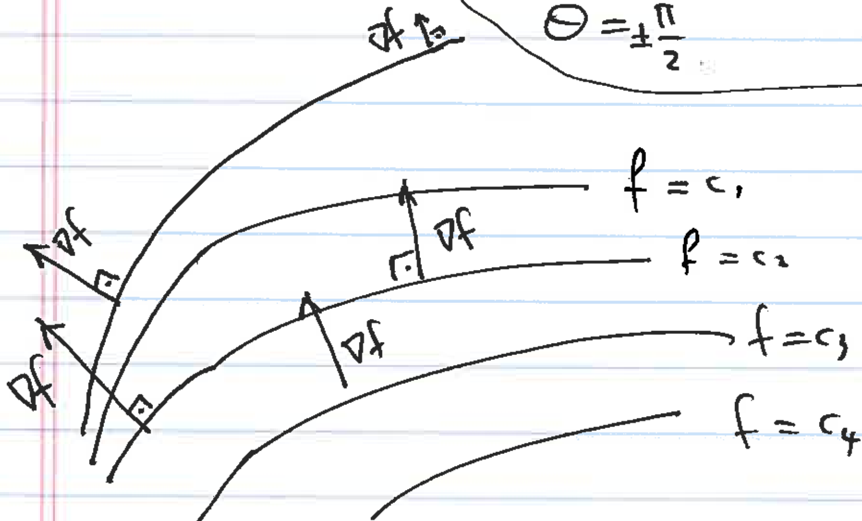
The directional derivative along the level sets $= 0$, since $f$ stays constant.

$$0 = (D_v f)(a) = |\nabla f(a)| \cdot |v| \cos \theta$$

$\cos \theta = 0$

$\theta = \pm \frac{\pi}{2}$

$\nabla f \perp$




Remark: when the level sets are closer to each other, $\nabla f$ is larger.

Ex

$$f(x,y,z) = x^2y + yz + xz;$$

find the directions of steepest ascend & steepest descend, at $(1,2,3)$.

$$\nabla f = (2xy + z,\ x^2 + z,\ y + x)$$

$$\nabla f(1,2,3) = (7, 4, 3) \rightsquigarrow \text{length} = \sqrt{74}$$

direction of steepest ascend $\dfrac{7i + 4j + 3k}{\sqrt{74}}$

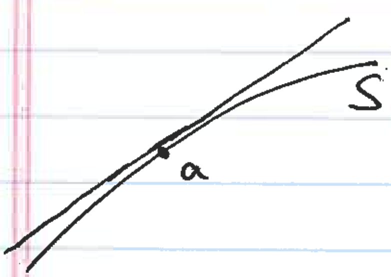
" " " descend $-($ " $)$

rate of increase $\sqrt{74}$; rate of decrease $-\sqrt{74}$

~~Ex~~ Consequence of (iii)

If $f: \underline{X}^{\text{open}} \subseteq \mathbb{R}^n \to \mathbb{R}$

$$a \in S = \{\vec{x} \in \underline{X} \mid f(\vec{x}) = c\}$$




If $f$ is diff'ble at $a$, and $\nabla f(a) \neq 0$. Then the equation of the tangent plane to $S$ at $a$

$$\nabla f(a) \cdot (\vec{x} - \vec{a}) = 0.$$

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Exc #18

Find the tangent plane the set defined by this eqⁿ.

$F := \boxed{2xz + yz - x^2y + 10 = 0}$ (Level set)

at the point $(1,-5,5)$

Solⁿ: $\nabla F = (2z - 2xy,\ z - x^2,\ 2x + y)$

$\nabla F(1,-5,5) = (20,\ 4,\ -3)$.

Tangent plane

$$(20, 4, -3) \cdot \big((x, y, z) - (1, -5, 5)\big) = 0$$

$$20x + 4y - 3z = (20, 4, -3) \cdot (1, -5, 5)$$

$$= 20 - 20 - 15 = -15$$

$$\boxed{20x + 4y - 3z = -15}$$ ✓

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#20 $x^2 - 2y^2 + 5xz = 7$ $\quad (-1, 0, -\frac{6}{5})$

a) old method:

$$z = \frac{7 - x^2 + 2y^2}{5x} = f(x,y)$$

$$\frac{\partial z}{\partial x} = \frac{(-2x)(5x) - 5\cdot(7 - x^2 + 2y^2)}{25x^2}$$

$$\frac{\partial z}{\partial x}(\underset{x}{-1}, \underset{y}{0}, -\tfrac{6}{5}) = \frac{(-10)\cdot(1) - 5(7 - 1 + 0)}{25} = \frac{-10-30}{25} = \frac{-40}{25} = -\frac{8}{5}$$

no need to write, since z is a dependent variable now.

$$\frac{\partial z}{\partial y} = \frac{4y}{5x}$$

$$\frac{\partial z}{\partial y}(-1,0) = 0$$

$$z = h(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

$$z = -\frac{6}{5} + \frac{-8}{5}(x+1) + 0\cdot(y-0)$$

$$z = -\frac{6}{5} - \frac{8}{5}x - \frac{8}{5} = -\frac{8}{5}x - \frac{14}{5}$$

$$5z = -8x - 14$$

$$5z + 8x = -14$$

#20 b Level set method

$$F(x,y,z) = x^2 - 2y^2 + 5xz$$

$$\nabla F = (2x + 5z, -4y, 5x)$$

$$\nabla F(-1, 0, -\tfrac{6}{5}) = (-2-6, 0, -5)$$

$$= (-8, 0, -5)$$

Implicit Eq$^n$

$\nabla F(a)\cdot(x-a)=0$

$$(-8, 0, -5)\cdot\left((x,y,z) - (-1, 0, -\tfrac{6}{5})\right) = 0$$

$$-8x - 5z = 8 + 6$$

$$8x + 5z = -14 \quad \left(\text{parts (a) \& (b) match}\right)$$

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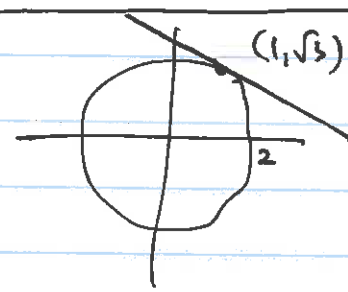
Calc I

$$x^2 + y^2 = 4$$




$$2x + 2yy' = 0$$

$$y' = -\frac{2x}{2y} = -\frac{F_x}{F_y}$$

$$F(x,y) = x^2 + y^2$$

$$F_x = 2x$$

$$F_y = 2y$$