2.6

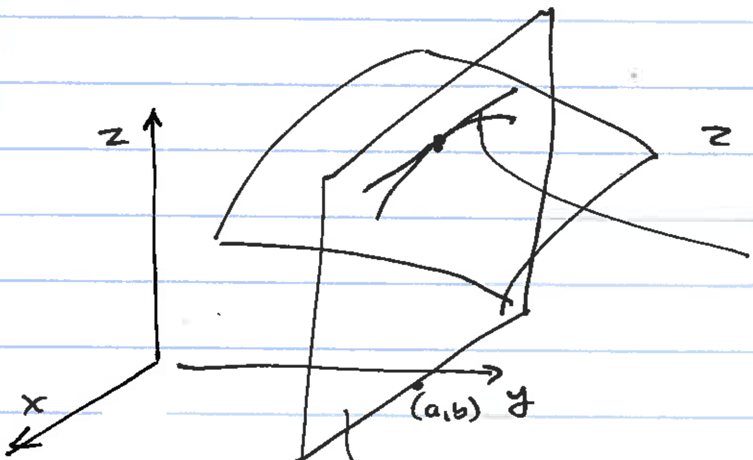
I Directional Derivatives.

$$f : X^{\text{open}} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^1$$

$x_1, x_2 \ldots x_n$

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots \frac{\partial f}{\partial x_n} \right) \quad \text{vector}$$

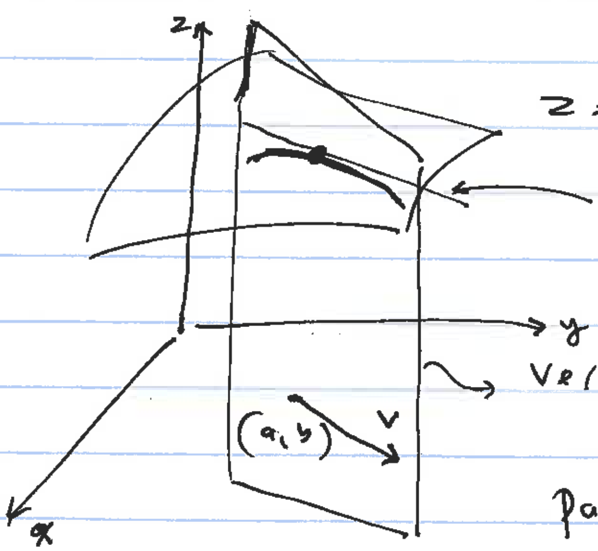
$$Df = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & & \frac{\partial f}{\partial x_n} \end{bmatrix} \quad \text{matrix.}$$



$z = f(x,y)$

$\frac{\partial f}{\partial x}(a,b)$ = slope of tangent line to the section $z = f(x,b)$ at $x = a$

$y = b$
// to x-axis



$z = f(x,y)$

slope $= (D_v f)(a,b)$

vertical plane, passing thru $(a,b,0)$ parallel to a $\underbrace{\text{direction}}_{\text{unit vector}}$ v

<u>Defn</u> Let $f: X^{open} \subseteq \mathbb{R}^n \to \mathbb{R}$
$\vec{a} \in X$, $\vec{v}$ be a unit direction vector

$$(D_v f)(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h}$$ if this limit exists.

<u>Theorem</u>: Let $f: X^{open} \subseteq \mathbb{R}^n \to \mathbb{R}$
$\vec{a} \in X$, $\|v\| = 1$,
* <u>Let f be diffble at a.</u>

Then $(D_v f)(\vec{a}) = \nabla f(a) \cdot \vec{v}$.

<u>Examples</u>:
p 173 #2 $f(x,y) = e^y(\sin x)$ $\vec{a} = (\frac{\pi}{3}, 0)$

$$\vec{u} = \frac{3i - j}{\sqrt{10}} \qquad |u| = 1$$

$$(D_u f)(\tfrac{\pi}{3}, 0) = \nabla f(\tfrac{\pi}{3}, 0) \cdot u$$

$$\frac{\partial f}{\partial x} = e^y \cos x \qquad \frac{\partial f}{\partial x}(\tfrac{\pi}{3}, 0) = e^0 \cos\tfrac{\pi}{3} = \tfrac{1}{2}$$

$$\frac{\partial f}{\partial y} = e^y \sin x \qquad \frac{\partial f}{\partial y}(\tfrac{\pi}{3}, 0) = e^0 \sin\tfrac{\pi}{3} = \tfrac{\sqrt{3}}{2}$$

$$(D_u f)(\tfrac{\pi}{3}, 0) = \underbrace{(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2})}_{\nabla f} \cdot \underbrace{(3, -1)\tfrac{1}{\sqrt{10}}}_{u} = \frac{1}{\sqrt{10}}\left(\frac{3}{2} - \frac{\sqrt{3}}{2}\right)$$

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$$f(x,y,z) = \frac{x e^{y}}{3z^2+1} \qquad \vec{a} = (2,-1,0)$$

$$u = i - 2j + 3k$$

Directional derivative of f at a, in a direction parallel to u.

each at $(2,-1,0)$

$$\frac{\partial f}{\partial x} = \frac{e^{y}}{3z^2+1} \qquad \frac{\partial f}{\partial x}(2,-1,0) = \frac{1}{e}$$

$$\frac{\partial f}{\partial y} = \frac{x e^{y}}{3z^2+1} \qquad \frac{\partial f}{\partial y}(2,-1,0) = \frac{2}{e}$$

$$\frac{\partial f}{\partial z} = x e^{y} \cdot \frac{-6z}{(3z^2+1)^2} \qquad \frac{\partial f}{\partial z}(2,-1,0) = 0$$

$$\nabla f(2,-1,0) = \left(\frac{1}{e}, \frac{2}{e}, 0\right)$$

$$v = \frac{u}{|u|} = \frac{i - 2j + 3k}{\sqrt{14}}$$

$$(D_v f)(a) = \left(\frac{1}{e}, \frac{2}{e}, 0\right) \cdot (1,-2,3)\frac{1}{\sqrt{14}}$$

$$= \frac{-3}{\sqrt{14} \cdot e}$$

Exc
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$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) $$\frac{\partial f}{\partial x}(0,0) = \frac{d}{dx} f(x,0)\Big|_{x=0}$$

$$= \frac{d}{dx} 0 = 0.$$

$$\frac{\partial f}{\partial y}(0,0) = 0$$

$$(\nabla f)(0,0) = (0,0)$$

(b) $\vec{v} = (a,b) = a\vec{i} + b\vec{j}$ $\quad a^2+b^2=1$ unit vector

$$(D_{ai+bj} f)(0,0) = \lim_{h\to 0} \frac{f((0,0)+h(a,b)) - f(0,0)}{h}.$$

$$= \lim_{h\to 0} \frac{f(ah,bh) - 0}{h.} = \lim_{h\to 0} \left(\frac{\dfrac{ah \cdot bh}{\sqrt{(ah)^2+(bh)^2}} - 0}{h.} \right)$$

$$= \lim_{h\to 0} \frac{ab\ h^2}{h\sqrt{a^2h^2+b^2h^2}} = \lim_{h\to 0} \frac{ab\ h^2}{h\ \underbrace{\sqrt{a^2+b^2}}_{1}\ \sqrt{h^2}}$$

$$= \lim_{h\to 0} \frac{ab\ h^2}{h\ |h|} = \lim_{h\to 0} ab \cdot \frac{h}{|h|}$$

⑤

$$\lim_{h\to 0} ab\frac{h}{|h|} = 0 \quad \text{Only when } ab = 0$$

$$\lim_{h\to 0} ab\frac{h}{|h|} \ \underline{\underline{\text{DNE}}} \quad \text{if } ab \neq 0$$

Recall $\lim_{h\to 0^-}\frac{h}{|h|} = -1$; $\lim_{h\to 0^+}\frac{h}{|h|} = +1$

If $ab \neq 0$; $(D_v f)(a,b) \neq \nabla f(a,b)\cdot(v)$

happens when $ai+bj \neq i$ and $ai+bj \neq j$

This $\underline{\underline{\text{Fails}}}$ since f is not diff'ble at $(0,0)$

Ⅱ Steepest ascend / descend

Assume $f : X^{open} \subseteq \mathbb{R}^n \to \mathbb{R}$:

$a \in X$, $\underline{f \text{ is diff'ble at } \vec{a}}$.

$\|v\| = 1$

$$(D_v f)(a) = \nabla f(a)\cdot v = |\nabla f(a)|\cdot|v|\cdot\cos\theta$$

where θ is the angle between $\nabla f(a)$ & $\vec{v}$.

When is $\cos\theta$ largest? when $\theta = 0$; $\cos 0 = 1$

When is $\cos\theta$ smallest? when $\theta = \pi$; $\cos\pi = -1$

$$(D_v f)(a) = |\nabla f(a)| \cdot \underbrace{|v|}_{1} \cdot \cos\theta$$

$(D_v f)(a)$ is largest when $\cos\theta = 1$, $\theta = 0$

$v \parallel \nabla f(a)$

& in the same direction

When $\theta = 0$ $(D_v f)(a) = |\nabla f(a)|$

$(D_v f)(a)$ is smallest when $\cos\theta = -1$, $\theta = \pi$

$v \parallel \nabla f(a)$

& in opposite direction

when $\theta = \pi$

$(D_v f)(a) = -|\nabla f(a)|$.