9/26/16 ①

2.5 <u>Chain Rule</u> Open form.

<u>Ex 1</u>

$$g(t) = (t^2, t^3) = (x,y)$$
$$z = f(x,y) = \cos x^2 y + \frac{x}{y}$$

$$\begin{tikzcd} & x \arrow[dr] & \\ t \arrow[ur, "g"] \arrow[dr] & & z \\ & y \arrow[ur, "f"'] & \end{tikzcd}$$

[Caution: Textbook
- is backwards
- has no arrows]

$$\mathbb{R}^1 \longrightarrow \underset{y \neq 0}{\mathbb{R}^2} \longrightarrow \mathbb{R}^1$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= \left(2xy(-\sin x^2y) + \frac{1}{y}\right)\cdot 2t \; +$$
$$+ \left(x^2(-\sin x^2 y) - \frac{x}{y^2}\right)\cdot 3t^2$$

- $$\left.\frac{dz}{dt}\right|_{t=2} = \left(64(-\sin 144) + \frac{1}{8}\right)\cdot 4 + \left(16(-\sin 144) - \frac{1}{16}\right)\cdot 1$$

$t = 2 \implies x = 4$ since $x = t^2$

$\qquad\qquad\; y = 8 \qquad\quad\; y = t^3$

Write As a function of t only $\quad x = t^2 \quad y = t^3$

- $$\frac{dz}{dt} = \left(2t^5(-\sin t^7) + \frac{1}{t^3}\right)\cdot 2t + \left(t^4(-\sin t^7) - \frac{t^2}{t^6}\right)3t^2$$

Ex 2

$$x = u + 3v^2 \qquad s = x^2 - 3xy$$
$$y = u^2 - 4v \qquad t = x^3 + e^y$$

$$\begin{array}{ccccc} u,v & & x,y & & s,t \\ \mathbb{R}^2 & \xrightarrow{F} & \mathbb{R}^2 & \xrightarrow{G} & \mathbb{R}^2 \end{array}$$

$$\begin{tikzcd} u \arrow[r] \arrow[dr] & x \arrow[r] \arrow[dr] & s \\ v \arrow[r] \arrow[ur] & y \arrow[r] \arrow[ur] & t \end{tikzcd}$$

$$\frac{\partial s}{\partial u} = \frac{\partial s}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial s}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial s}{\partial v} = \frac{\partial s}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial s}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial t}{\partial u} = \frac{\partial t}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial t}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial t}{\partial v} = \frac{\partial t}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial t}{\partial y}\frac{\partial y}{\partial v}$$

compare

$$F(u,v) = (x,y) \qquad (s,t) = (G \circ F)(u,v)$$
$$G(x,y) = (s,t)$$

$$D(G \circ F) = DG \cdot DF$$

$$\begin{bmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$$\overbrace{\begin{aligned} x &= u + 3v^2 \\ y &= u^2 - 4v \end{aligned}}^{F} \qquad \overbrace{\begin{aligned} s &= x^2 - 3xy \\ t &= x^3 + e^y \end{aligned}}^{G}$$

$$\frac{\partial x}{\partial u} = 1 + 0 \qquad \frac{\partial s}{\partial x} = 2x - 3y$$

$$\frac{\partial x}{\partial v} = 6v \qquad \frac{\partial s}{\partial y} = -3x$$

$$\frac{\partial y}{\partial u} = 2u \qquad \frac{\partial t}{\partial x} = 3x^2$$

$$\frac{\partial y}{\partial v} = -4 \qquad \frac{\partial t}{\partial y} = e^y$$

$$DF = \begin{bmatrix} 1 & 6v \\ 2u & -4 \end{bmatrix} \qquad \begin{bmatrix} 2x-3y & -3x \\ 3x^2 & e^y \end{bmatrix} = DG$$

$$F(u,v) = (x,y) \qquad G(x,y) = (s,t)$$

$$D(G \circ F) = DG \cdot DF$$

$$\begin{bmatrix} 2x-3y & -3x \\ 3x^2 & e^y \end{bmatrix} \begin{bmatrix} 1 & 6v \\ 2u & -4 \end{bmatrix}$$

$$= \begin{bmatrix} (2x-3y) + (-3x)(2u) & (2x-3y)6v + 12x \\ 3x^2 + 2u e^y & 18x^2 v - 4e^y \end{bmatrix}$$

$$\frac{\partial t}{\partial u}$$

Ill-posed questions.

Ex $\qquad F(x,y,t) = x^2 t + xyt^2.$

$$x = t^2$$
$$y = t^3$$

$$\frac{d}{dt} F(x,y,t) = ?$$

using t for two different roles.

$$\begin{tikzcd} & x \arrow[dr] & \\ t \arrow[ur] \arrow[r] \arrow[dr] & y \arrow[r] & F \\ & \bar{t} \arrow[ur] & \end{tikzcd}$$

use $t, \bar{t}$ to seperate the roles.

$$\frac{\partial x}{\partial \bar{t}} = 0 \qquad \frac{\partial x}{\partial t} = 2t$$

$$\frac{d}{dt} F(x,y,t) = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial F}{\partial \bar{t}} \cdot \frac{\partial \bar{t}}{\partial t}$$

$$= (2x\bar{t} + y\bar{t}^2) \cdot 2t +$$
$$+ (x\bar{t}^2) \cdot 3t^2 +$$
$$+ (x^2 + 2xy\bar{t}) \cdot 1$$

Ex #30 p156

$$z = f(x,y)$$

Show that $$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

where $\left\{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right.$

Soln.

$$\begin{tikzcd} r \arrow[r] \arrow[dr] & x \arrow[dr] & \\ \theta \arrow[ur] \arrow[r] & y \arrow[r] & z. \end{tikzcd}$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cdot\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y} r\cos\theta$$

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)^2 +$$

$$+ \frac{1}{r^2}\left(\frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y} r\cos\theta\right)^2.$$

$$= \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\,\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta +$$

$$+ \frac{1}{\cancel{r^2}}\left(\left(\frac{\partial z}{\partial x}\right)^2\cancel{r^2}\sin^2\theta - 2\,\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\,\underline{\sin\theta\cos\theta}\,\cancel{r^2} + \left(\frac{\partial z}{\partial y}\right)^2\cancel{r^2}\cos^2\theta\right)$$

Factor & cancel r^2's.

PTO

$$= \left(\frac{\partial z}{\partial x}\right)^2 \cos^2\theta + \cancel{2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta} + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2\theta \; +$$

$$+ \left(\frac{\partial z}{\partial x}\right) \sin^2\theta - \cancel{2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta} + \left(\frac{\partial z}{\partial y}\right)^2 \cos^2\theta$$

$$= \left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right)\left(\underbrace{\cos^2\theta + \sin^2\theta}_{1}\right)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \quad \#$$