9/23/16 ①

2.5 Chain Rule

Recall

$$\begin{array}{ccccc} I & \xrightarrow{g} & J & \xrightarrow{f} & \mathbb{R}^1 \\ \cap\!| & & \cap\!| & & \\ \mathbb{R}^1 & & \mathbb{R}^1 & & \\ u & & x & & y \end{array}$$

$x = g(u)$
$y = f(x)$
$y = (f \circ g)(u)$

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du} \qquad \frac{dy}{du} = f'(g(u)) \cdot g'(u)$$

Theorem Let

$$\begin{array}{ccccc} U^{\text{open}} & \xrightarrow{G} & V^{\text{open}} & \xrightarrow{F} & \mathbb{R}^k \\ \cap\!| & & \cap\!| & & \\ \mathbb{R}^n & & \mathbb{R}^m & & \end{array}$$

Let G be diff'ble at $a \in U$;
Let F be diff'ble at $G(a) \in V$.
Then $F \circ G$ is diff'ble at a and

$$D(F \circ G)(a) = DF(G(a)) \cdot DG(a)$$

$(k \times m) \cdot (m \times n)$

$k \times n.$ $\qquad$ $k \times n$

Exc # 28 p 156

$g : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$g(1,-1,3) = (2,5)$

$Dg(1,-1,3) = \begin{bmatrix} 1 & -1 & 0 \\ 4 & 0 & 7 \end{bmatrix}.$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$f(x,y) = (2xy,\ 3x - y + 5)$

$D(f \circ g)(1,-1,3) = ?$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \mathbb{R}^2 \arrow[r, "f"] & \mathbb{R}^2 \end{tikzcd}$$

$(1,-1,3) \longmapsto (2,5)$

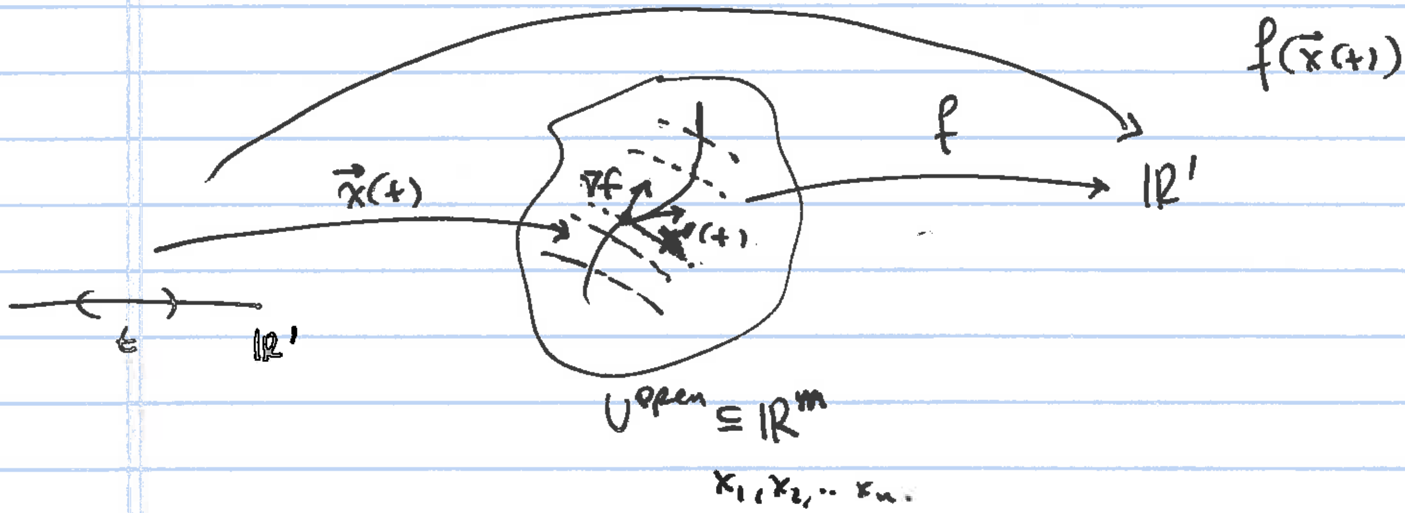
$D(f \circ g)(1,-1,3)) = Df(2,5) \cdot Dg(1,-1,3)$

$Df = \begin{bmatrix} 2y & 2x \\ 3 & -1 \end{bmatrix}$

$Df(\underset{\underset{x}{\shortparallel}}{2}, \underset{\underset{y}{\shortparallel}}{5}) = \begin{bmatrix} 10 & 4 \\ 3 & -1 \end{bmatrix}$

$D(f \circ g)(1,-1,3) = \underset{2\times 2}{\begin{bmatrix} 10 & 4 \\ 3 & -1 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 1 & -1 & 0 \\ 4 & 0 & 7 \end{bmatrix}} = \underset{2\times 3}{\begin{bmatrix} 26 & -10 & 28 \\ -1 & -3 & -7 \end{bmatrix}}$

Crucial Case of (open form)



$$\underbrace{D\left(f(\vec{x}(t)\right)}_{1\times 1} = \underbrace{\underbrace{Df\,(x(t))}_{1\times m} \cdot \underbrace{D\vec{x}(t)}_{m\times 1}}_{1\times 1}$$

$\vec{x} = (x_1(t), x_2(t), \dots, x_n(t))$

$\vec{x}' = (x_1', x_2', \dots x_n')$

$x_i' = \frac{\partial x_i}{\partial t}$

$$= \Big[\underbrace{\nabla f\,(x(t))}_{\text{gradient of } f} \cdot \underbrace{x'(t)}_{\text{velocity of } x(t)}\Big]$$

$$= \left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \frac{\partial f}{\partial x_3} \cdots \frac{\partial f}{\partial x_n}\right] \cdot \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$$

$$\frac{df(x(t))}{dt} = \underbrace{\frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial f}{\partial x_3}\frac{dx_3}{dt} + \cdots + \frac{\partial f}{\partial x_m}\frac{dx_m}{dt}}_{\#\text{ of terms} = m \;=\; \dim \mathbb{R}^m \text{ middle domain}}$$