Quiz #2, Sept 23
1.4; 1.5; 2.1

2.4 Properties of derivatives

$F, G : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$
$f, g : X \subseteq \mathbb{R}^n \to \mathbb{R}^1$

- $$\frac{\partial}{\partial x_i}(f+g) = \frac{\partial f}{\partial x_i} + \frac{\partial g}{\partial x_i}$$

$$\boxed{D(F+G) = DF + DG}$$ as $m \times n$ matrices

- $$\frac{\partial}{\partial x_i}(f \cdot g) = \frac{\partial f}{\partial x_i} \cdot g + f \cdot \frac{\partial g}{\partial x_i}$$

$$\boxed{\underbrace{D(f \cdot g)}_{1\times n} = \underbrace{(Df)}_{1\times n}\underbrace{g}_{\text{real}} + f\underbrace{(Dg)}_{1\times n}}$$

- $$\frac{\partial}{\partial x_i}(c \cdot f) = c\,\frac{\partial f}{\partial x_i} \qquad c \in \mathbb{R}$$

$$\boxed{D(c \cdot f) = c\,Df}$$ as $1 \times n$ matrices.

Ex
$$F(x,y) = (x^2+y,\ x-y,\ xy)$$
$$G(x,y) = (2x,\ y^2,\ x+y)$$

$$DF = \begin{bmatrix} 2x & 1 \\ 1 & -1 \\ y & x \end{bmatrix} \qquad DG = \begin{bmatrix} 2 & 0 \\ 0 & 2y \\ 1 & 1 \end{bmatrix}$$

2

$$F+G = (x^2+y+2x,\ x-y+y^2,\ xy+x+y)$$

$$DF+DG = \begin{bmatrix} 2x & 1 \\ 1 & -1 \\ y & x \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2y \\ 1 & 1 \end{bmatrix}$$

Check. = ✓

Verifying and identity

$$D(F+G) = \begin{bmatrix} 2x+2 & 1 \\ 1 & -1+2y \\ y+1 & x+1 \end{bmatrix}$$

Ex) $F=(g,h)$ vector valued, f real valued $\Big\}$ both on $\mathbb{R}^2_{x,y}$

$D(f\cdot F)$ is a 2×2 matrix

$$f\cdot F = f(g,h) = (fg, fh)$$

$$D((fg,fh)) = \begin{bmatrix} (fg)_x & (fg)_y \\ (fh)_x & (fh)_y \end{bmatrix}$$

$$= \begin{bmatrix} f_x g + f\cdot g_x & f_y g + f\cdot g_y \\ f_x h + f\cdot h_x & f_y h + f\cdot h_y \end{bmatrix}$$

$$= f \underbrace{\begin{bmatrix} g_x & g_y \\ h_x & h_y \end{bmatrix}}_{D((g,h))=DF} + \underbrace{\begin{bmatrix} f_x g & f_y g \\ f_x h & f_y h \end{bmatrix}}_{2\times 2}$$

$$= f \underbrace{\begin{bmatrix} g_x & g_y \\ h_x & h_y \end{bmatrix}}_{DF} + \underbrace{\overbrace{\begin{bmatrix} g \\ h \end{bmatrix}}^{F}}_{(2\times 1)} \underbrace{\overbrace{\begin{bmatrix} f_x & f_y \end{bmatrix}}^{Df}}_{(1\times 2)}$$

$\underbrace{(2\times 1)\ (1\times 2)}_{2\times 2 \text{ matrix.}}$

General Case

$$\boxed{D(f\cdot F) = f \cdot DF + F \cdot Df}$$

$f : \mathbb{R}^n \to \mathbb{R}^1$

$F : \mathbb{R}^n \to \mathbb{R}^m$

$m\times n$. matrix

real valued $\quad m\times n$ — $\underbrace{}_{m\times n}$

$m\times 1 \quad 1\times n$ — $\underbrace{}_{m\times n}$

$1\times n$

$$\boxed{D(F\cdot G) = \underbrace{G^T \cdot DF + F^T \cdot DG.}}$$

$1\times m \quad m\times n \quad 1\times m \quad m\times n$

$F, G : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$

row matrix

$F \cdot G : \mathbb{R}^n \to \mathbb{R}^1$

Dot product.

matrix multiplication

Higher order Derivatives

Def

$$\frac{\partial^2 f}{\partial x \partial y} = \underbrace{\frac{\partial}{\partial x}\underbrace{\left(\frac{\partial}{\partial y}\right)}_{\text{do first}}}_{\text{do second}} f$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f\right)$$

$$(f_{xy}) = (f_x)_y$$

$\frac{\partial}{\partial x}$ first $\qquad \frac{\partial}{\partial y}$ next

Ex $\quad f(x,y,z) = x^2y^3 + yz$.

Find all first & Second order partial derivatives

$$f_x = 2xy^3$$

$$f_y = 3y^2x^2 + z$$

$$f_z = y$$

$$f_{xx} = 2y^3$$

$$f_{xy} = (f_x)_y = 6xy^2$$

$$f_{xz} = 0$$

$$f_{yx} = (f_y)_x = 6xy^2$$

$$f_{yy} = 6yx^2$$

$$f_{yz} = (f_y)_z = 1$$

$$f_{zx} = (f_z)_x = 0$$

$$f_{zy} = 1$$

$$f_{zz} = 0$$

$$D(\nabla f) = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} 2y^3 & 6xy^2 & 0 \\ 6xy^2 & 6yx^2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Thm: If $f: \underline{X}^{open} \subseteq \mathbb{R}^2 \to \mathbb{R}$ has continuous first and second order derivatives, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Caution: This is not always true

cf. Exc 30 p 142

Ex

$$\frac{\partial^2}{\partial x \partial y}(x^3 + xy, xz)$$

$$= \frac{\partial}{\partial x} \underbrace{\frac{\partial}{\partial y}(x^3 + xy, xz)}$$

$$= \frac{\partial}{\partial x}(x, 0)$$

$$= (1, 0)$$