(2.3) Continue.

How do we see if a function is diff'ble?

Harder to use.

Thm: Let $f: X^{\text{open}} \subseteq \mathbb{R}^2 \to \mathbb{R}$.

If $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are continuous on X, then

f is diff'ble on X.

Easier to use

Thm: Let $f, g: X^{\text{open}} \subseteq \mathbb{R}^2 \to \mathbb{R}$, and both be diff'ble at $(a,b) \in X$.

Then (i) $f+g$ is diff'ble at (a,b)

(ii) $f \cdot g$ " " " "

(iii) $f-g$ " " " "

(iv) $c \cdot f$ " " " " $\forall c \in \mathbb{R}$

(v) f/g " " " ", if $g(a,b) \neq 0$ ($g(x,y) \neq 0$ near (a,b))

Thm: (informally)

(We'll see it (in 2.5)) Compositions of diff'ble functions are diff'ble.

#36 $f(x,y) = \left(\frac{xy^2}{x^2+y^4}, \frac{x}{y} + \frac{y}{x} \right)$

D = Domain of $f = \{(x,y) \mid x \neq 0 \text{ and } y \neq 0\}$

x is diff'ble on D, y^2 is diff'ble on D $\Rightarrow xy^2$ is diff'ble on D

x^2+y^4 " " " $\Rightarrow \frac{xy^2}{x^2+y^4}$ is diff'ble on D.

$\frac{x}{y}$ diff'ble on D since x & y are, $y \neq 0$

$\frac{y}{x}$ " " " y & x are, $x \neq 0$

$\frac{x}{y} + \frac{y}{x}$ " " " $\frac{x}{y}, \frac{y}{x}$ are

$\left(\frac{xy^2}{x^2+y^4}, \frac{x}{y} + \frac{y}{x}\right)$ is diff'ble on D, since each component is a diff'ble function of (x,y) on D.

<u>Defn</u> Let $f: \underset{x_1, x_2, \dots x_n}{\underline{X} \subseteq \mathbb{R}^n} \to \mathbb{R}$.

The gradient of f, $\vec{\nabla} f = \underbrace{\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots \frac{\partial f}{\partial x_n}\right)}_{n = \#\text{ variables}}$

<u>Defn</u> Let $F: \underset{x_1 \dots x_n}{\underline{X} \subseteq \mathbb{R}^n} \to \mathbb{R}^m$ $F = (F_1, F_2, \dots F_m)$

The (first) derivative matrix:

$$\underset{m \times n}{DF} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & & \frac{\partial F_2}{\partial x_n} \\ \vdots & & & & \\ \frac{\partial F_m}{\partial x_1} & \cdots & & & \frac{\partial F_m}{\partial x_n} \end{bmatrix}$$

Ex $f(x,y,z) = x^2y + xz + e^y z^2$: $\mathbb{R}^3 \to \mathbb{R}^1$

vector: $\nabla f = (2xy + z,\ x^2 + e^y z^2,\ x + 2ze^y)$

$$Df = \begin{bmatrix} 2xy + z & x^2 + e^y z^2 & x + 2ze^y \end{bmatrix}$$

1×3 matrix.

Ex $F(x,y,z) = (x^2 + y,\ x^3y^2 + 3z)$

$$\underset{x,y,z}{\mathbb{R}^3} \longrightarrow \mathbb{R}^2$$

$$\underset{2\times 3}{DF} = \begin{bmatrix} 2x & 1 & 0 \\ 3x^2y^2 & 2x^3y & 3 \end{bmatrix}$$

Ex $G(x,y) = (x + 3y,\ 2x - y,\ x + 6y)$

$$\underset{x,y}{\mathbb{R}^2} \longrightarrow \mathbb{R}^3$$

$$\underset{3\times 2}{DG} = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 1 & 6 \end{bmatrix}$$

(Matrix of the linear transformation G

Exc #20 p 132

$f(x,y,z) = \sin xyz \qquad \vec{a} = (\pi, 0, \frac{\pi}{2})$

$\nabla f(\vec{a}) = ?$

$\nabla f = (yz\cos xyz, \; xz\cos(xyz), \; xy\cos xyz)$

$\nabla f(a) = \left(0, \; \frac{\pi^2}{2}, \; 0\right).$

Exc. #28 p132

$f(x,y,z) = (2x - 3y + 5z, \; x^2 + y, \; \ln yz)$

$a = (3, -1, -2)$

$Df(a) = ?$

$$Df = \begin{array}{cc} \begin{bmatrix} 2 & -3 & 5 \\ 2x & 1 & 0 \\ 0 & \frac{z}{yz} & \frac{y}{yz} \end{bmatrix} & \begin{array}{c} f_1 \\ f_2 \\ f_3 \end{array} \\ \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{array} & \end{array}$$

$$Df(3,-1,-2) = \begin{bmatrix} 2 & -3 & 5 \\ 6 & 1 & 0 \\ 0 & -1 & -\frac{1}{2} \end{bmatrix}$$

<u>Def</u> Let $F : \overset{\text{open}}{X} \subseteq \underset{x_1, x_2 \ldots x_n}{\mathbb{R}^n} \longrightarrow \mathbb{R}^m$ $\qquad F = (F_1, F_2 \ldots F_m)$

Let $\vec{a} \in X$.

F is called diff'ble at a if

(i) All $\dfrac{\partial F_i}{\partial x_j}(a)$ exist, $\quad i = 1, \ldots m$, $j = 1, \ldots n$

and

(ii) $$\underbrace{H(\vec{x})}_{\substack{m\times 1\\ \text{column}\\ \text{matrix}}} = \underbrace{F(a)}_{\substack{m\times 1\\ \text{column}\\ \text{matrix}}} + \underbrace{\underbrace{DF(\vec{a})}_{m\times n} \cdot \underbrace{(x-a)}_{n\times 1 \text{ column matrix}}}_{m\times 1 \text{ '' ''}}$$

is a linear approximation of $F(\vec{x})$ s.t

$$\lim_{\vec{x} \to \vec{a}} \frac{\| F(x) - H(x) \|}{\| x - a \|} = 0.$$

(iii) if f is diff'ble at $\vec{a}$ then

$H(\vec{x})$ is called the tangent plane approx. of $F(x)$, at $\vec{x} = \vec{a}$

$H(\vec{x})$ is also called the first degree Taylor polynomial for F at $\vec{a}$.

Ex $F(x,y,z) = (x^2+y^2,\ xyz)$

$a = (1, 2, 0)$.

$$\underset{2\times 3}{DF} = \begin{bmatrix} 2x & 2y & 0 \\ yz & xz & xy \end{bmatrix}$$

$$DF(a) = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

$$F(a) = F(1,2,0) = (5, 0)$$

write as a 2×1 column matrix.

$$H(x,y,z) = \begin{bmatrix} 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \\ z-0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 2(x-1)+4(y-2)+0(z-0) \\ 0(x-1)+0(y-2)+2(z-0) \end{bmatrix}$$

$$= \begin{bmatrix} 5+2(x-1)+4(y-2) \\ 0+2(z-0) \end{bmatrix}$$

In this set up we also write $F(\vec{x})$ as a column matrix:

$$F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x^2+y^2 \\ xyz \end{bmatrix}$$

H is the Tangent plane approx. of F, at $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$