(2.3)

- $f: \underset{\substack{\cup \\ a}}{\mathbb{R}^1} \longrightarrow \mathbb{R}^1$ single variable function.

$$\begin{tikzcd}
f'(a) \text{ exists} \arrow[r, Leftrightarrow] \arrow[dr, Rightarrow] & f \text{ is diff'ble at } a. \arrow[d, Rightarrow] \\
& f \text{ is continuous at } a.
\end{tikzcd}$$

- $f: X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^1$

$n \geq 2$, multivariable function

$\vec{a} \in X$

$$\begin{tikzcd}
\text{All } \dfrac{\partial f}{\partial x_1}(\vec{a}), \dots \dfrac{\partial f}{\partial x_n}(a) \text{ exist} \arrow[dr, "\not\Rightarrow", "\text{no}"'] & f \text{ is diff'ble at } a. \arrow[l, Rightarrow, "\text{defn.}"', "\not\Rightarrow\ \text{no}"] \arrow[d, shift right, Rightarrow] \\
& f \text{ continuous at } a \arrow[u, shift right, "\not\Rightarrow\ \text{no}\ ⊛⊛"']
\end{tikzcd}$$

Counter example $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq 0 \\ 0 & \text{if } x=y=0 \end{cases}$

⊛⊛ Counterexample $f(x,y) = |x|$ not diff'ble at $(0,0)$.

Ex $f(x,y) = x^2 + y^2 : \mathbb{R}^2 \to \mathbb{R}$.

$\vec{a} = (1,3)$

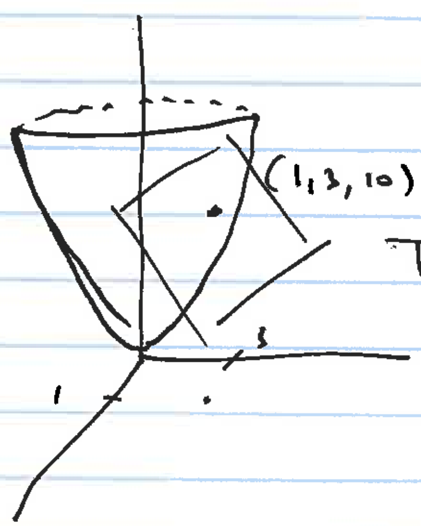
$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial x}(1,3) = 2$

$\frac{\partial f}{\partial y} = 2y \qquad \frac{\partial f}{\partial y}(1,3) = 6$

$$h(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

$$z = h(x,y) = 10 + 2(x-1) + 6(y-3)$$

① Tangent plane at $(1,3,10)$



Tangent plane :

$$z = 10 + 2(x-1) + 6(y-3)$$
$$z = 10 + 2x - 2 + 6y - 18$$
$$z = 2x + 6y - 10$$

② Need to check:

$$\lim_{(x,y)\to(1,3)} \frac{\| f(x,y) - h(x,y) \|}{\| (x,y) - (1,3) \|} = 0$$

$$f(x,y) - h(x,y) = x^2 + y^2 - (2x + 6y - 10)$$
$$= (x^2 - 2x + 1) + (y^2 - 6y + 9)$$
$$= (x-1)^2 + (y-3)^2$$

$$\|(x,y) - (1,3)\| = \|(x-1, y-3)\|$$
$$= \sqrt{(x-1)^2 + (y-3)^2}$$

$$\lim_{(x,y)\to(1,3)} \frac{\|f(x,y) - h(x,y)\|}{\|(x,y) - (1,3)\|} = \lim_{(x,y)\to(1,3)} \frac{(x-1)^2 + (y-3)^2}{\sqrt{(x-1)^2 + (y-3)^2}}$$

$$= \lim_{(x,y)\to(1,3)} \sqrt{(x-1)^2 + (y-3)^2} = 0$$

3 Approximate $(1.003)^2 + (3.004)^2$ by using the tangent plane approximation:

$$z = 10 + 2(x-1) + 6(y-3)$$
$$z = 10 + 2(1.003 - 1) + 6(3.004 - 3)$$
$$z = 10 + 0.006 + 0.024$$
$$z = 10.030 \text{ approximates}$$
$$(1.003)^2 + (3.004)^2 = 10.030025$$

$|\text{error}| \leq 0.00003$

Ex 2 $\sqrt{x^2+y^3}$ Find Tangent plane at $(1,2)$ Why?

approximate $\sqrt{(1.04)^2+(1.999)^3}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\,\frac{2x}{\sqrt{x^2+y^3}} \qquad \frac{\partial f}{\partial x}(1,2) = \frac{1}{\sqrt{1+8}} = \frac{1}{3}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}\,\frac{3y^2}{\sqrt{x^2+y^3}} \qquad \frac{\partial f}{\partial y}(1,2) = \frac{1}{2}\cdot\frac{12}{3} = 2$$

$f(1,2) = 3$

Tangent plane approx:

$$h(x,y) = 3 + \frac{1}{3}(x-1) + 2(y-2)$$

$$\circledast \quad \sqrt{(1.04)^2+(1.999)^3} \approx 3 + \frac{1}{3}(.04) + 2(-0.001)$$

$$= 3 + 0.01\bar{3} - 0.002$$

$$= 3.011\bar{3} \quad \text{approximates } \circledast$$

Correction

$$\begin{array}{rr} + & 3.0000 \\ + & 0.013\bar{3} \\ - & 0.0020 \\ \hline & 3.011\bar{3} \end{array}$$

Prop Let $f: \overline{X} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be diff'ble at $\vec{a} \in \overline{X}$. Then f is continuous at $\vec{a} = (a,b)$.

Proof:

$$\lim_{(x,y)\to(a,b)} \frac{\| f(x,y) - h(x,y) \|}{\| (x,y) - (a,b) \|} = 0 \qquad \text{Def}^{n} \text{ of diff'ble}$$

$$\lim_{(x,y)\to(a,b)} \| f(x,y) - h(x,y) \| = \lim_{(x,y)\to(a,b)} \underbrace{\frac{\| f(x,y) - h(x,y) \|}{\| (x,y) - (a,b) \|}}_{\to 0} \underbrace{\| (x,y) - (a,b) \|}_{\to 0}$$

$$= 0.$$

So:

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) =$$

$$= \lim_{(x,y)\to(a,b)} f(a,b) + \underbrace{\underbrace{\frac{\partial f}{\partial x}}_{\text{fixed}}(a,b)\underbrace{(x-a)}_{\to 0} + \underbrace{\frac{\partial f}{\partial y}}_{\text{fixed}}(a,b)\underbrace{(y-b)}_{\to 0}}_{\to 0}$$

$$= f(a,b)$$

We conclude that

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b) \qquad \text{continuous at } (a,b).$$

Learn this proof. It may show up in an exam.