(2.2) Continuity

#41) Is $\cos\left(\dfrac{x^2-y^2}{x^2+1}\right)$ continuous?

- composition of continuous functions is continuous
- quotient of continuous functions is continuous if denom $\neq 0$.

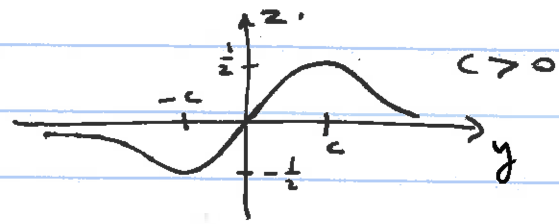
Ex $\dfrac{x^2-y^2}{x^2-1}$ is continuous on $\{(x,y) \mid x \neq \pm 1\}$

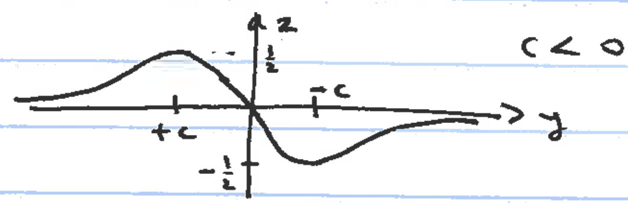
$$\begin{cases} \dfrac{x^2-y^2}{x^2-1} & \text{if } x \neq \pm 1 \\ c_1 & \text{if } x = 1 \\ c_2 & \text{if } x = -1 \end{cases}$$ not continuous

II $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

$f(0,y) = 0$ continuous in y

$f(c,y) = \dfrac{cy}{c^2+y^2}$ continuous in y





$f(x,0) = 0$ continuous in x

$f(x,c) = \dfrac{cx}{c^2+x^2}$ continuous in x.

*** Every section of the graph of f by planes $x=c$ and planes $y=c$ are continuous.

BUT $f(x,y)$ is **not** continuous in 2 variables.

since $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ DNE.

(2.2) Open and closed sets.

$n=1$ $[a,b]$ boundary $=\{a,b\}\subseteq[a,b]$

(a,b) boundary $=\{a,b\}$, $\{a,b\}\cap(a,b)=\emptyset$.

$[a,b)$ boundary $\{a,b\}$ $\{a,b\}\not\subseteq[a,b)$

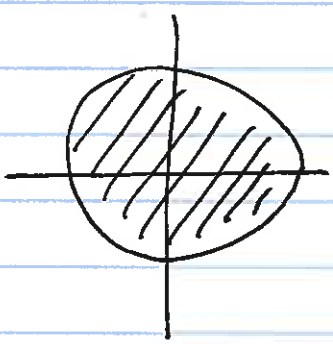
$\{a,b\}\cap[a,b)\neq\emptyset$

all n in $\mathbb{R}^n$

If a set S contains all of its boundary then S is called closed.

If a set S contains no points of its boundary then it is called open.

Ex1 $\{(x,y)\mid x^2+y^2\leq 1\}=S$

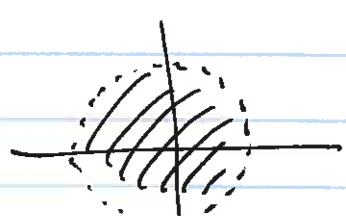
$S=$ 

Boundary of S $=\{(x,y)\mid x^2+y^2=1\}$

Boundary of $S\subseteq S$

S is closed

$T=\{(x,y)\mid x^2+y^2<1\}$

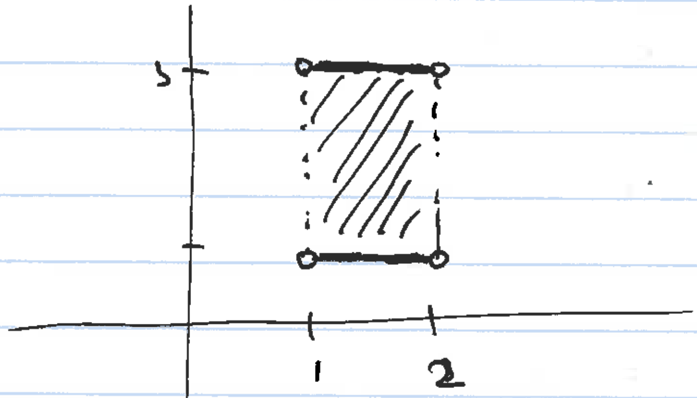


open.

Boundary of $T=\{(x,y)\mid x^2+y^2=1\}$ $(\text{Bd }T)\cap T=\emptyset$

Ex 2

$$\left\{ (x,y) \;\middle|\; \begin{array}{c} 1 < x < 2 \\ 1 \leq y \leq 3 \end{array} \right\} = \mathbb{X}$$



Boundary of $\mathbb{X}$

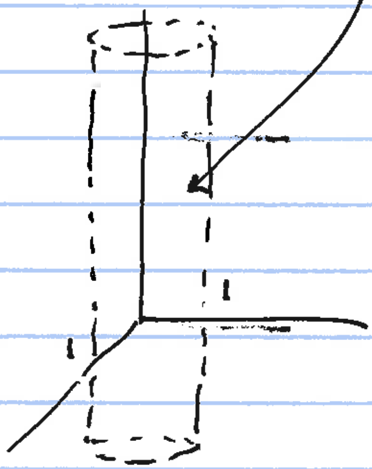
$$\mathrm{Bd}\,\mathbb{X} = \left\{ (x,y) \;\middle|\; \begin{array}{l} (x = 1 \text{ and } 1 \leq y \leq 3) \\ \quad \text{OR} \\ (x = 2 \text{ and } 1 \leq y \leq 3) \\ \quad \text{OR} \\ (y = 1 \text{ and } 1 \leq x \leq 2) \\ \quad \text{OR} \\ (y = 3 \text{ and } 1 \leq x \leq 2) \end{array} \right\}$$

$\mathbb{X}$ is not closed : $\mathrm{Bd}\,\mathbb{X} \not\subseteq \mathbb{X}$

$\mathbb{X}$ is not open : $(\mathrm{Bd}\,\mathbb{X}) \cap \mathbb{X} \neq \emptyset$.

Ex $\{(x,y,z) \mid x^2+y^2 \lneq 1\} = K$

only inside



open set

not closed

Boundary of K

$= \{(x,y,z) \mid x^2+y^2=1\}$

open solid cylinder

2.3 Partial Derivatives

Defn Let $f : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$

$x_1, x_2 \ldots x_n$

$f(x_1, x_2 \ldots x_n)$

leave as a variable

$$f(\underbrace{a_1, a_2, \ldots a_{i-1}}_{\text{fixing/ plugging in numbers}}, x_i, \underbrace{a_{i+1}, \ldots a_n}_{})$$

is called a partial function.

Defn $$\frac{\partial f}{\partial x_i}(a_1, a_2, \ldots, a_n) = \frac{d}{dx_i} f(a_1, a_2 \ldots, a_{i-1}, x_i, a_{i+1} \ldots)\Big|_{x_i = a_i}$$

partial derivatives

$$\frac{\partial f}{\partial x_i} = D_{x_i} f = f_{x_i}$$ (Caution, different notations.)

Ex $f(x,y) = x^2 y + e^x y^3$

Find $\frac{\partial f}{\partial x}(0,2)$, $\frac{\partial f}{\partial y}(0,2)$.

Sol'n

$$\frac{\partial f}{\partial x}(0,2) = \frac{d}{dx} f(x,2)\Big|_{x=0} = \frac{d}{dx}(2x^2 + 8e^x)\Big|_{x=0}$$

$$= 4x + 8e^x\Big|_{x=0}$$

$$= 0 + 8 = 8$$

$$\frac{\partial f}{\partial y}(0,2) = \frac{d}{dy} f(0,y)\Big|_{y=2}$$

$$= \frac{d}{dy} y^3 \Big|_{y=2} = 3y^2\Big|_{y=2} = 12.$$

Fast / for "nice" functions

$$f(x,y) = x^2 y + e^x y^3$$

Consider y as constant $\quad \frac{\partial f}{\partial x} = 2xy + e^x y^3; \quad \frac{\partial f}{\partial x}(0,2) = 8$

Consider x as constant $\quad \frac{\partial f}{\partial y} = x^2 + e^x 3y^2; \quad \frac{\partial f}{\partial y}(0,2) = 12$