Ex
$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0.$$

Why?

$$x^2 \leq x^2+y^2$$

$$0 \leq \frac{x^2}{x^2+y^2} \leq 1 \qquad \text{except when } (x,y)=(0,0).$$

$$0 \leq \left|\frac{x^3}{x^2+y^2}\right| \leq |x|$$

Formal: Let $\varepsilon > 0$ be given. Take $\delta = \varepsilon$.

$\circledast$ $\forall (x,y) \in \mathbb{R}^2$ $\quad 0 < \|(x,y) - (0,0)\| < \delta$

one has

$$|x-0| \leq \|(x-0, y-0)\| < \delta$$

$$\left|\frac{x^3}{x^2+y^2} - 0\right| \leq |x| \leq \|(x,y)\| < \delta = \varepsilon.$$

Informal

$$\underset{\downarrow\, 0}{0} \leq \left|\frac{x^3}{x^2+y^2}\right| \leq \underset{\downarrow\, 0}{|x|} \qquad \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0.$$

"Squeeze", "sandwich" type thm will imply the result

Q: How does one prove it? Ans: as in $\circledast$ above.

p114

Exercises

7) $$\lim_{(x,y,z)\to(0,0,0)} x^2+2xy+yz+z^3+2 = 2.$$

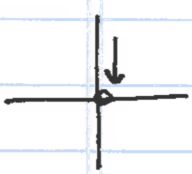
10) $$\lim_{(x,y)\to(0,0)} \frac{e^x e^y}{x+y+2} = \frac{e^0 e^0}{0+0+2} = \frac{1}{2}$$

12) $$\lim_{(x,y)\to(-1,2)} \frac{2x^2+y^2}{x^2+y^2} = \frac{2+4}{1+4} = \frac{6}{5}$$

8) $$\lim_{(x,y)\to(0,0)} \frac{|y|}{\sqrt{x^2+y^2}}$$ DNE because ✓

$(x,0)\to(0,0)$ $$\lim_{(x,0)\to(0,0)} \frac{0}{\sqrt{x^2}} = \lim 0 = 0$$

$\neq$

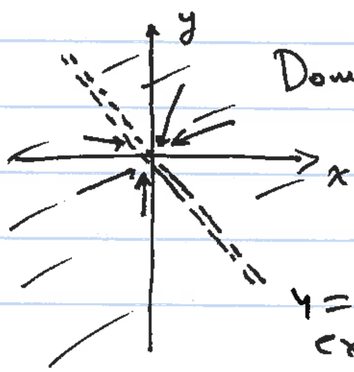


$(0,y)\to(0,0)$ $$\lim_{(0,y)\to(0,0)} \frac{|y|}{\sqrt{y^2}} = 1$$

13) $$\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x+y} = \lim \frac{(x+y)^2}{x+y} = \lim x+y$$
$$= 0.$$

If $x+y\neq 0$

Domain of



$y=-x$ excluded

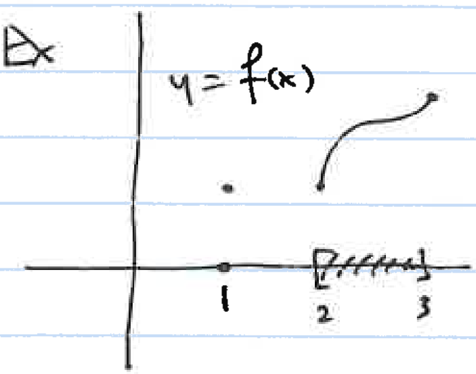
Defn Let $f : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $a \in X$.
Then we say that f is continuous at a if either

(i) a is an isolated point of the domain X

OR

(ii) $f(a) = \lim_{x \to a} f(x)$ when RHS is meaningful that is:
$\underbrace{\forall \delta > 0 \ \exists x \in X \text{ s.t. } 0 < \|x - a\| < \delta}_{\text{"a is not isolated"}}$.

Ex



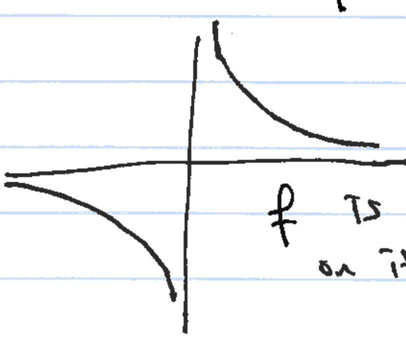
Domain $= \{1\} \cup [2, 3]$.

$\lim_{x \to 1} f(x)$ is not meaningful to consider since $x \neq 1$, $x \to 1$ can't happen in $\{1\} \cup [2, 3]$.

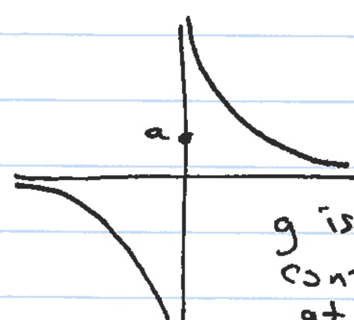
f is automatically continuous at 1, since 1 is an isolated point of the domain $\{1\} \cup [2, 3]$.

Ex $f(x) = \frac{1}{x} : \mathbb{R} - \{0\} \longrightarrow \mathbb{R} - \{0\}$

$$g(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ a & \text{if } x = 0 \end{cases}, \quad g : \mathbb{R} \longrightarrow \mathbb{R}$$



f is continuous on its domain



g is not continuous at 0.

Exc # 46 p 115

$$g(x,y) = \begin{cases} \dfrac{x^3 + xy^2 + 2x^2 + 2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ c & \text{if } (x,y) = (0,0) \end{cases}$$

What is c, in order g to be continuous?

g is continuous on $\mathbb{R}^2 - \{(0,0)\}$ by the limit theorem.

$c = 2$ is needed for the continuity at $(0,0)$:

$$0 \le \left| \frac{x^3}{x^2+y^2} \right| \le |x| \Big| \underbrace{\frac{x^2}{x^2+y^2}}_{\le 1} \Big| \le |x| \to 0 \quad \text{so} \quad \frac{x^3}{x^2+y^2} \to 0$$

$$0 \le \left| \frac{xy^2}{x^2+y^2} \right| \le |x| \left| \frac{y^2}{x^2+y^2} \right| \le |x| \to 0 \quad \text{so} \quad \frac{xy^2}{x^2+y^2} \to 0$$

$$\frac{2x^2+2y^2}{x^2+y^2} = 2 \longrightarrow 2$$

So $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^3+xy^2+2x^2+2y^2}{x^2+y^2} = 2$.

PTO

A shorter way:

$$\frac{x^3+xy^2+2x^2+2y^2}{x^2+y^2} = \frac{(x^2+y^2)(x+2)}{x^2+y^2} = x+2$$

if $x^2+y^2 \neq 0$.

So $$\lim_{(x,y)\to(0,0)} \frac{x^3+xy^2+2x^2+2y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} x+2 = 2.$$