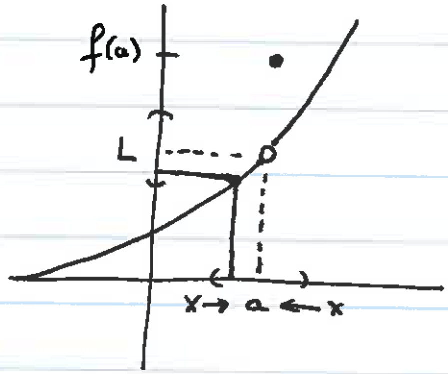
2.2

Review $f: A \subseteq \mathbb{R}^1 \longrightarrow \mathbb{R}^1$

$$\lim_{x \to a} f(x) = L$$



Informally Given an error margin $\varepsilon > 0$

One needs to find a small vicinity of a, $(a-\delta, a+\delta)$, s.t. $x \in (a-\delta, a+\delta)$ and $x \neq a$ will yield $f(x)$ within ε margin of error about L.

Same definition works for $f: \underline{X} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

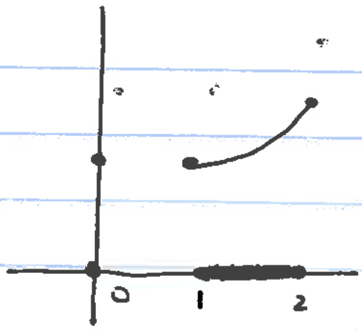
__Defn__ Let $f: \underline{X} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $a \in \mathbb{R}^n$

$$\lim_{x \to a} f(x) = L \in \mathbb{R}^m$$

$$\iff \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in \underline{X} \quad 0 < \|x - a\| < \delta \implies \|f(x) - L\| < \varepsilon.$$

__Caution 1__ In order to have $\lim_{x \to a} f(x)$ make sense, one needs to have x's satisfying $x \in \underline{X}$ and $0 < \|x-a\| < \delta$ for all δ.

Ex

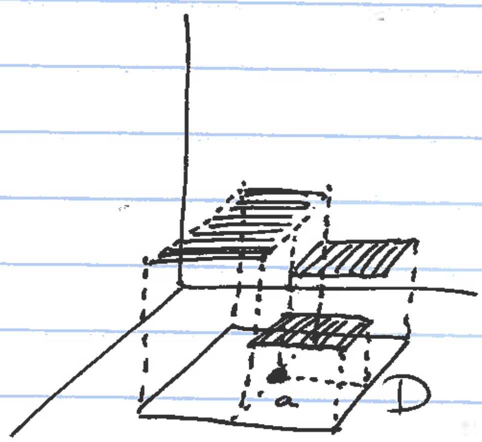


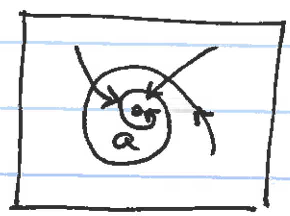
D = domain of f = $\{0\} \cup [1,2]$

$\lim_{x\to 0} f(x)$ has no meaning

since x cannot approach to 0 ($x \not\to 0$) in D.

<u>Caution 2</u> In $\mathbb{R}^n$, $n>1$ left and/or right limits do not make sense:





$D \subseteq \mathbb{R}^2$.

There are many ways, directions, to approach a.

Ex. 1 Let $f(x,y) = 3x + 4y - 6 \; : \mathbb{R}^2 \to \mathbb{R}^1$

Show that $\lim_{(x,y)\to(2,1)} f(x,y) = 4.$

That is: we want to show that

$\forall \varepsilon > 0 \; \exists \delta$ (depending on ε) s.t.

$$0 < \|(x,y) - (2,1)\| < \delta \implies \|f(x,y) - 4\| < \varepsilon.$$

Let $\varepsilon > 0$ be given arbitrarily, choose $\delta = \frac{\varepsilon}{7}$.

Take $(x,y) \in \mathbb{R}^2$ s.t. $\|(x,y) - (2,1)\| < \delta$

$$\|(x-2, y-1)\| < \delta$$

$$\sqrt{(x-2)^2 + (y-1)^2} < \delta$$

$$(x-2)^2 + (y-1)^2 < \delta^2$$

$$(x-2)^2 \le (x-2)^2 + (y-1)^2$$

$$(x-2)^2 < \delta^2$$

$$|x-2| < \delta.$$

similarly $|y-1| < \delta.$

Then

$$|f(x,y) - 4| = |(3x + 4y - 6) - 4| = |3x + 4y - 10|$$

$$= |3x - 6 + 4y - 4| = |3(x-2) + 4(y-1)|$$

$$\le 3|x-2| + 4|y-1| < 3\cdot\delta + 4\cdot\delta = 7\delta = \varepsilon.$$

THM Let $\begin{cases} a \in \mathbb{R}^n \\ F, G : \underline{X} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m \\ f, g : \underline{X} \subseteq \mathbb{R}^n \to \mathbb{R}^1 \end{cases}$ be s.t.

$$\lim_{x \to a} F(\vec{x}) = L \in \mathbb{R}^m \; ; \qquad \lim_{x \to a} f(x) = c \in \mathbb{R}$$

$$\lim_{x \to a} G(x) = M \in \mathbb{R}^m \; ; \qquad \lim_{x \to a} g(x) = d \in \mathbb{R}.$$

Then:

$$\lim_{x \to a} F(x) + G(x) = L + M \in \mathbb{R}^m$$

$$\lim_{x \to a} f(x) F(x) = cL \in \mathbb{R}^m$$

$\forall k \in \mathbb{R}$ $$\lim_{x \to a} k \cdot F(x) = k \cdot L \in \mathbb{R}^m$$

$$\lim_{x \to a} f(x) g(x) = cd \in \mathbb{R}^1$$

- dot product $$\lim_{x \to a} F(x) \cdot G(x) = L \cdot M \in \mathbb{R}^1$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{c}{d} \text{ if } d \neq 0,\ g(x) \neq 0 \text{ on } \underline{X}.$$

Def $\Rightarrow$ $\left.\begin{array}{l} \displaystyle\lim_{(x,y) \to (a,b)} x = a. \\ \displaystyle\lim_{(x,y) \to (a,b)} y = b \end{array}\right\}$ Why?

Ans. $$|x - a| \leq \sqrt{(x-a)^2 + (y-b)^2} = |(x,y) - (a,b)| < \delta$$

Take $\varepsilon = \delta$.

Consequences:

1. For any polynomial function $p(x_1, x_2 \cdots x_n)$,

$$\lim_{(x_1, x_2, \cdots x_n) \to (a_1, a_2 \cdots a_n)} p(x_1, x_2 \cdots x_n) = p(a_1, a_2 \cdots a_n)$$

Exs
$$\lim_{(x,y,z) \to (1,5,2)} x^2 y^3 + xyz + z^2 = 1^2 \cdot 5^3 + 10 + 4$$
$$= 139$$

2. Similarly for rational functions, as long as the denominator is not 0 at the limit.

Ex
$$\lim_{(x,y) \to (4,2)} \frac{x^2 - y^3 + 5xy}{x^2 + y^2} = \frac{16 - 8 + 40}{16 + 4} = \frac{48}{20}$$
$$= \frac{12}{5}$$

3. Thm $F : \underset{\substack{x_1, \cdots x_n \\ \text{variables}}}{\underbrace{X}} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$F = (F_1, F_2 \cdots F_m)$

$\uparrow$ components.

$$\lim_{x \to a} F(x) = L = (L_1, L_2 \cdots L_m)$$
$$\iff \forall i: \lim_{x \to a} F_i(x) = L_i$$

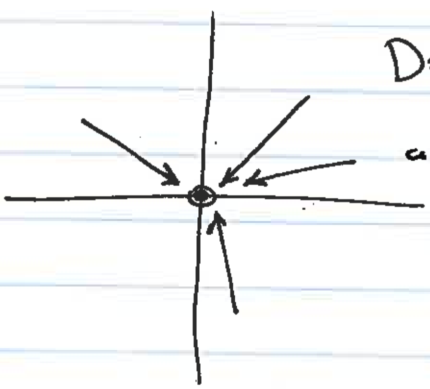
Ex
$$\lim_{(x,y) \to (0,1)} \left(e^x y + 5 ,\ x \sin(y-2),\ \frac{x^2 + y}{x^2 + y^2} \right)$$

$$= \left(\lim_{(x,y) \to (0,1)} e^x y + 5,\ \lim_{(x,y) \to (0,1)} x \sin(y-2),\ \lim_{(x,y) \to (0,1)} \frac{x^2 + y}{x^2 + y^2} \right) = (6, 0, 1$$

Ex 1 $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

$$\lim_{(x,y)\to(3,4)} = \frac{12}{9+16} = \frac{12}{25}$$

$\lim_{(x,y)\to(0,0)} f(x,y)$ <u>Doesn't exist</u>



Domain of $f = \mathbb{R}^2$

approach $(0,0)$ along the lines $y = mx$.

$(x, mx) \longrightarrow (0,0)$ m fixed

$x \neq 0,\ x \to 0$

$$f(x, mx) = \frac{x \cdot mx}{x^2 + (mx)^2} = \frac{mx^2}{x^2(1+m^2)}$$

$$= \frac{m}{1+m^2}.$$

If $m=1$, approach $(0,0)$ along (x,x):

$\lim_{x\to 0} f(x,x) = \frac{1}{2}$.

if $m=0$, approach $(0,0)$ along $(x,0)$, $\lim_{x\to 0} f(x,0) = 0$

if $m=-1$ approach $(0,0)$ along $(x,-x)$, $\lim_{x\to 0} f(x,-x) = \frac{-1}{2}$.

So $\lim_{(x,y)\to(0,0)} f(x,y)$ can't equal to any real number.