1.4 Cross product (Continue)

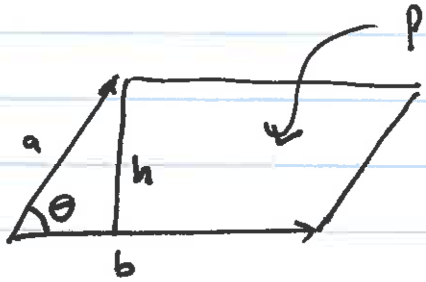
$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Satisfying:

- $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \sin\theta$; θ angle between a & b whenever both are non-zero
- $a \perp a \times b$, $b \perp a \times b$
- $a, b, a \times b$ satisfy Right hand Rule.

Area:

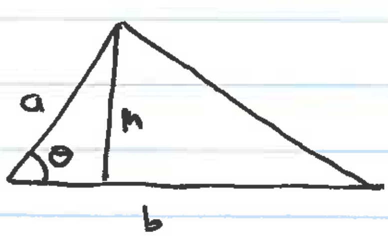


Area $= h \cdot b$

$h = a \sin\theta$

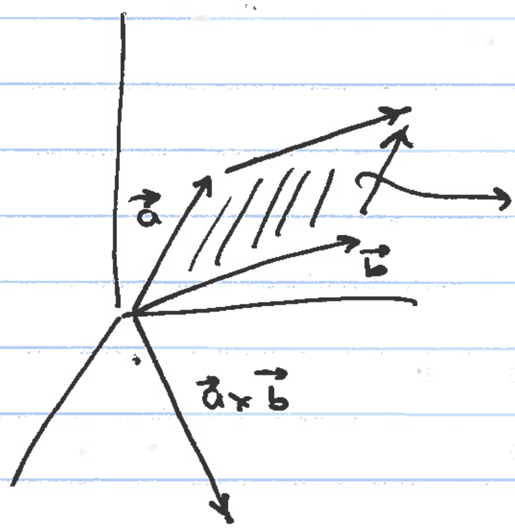
$a, b, h \in \mathbb{R}$.

Area $= a \cdot b \cdot \sin\theta$



Area of $\Delta = \frac{1}{2} a \cdot b \cdot \sin\theta$

$\vec{a}, \vec{b} \in \mathbb{R}^3$



$$\|a \times b\| = \|a\| \, \|b\| \sin\theta$$
= area of this parallelogram.

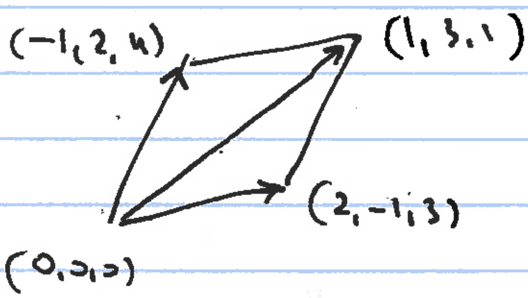
Ex Find the area of the parallelogram with vertices

$$(1,1,1),\ (3,2,-2),\ (0,3,5),\ (2,4,2).$$

move $(1,1,1)$ to the origin $(x,y,z)-(1,1,1)$

$$(0,0,0),\ \underbrace{(2,1,-3)}_{a},\ \underbrace{(-1,2,4)}_{b},\ \underbrace{(1,3,1)}_{a+b}$$

✱ Correction



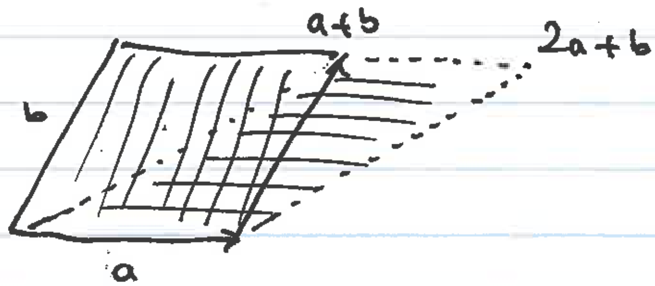
correct carry

$$\begin{vmatrix} i & j & k \\ -1 & 2 & 4 \\ 2 & \cancel{-1}\ {+1} & \cancel{3}\ {-3} \end{vmatrix}$$

$$\text{Area} = \|(-1,2,4)\times(2,+1,-3)\| = \|(-10,5,-5)\|$$
$$= \|(2,1,-3)\times(-1,2,4)\| = \sqrt{100+25+25} = \sqrt{150}$$

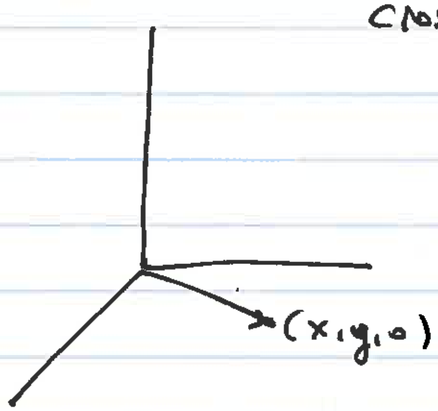
Obs

$$a \times (a+b) = \underbrace{a \times a}_{0} + a \times b$$



Same:
$\|a \times b\|$ area of ////
$\|a \times (a+b)\|$ area of ≡

Can you do $\underbrace{\times}_{\text{cross product}}$ in $\mathbb{R}^2$?



$(x,y) \in \mathbb{R}^2 \longmapsto (x,y,0) \in \mathbb{R}^3$

standard embedding of $\mathbb{R}^2$ into $\mathbb{R}^3$

$(x_1, y_1) \longmapsto (x_1, y_1, 0)$
$(x_2, y_2) \longmapsto (x_2, y_2, 0)$

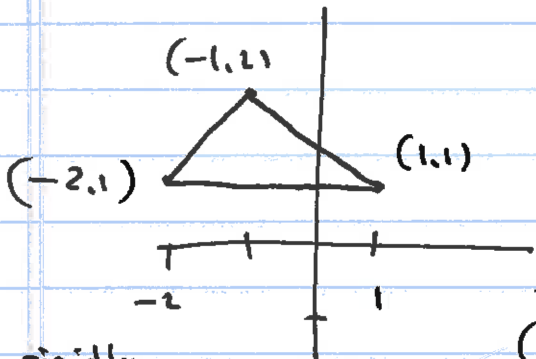
$$(x_1,y_1,0) \times (x_2,y_2,0) = \begin{vmatrix} i & j & k \\ x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \end{vmatrix}$$

$$= (0, 0, x_1y_2 - x_2y_1) \notin \mathbb{R}^2$$

Caution

Ex Find the area of the triangle with vertices

$$(1,1), \ (-2,1), \ (-1,2)$$



move: $(-2,1)$ to $(0,0)$

$$(3,0) \quad (0,0) \quad (1,1)$$

$$(1,1)-(-2,1) \qquad\qquad (1,1) = (-1,2)-(-2,1)$$

Move all rigidly
By using:
$(x,y)-(-2,1)$

using $\mathbb{R}^2$ standard inclusion into $\mathbb{R}^3$

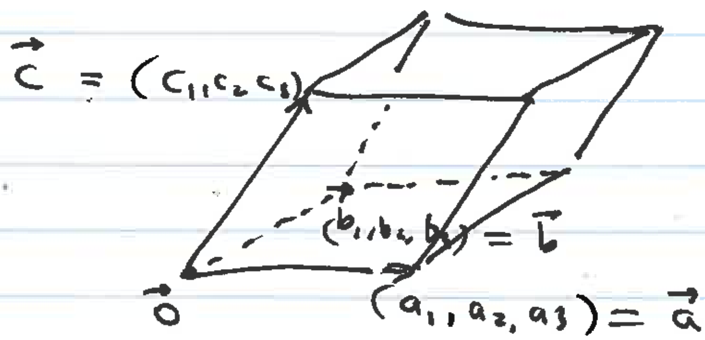
$$\text{Area } \Delta = \frac{1}{2} \| (3,0,0) \times (1,1,0) \| = \frac{1}{2} \| (0,0,3) \|$$

$$= 3/2 .$$

$$\begin{vmatrix} i & j & k \\ 3 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = (0,0,3)$$

Volume of parallelepiped

6 faces
Opposite faces are parallel to each other



$$\text{Volume} = \left| (\vec{a} \times \vec{b}) \cdot \vec{c} \right| = \left| \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \right|$$

$$= \left| \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \right|$$

$=$ by 2 switches or rows

1.4 Exc #18 Volume of parallelepiped

$$\vec{a} = 3i - j$$
$$\vec{b} = -2i + k$$
$$\vec{c} = i - 2j + 4k.$$

$$V = \left| (3i - j) \times (-2i + k) \cdot (i - 2j + 4k) \right|$$

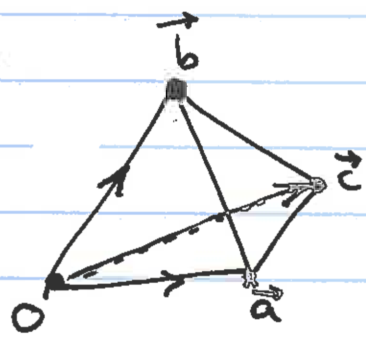
$$= \left| \begin{vmatrix} 3 & -1 & 0 \\ -2 & 0 & 1 \\ 1 & -2 & 4 \end{vmatrix} \right| = \left| 1 \begin{vmatrix} -2 & 1 \\ 1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} \right|$$

$$= \left| (-8-1) + 2(3-0) \right|$$

$$= \left| -9+6 \right| = 3.$$

Tetrahedron (4 faced solid) each is a triangle

(4 faces, 6 edges, 4 vertices)



The volume tetrahedon with vertices

$O, \vec{a}, \vec{b}, \vec{c}$ is

$$= \frac{1}{6} \left\| (\vec{a} \times \vec{b}) \cdot \vec{c} \right\|$$