Aug 29, 2016

1.3

Normalization If $\vec{a} \in \mathbb{R}^n$, $\vec{a} \neq 0$ then

$\dfrac{\vec{a}}{\|\vec{a}\|}$ is a unit vector, parallel to $\vec{a}$

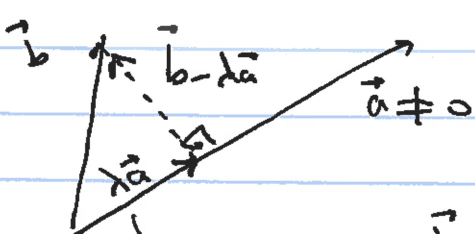
and in the same direction as $\vec{a}$.

$\vec{i} + 5\vec{j} + 3\vec{k}$

$\|\vec{i} + 5\vec{j} + 3\vec{k}\| = \sqrt{1+25+9} = \sqrt{35}$

$\dfrac{\vec{i} + 5\vec{j} + 3\vec{k}}{\sqrt{35}}$ is unit.

Orthogonal projection



$\text{proj}_{\vec{a}} \vec{b}$ parallel to $\vec{a}$

$\text{proj}_{\vec{a}} \vec{b} = \lambda \vec{a}$ for some $\lambda \in \mathbb{R}$.

$$0 = \underbrace{(\vec{b} - \lambda\vec{a})}_{\text{want } b - \lambda a \perp a} \cdot \vec{a} = (\vec{b} \cdot \vec{a}) - \lambda \cdot (\vec{a} \cdot \vec{a})$$

$$\lambda = \frac{b \cdot a}{a \cdot a}$$

$$\text{proj}_{\vec a}\vec b = \frac{b\cdot a}{a\cdot a}\,a \quad \text{parallel to } \vec a$$

$$b-\lambda a = b - \frac{b\cdot a}{a\cdot a}\,a \quad (\text{Gram-Schmidt})$$

Perpendicular to $\vec a$.

Ex Exc #12 (1.3)

$$\vec a = \vec i + \vec j$$

$$\vec b = 2\vec i + 3\vec j - \vec k$$

$$\text{proj}_a b = \frac{(i+j)\cdot(2i+3j+\vec k)}{(i+j)\cdot(i+j)}(i+j)$$

$$= \frac{5}{2}(i+j)$$

p 26 Exc #21 When $\text{proj}_a b = \text{proj}_b a$?

Soln : $\vec a, \vec b \neq 0$ needed for equality to make sense.

$$\frac{b\cdot a}{a\cdot a}\,a = \text{proj}_a b = \text{proj}_b a = \frac{b\cdot a}{b\cdot b}\,b$$

$$\frac{b\cdot a}{a\cdot a}\, a = \frac{b\cdot a}{b\cdot b}\, b.$$

$\boxed{a\cdot b = 0 \text{ is a solution}} \Rightarrow a \perp b.$

If $a\cdot b \neq 0$, we can cancel $a\cdot b$.

$$\frac{a}{\|a\|^2} = \frac{b}{\|b\|^2}.$$

$$\frac{1}{\|a\|^2}\cdot\|a\| = \left\|\frac{a}{\|a\|^2}\right\| = \left\|\frac{b}{\|b\|^2}\right\| = \|b\|\,\frac{1}{\|b\|^2}$$

$$\frac{1}{\|a\|} = \frac{1}{\|b\|}$$

$\Longrightarrow \quad \|a\| = \|b\|$ needed if $a\cdot b \neq 0$.

$$\frac{a}{\|a\|^2} = \frac{b}{\|b\|^2} = \frac{b}{\|a\|^2}$$

now cancel $\|a\|^2$.

$$\boxed{a = b}$$

Soln $\vec{a} = \vec{b}$ OR $a\cdot b = 0$.

1.4 Cross-product

Def Let $\vec{a}$ & $\vec{b} \in \mathbb{R}^3$, Then $\vec{a} \times \vec{b} \in \mathbb{R}^3$ is defined to be the unique vector in $\mathbb{R}^3$ satisfying

(i) $\|\vec{a} \times \vec{b}\| = \|a\| \|b\| \sin\theta$

where θ is the angle between a & b.

(unless a or $b = 0$)

(ii) $\vec{a} \times \vec{b} \perp \vec{a}$

$\vec{a} \times \vec{b} \perp \vec{b}$ (" ")

(iii) $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$ satisfy RHR (" ")

right hand rule

Prop

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Ex

$$\begin{vmatrix} 2 & 5 \\ -1 & 6 \end{vmatrix} = 2 \cdot 6 - (-1)(5) = 17.$$

$$\begin{vmatrix} 1 & 2 & 5 \\ -1 & 0 & 6 \\ 1 & 1 & 2 \end{vmatrix} = -2 \begin{vmatrix} -1 & 6 \\ 1 & 2 \end{vmatrix} + 0 \begin{vmatrix} 1 & 5 \\ 1 & 2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 5 \\ -1 & 6 \end{vmatrix}$$

$$= -2(-2-6) - 1(6+5) = 16 - 11 = 5$$

OR

$$\left|\begin{array}{ccc|cc} 1 & 2 & 5 & 1 & 2 \\ -1 & 0 & 6 & -1 & 0 \\ 1 & 1 & 2 & 1 & 1 \end{array}\right.$$

$$= (0 + 12 - 5) - (0 + 6 - 4)$$

$$= 7 - 2 = 5.$$

Ex $(1,3,5) \times (7,1,6) = \begin{vmatrix} i & j & k \\ 1 & 3 & 5 \\ 7 & 1 & 6 \end{vmatrix}$

$$= i\begin{vmatrix} 3 & 5 \\ 1 & 6 \end{vmatrix} - j\begin{vmatrix} 1 & 5 \\ 7 & 6 \end{vmatrix} + k\begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix}$$

$$= 13i + 29j - 20k.$$

Ex Find all unit vectors $\perp$ $(3,-1,0)$ & $(1,2,4)$

$$(3,-1,0) \times (1,2,4) = \begin{vmatrix} i & j & k \\ 3 & -1 & 0 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= -4i - 12j + 7k. = (-4,-12,7)$$

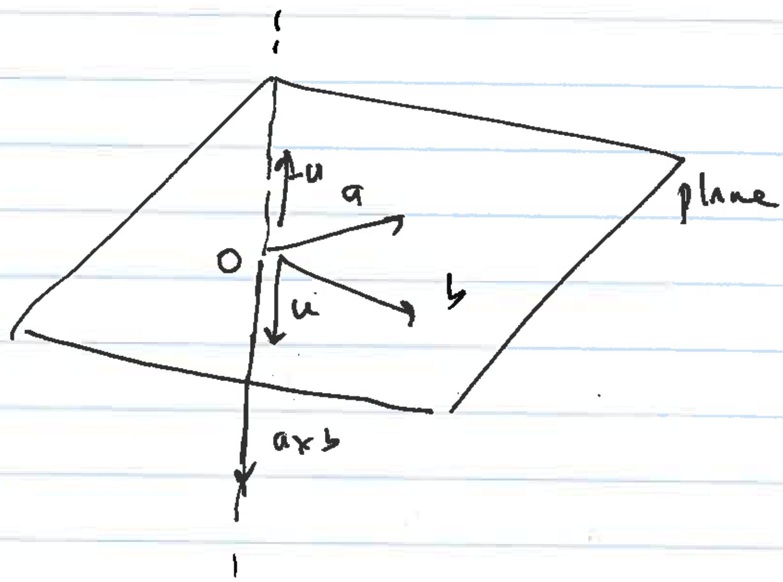
Recall $a \perp b \Rightarrow a \cdot b = 0$

$$\begin{cases} (-4,-12,7) \cdot (3,-1,0) = -12 + 12 + 0 = 0 \\ (-4,-12,7) \cdot (1,2,4) = -4 - 24 + 28 = 0 \end{cases}$$

$(-4, -12, 7)$ is not a unit vector.

$$\|(-4,-12,7)\| = \sqrt{16 + 144 + 49} = \sqrt{209}$$

$$\pm\vec{u} = \pm \frac{(-4, -12, 7)}{\sqrt{209}} \quad \text{unit and } \perp (3,-1,0),\ \perp (1,2,4)$$



Properties of × product: Which ones are correct??

$\forall a, b, c \in \mathbb{R}^3$

False $\cancel{a \times b = b \times a}$ Correct: $a \times b = -b \times a$

Correct:
- ✓ $a \times (b + c) = a \times b + a \times c$
- ✓ $(a+b) \times c = a \times c + b \times c$
- ✓ $k(a \times b) = (ka) \times b$

False $\cancel{(a \times b) \times c = a \times (b \times c)}$ False

$$0 = \underbrace{(i \times i)}_{0} \times j \neq i \times (i \times j) = i \times k = -j$$

$\neq -j$