Additional Notes about functions,

<u>Defn</u> A function f from a set D <u>into</u> a set Y is a rule that assigns a unique number $f(x)$ in Y to each element x in D.

- D is called Domain
- Y is called Codomain

The set of all values $f(x)$ where x varies through all of D is called Range

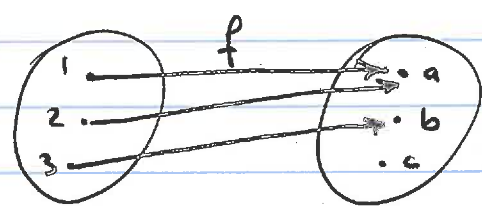
- $\text{Range} = \{ f(x) \mid x \in D \}$.

Notation: We write $f: D \longrightarrow Y$.

Even though every element x in D is assigned to a value in Y, sometimes the following may happen:

① Not every value in Y is assigned to a value^x from D

② Some values in Y may be assigned to more than one value x in D



If we donot want ① to happen, we require:

<u>Defn</u> A function $f: D \longrightarrow Y$ is called <u>onto</u> if for every y in Y, there is x in D s.t. $f(x) = y$

In Other words: Codomain = Range

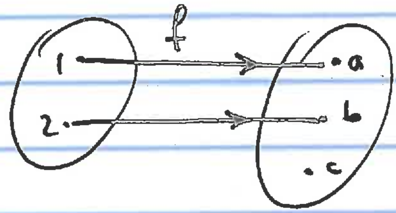
If we do not want ② to happen, we require:

<u>Defn</u> A function $f: D \to Y$ is called one-to-one if for all $x_1, x_2 \in D$ ($f(x_1) = f(x_2) \implies x_1 = x_2$.)

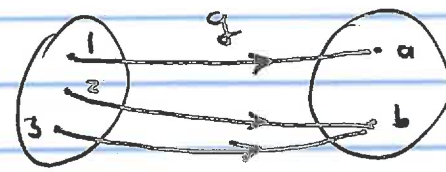
In other words: Horizontal line test holds.

When can we find an inverse of a function?

In the following examples, reversing arrows will not give a function:



f is one-to-one but not onto



g is onto but not one-to-one.

<u>Def</u> Let f be function $f: D \to Y$.

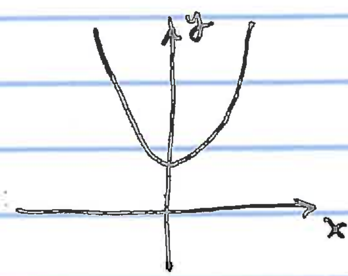
If ① f is onto Y and

② f is one-to-one, then there is an inverse function $f^{-1}: Y \longrightarrow D$ defined by

$$f(x) = y \iff f^{-1}(y) = x$$

In order to f^{-1} exist, f must be one-to-one and onto.

<u>Ex</u> $f(x) = x^2 + 1$



$f: \mathbb{R} \to \mathbb{R}$ is neither 1-1 nor onto $\mathbb{R}$.

If we restrict the domain to $[0, \infty)$ then we have a one-to-one function

$$g(x) = x^2 + 1 : [0, \infty) \longrightarrow \mathbb{R}.$$

This function still is not invertible, since it is not onto $\mathbb{R}$.

But, $\text{Range}(g) = [1, \infty)$; so we define a new function

$$h(x) = x^2 + 1 : [0, \infty) \longrightarrow [1, \infty)$$

which is 1–1 and onto $[1, \infty)$.

There is : $h^{-1} : [1, \infty) \longrightarrow [0, \infty)$

$h^{-1}(y) = \sqrt{y - 1}$ is the inverse of h.

CAUTION: The definition we have above has:

$$f : \underset{\text{domain}}{D} \longrightarrow \underset{\text{codomain}}{Y}$$

$$\{ f(x) \mid x \in D \} = C \longleftarrow \text{called range}.$$

There are books which use the words a little differently:

$$f : \underset{\text{domain}}{D} \longrightarrow \underset{\text{range}}{Y} \qquad \{ f(x) \mid x \in D \} = \underset{\text{image}}{C}$$

But this doesnt effect the notions of one-to-one and onto.