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(P1)

(7.1) Continue

A symmetric $n \times n$

- Has n real eigenvalues (counting w/ multiplicities)
- dim eigenspace for λ_i = algebraic multiplicity of λ_i
- Eigenspaces of different eigenvalues are orthogonal to each other
- A is orthogonally diagonalizable:

$$A = PDP^T$$

D diagonal

P orthogonal $PP^T = P^TP = I$

Procedure for orthogonally diagonalize A (symmetric)

- Find eigenvalues
- Find basis for each eigenspace
- Within each eigenspace use Gram-Schmidt to orthogonalize basis.
- Orthonormalize ($v \rightarrow \frac{v}{\|v\|}$)

$$P = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$$

o.n. basis consisting eigenvectors of A

7.1 #24 Given $A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}$

$$\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -20 \\ 20 \\ 10 \end{bmatrix} = 10 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\lambda_1 = 10.$$

$$\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 1.$$

A is 3x3

$$\det A = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$$

why??

every For any $n \times n$ matrix A :

$\det A = \lambda_1 \cdots \lambda_n$ provided that all eigenvalues in \mathbb{C} (with multiplicity) are used: $\lambda_1, \dots, \lambda_n$

Reason: $\det(A - \lambda I) = \text{char. poly} = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)$

plus in $\lambda = 0$

$$\det A = \lambda_1 \lambda_2 \lambda_3.$$

3x3

$$\begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & \ddots \\ & & & a_{nn} \end{bmatrix}$$

$$a_{11} + a_{22} + \dots + a_{nn} \stackrel{\text{defn}}{=} \text{trace } A$$

$$\text{trace } A = \lambda_1 + \lambda_2 + \dots + \lambda_n$$

Also true for all $n \times n$ matrices A provided that all eigenvalues (in \mathbb{C}) are used (with multiplicity) $\lambda_1, \lambda_2, \dots, \lambda_n$.

$$\det A = \begin{vmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{vmatrix} = 10 = \lambda_1 \lambda_2 \lambda_3$$

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ 10 & 1 & 1 \end{matrix}$

$$\text{Trace} = 12 = \lambda_1 + \lambda_2 + \lambda_3$$

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ 10 & 1 & 1 \end{matrix}$

$$\lambda_3 = 1$$

$$\lambda=1$$

$$A - \lambda I = \begin{bmatrix} 4 & -4 & -2 \\ -4 & 4 & 2 \\ -2 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} x_2 + \frac{1}{2}x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Basis for eigenspace $\lambda=1$

$$\lambda=10$$

$$A - 10I = \begin{bmatrix} -5 & & \\ & -5 & \\ & & -5 \end{bmatrix}$$

No need

$$\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \text{ was given.}$$

Basis

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

u_1 u_2 u_3

$$u_3 \perp u_1$$

$$u_3 \perp u_2$$

G-S on $\{u_1, u_2\}$.

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 2 \end{bmatrix}$$

(p4)

orthogonal basis

$$\begin{matrix} \nearrow l_1 = \sqrt{2} & l_2 = \sqrt{18} & l_3 = 3 \\ \downarrow \end{matrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.$$

orthonormal basis

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{bmatrix}, \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{18} & -2/3 \\ 1/\sqrt{2} & -1/\sqrt{18} & 2/3 \\ 0 & 4/\sqrt{18} & 1/3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

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7.2 Defn Let A be $n \times n$ symmetric matrix

Define $Q: \mathbb{R}^n \rightarrow \mathbb{R}$

$$Q(\vec{x}) = \vec{x}^T A \vec{x}$$

Q is called the quadratic form associated to A .

prob #2

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$Q\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= 4x_1^2 + 2x_2^2 + 1 \cdot x_3^2 + 6x_1x_2 + 0 \cdot x_1x_3 + 2x_2x_3$$

#4 $Q\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = 20x_1^2 + 15x_1x_2 - 10x_2^2$ in \mathbb{R}^2

$$A = \begin{bmatrix} 20 & 7\frac{1}{2} \\ 7\frac{1}{2} & -10 \end{bmatrix}$$

Caution: $Q' = 20x_1^2 + 15x_1x_2 - 10x_2^2$ in \mathbb{R}^3

$$A' = \begin{bmatrix} 20 & 7\frac{1}{2} & 0 \\ 7\frac{1}{2} & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

CHANGE of VARIABLES (For Quadratic Forms)

$$\vec{x} = P\vec{y} \quad (\text{Linear change of variables})$$

$$Q(x) = x^T A x = (P y)^T A (P y) = y^T \underbrace{P^T A P}_{} y$$

Recall orthogonal diagonalization for symmetric A

$$P D P^T = A$$

$$P P^T = I$$

$$P^{-1} = P^T$$

$$D = P^{-1} A (P^T)^{-1}$$

$$(P^T)^{-1} = P$$

$$D = P^T A P$$

If we choose P to orthogonally diagonalize A

$$Q(x) = Q(Py) = y^T D y$$

PRINCIPAL AXIS THEOREM:

Let A be an $n \times n$ symmetric matrix.

Then there is an orthogonal change of variables $x = Py$ that transforms the quadratic form $x^T A x$ into $y^T D y$ which has no cross-product terms.

Since D is diagonal.