Tail Dependence Coefficients

A Decision Problem

Shyamalkumar, N. D.[1]

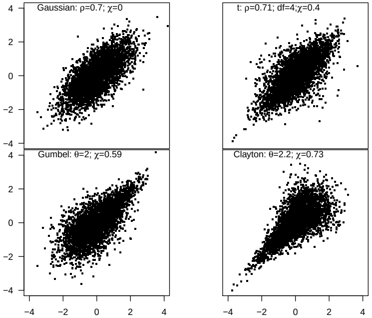
(Work done with Siyang Tao[1])

[1]Department of Statistics and Actuarial Science
The University of Iowa

IME (2017) - Vienna, Austria

Tail (Extreme) Dependence

- Dependence exhibited in tails
    - Dependence in the middle can be different from the tail
    - Dependence in tail more relevant in the age of extremes
        - Financial Crises
    - An area of active practical and academic interest
- See McNeil et al. (2015)



Measures of Dependence

- Some Bivariate Measures
  - Linear Correlation Coefficient (ρ)
  - Kendall's τ - $\rho(\mathrm{sgn}(X_1 - \tilde{X}_1), \mathrm{sgn}(X_2 - \tilde{X}_2))$
  - Spearman's ρ_S - $\rho(F_1(X_1), F_2(X_2))$
- The matrix of bivariate measures serves as a *d*-dimensional measure
- The tails contribute little to the above measures

Tail Dependence Coefficient

- The (lower) Tail Dependence Coefficient of two continuous random variables $X_1 \sim F_1$ and $X_2 \sim F_2$ is defined by

$$\chi(X_1, X_2) = \lim_{u \downarrow 0} \frac{\mathbb{P}(F_1(X_1) \leq u, F_2(X_2) \leq u)}{u},$$

given the limit exist.

- A motivation for the above is

$$\begin{aligned} \lim_{u \downarrow 0} \rho(I_{U_1 \leq u}, I_{U_2 \leq u}) &= \lim_{u \downarrow 0} \frac{\mathbb{E}\left(I_{U_1 \leq u} I_{U_2 \leq u}\right) - u^2}{u(1-u)} \\ &= \lim_{u \downarrow 0} \frac{\mathbb{P}(U_1 \leq u, U_2 \leq u)}{u} \end{aligned}$$

The Realization Problem

- Given $T_{d\times d} \in [0,1]^d$, does it correspond to the matrix of tail dependence coefficients (TDM) of some $\mathbf{X}$?
  - Embrechts et al. (2016)
    - Study a related class - Bernoulli Compatible matrices (BCM)
    - Establish a connection between BCMs and TDMs
  - Fiebig et al. (2017)
    - Study TDM and its extension to stochastic processes
    - Establish detailed geometric properties
  - Our contributions:
    - Determine the `computational complexity` of the realization problem for BCM and TDM
    - An algorithm that works for *d* up to the 20's or larger (in the presence of symmetry)

Bernoulli Compatible Matrices

- From Embrechts et al. (2016):
  - Matrices of the form $\mathbb{E}(\mathbf{ZZ}')$, $\mathbf{Z} \in \{0,1\}^d$ are called Bernoulli Compatible Matrices (BCMs)
- For $A_i := \{Z_i = 1\}$, we have $\mathbb{E}(\mathbf{ZZ}')$ is $1-1$ fn. of

$$\tilde{p}_d := (1, \mathbb{P}(A_1), \ldots, \mathbb{P}(A_d), \mathbb{P}(A_1A_2), \mathbb{P}(A_1A_3), \ldots, \mathbb{P}(A_{d-1}A_d))^\top$$

- For $q_{i,j,k} := \mathbb{P}\left(A_1^i A_2^j A_3^k\right)$, for $i, j, k = 0, 1$, define

$$\tilde{q}_3 = (q_{0,0,0}, q_{0,0,1}, \ldots, q_{1,1,1})^\top$$

- Exists a $(\binom{d}{2} + d + 1) \times 2^d$ matrix C_d with elements in $\{0,1\}$ such that $C_d\tilde{q}_d = \tilde{p}_d$.

Equivalent Problem

Proposition

A matrix $A_{d\times d}$ is BCM iff there exists a vector $\tilde{x} \geq 0$ such that $C_d\tilde{x}_d = \tilde{p}_d$.

- Note that C_d is $\left(\binom{d}{2} + d + 1\right) \times 2^d$
- Symmetry in a problem helps reduce the dimensionality

Illustrative Example 1

- For what values of *a* and *b* is this a BCM?

$$\begin{pmatrix} a & \cdots & b \\ \vdots & \ddots & \vdots \\ b & \cdots & a \end{pmatrix} = (a-b) * I_{d\times d} + bJ_{d\times d}$$

- For fixed $a \in [0,1]$ seek set of *b*'s
  - Clearly $b \leq a$, with upper bound being sharp
  - Since $aJ_{d\times d}$ is a BCM, suffices to find lower bound b_a

$$\alpha(a-b_a) * I_{d\times d} + (\alpha * b_a + (1-\alpha) * a)J_{d\times d}, \quad \forall\alpha \in [0,1]$$

- Familiar setting in the context of linear correlation
  - Hogg and Craig (1978): Variance of sum; $\rho \geq -1/d$
  - The above argument yields sharp lower bound, but only when $ad \in \mathbb{Z}^+$

Example 1: Solution

- $\tilde{p} = (1, a, a, a, b, b, b)^\top$ is invariant w.r.t. S_3
- We can assume that the solution is invariant w.r.t. S_3
- This reduces dimension as the problem becomes
- $$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_1 \\ x_1 \\ x_2 \\ x_2 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ a \\ a \\ a \\ b \\ b \\ b \end{pmatrix}$$

$$\begin{aligned} \text{minimize} \quad & \sum_{k=2}^{d} \binom{d-2}{k-2} x_k \\ \text{subject to} \quad & \sum_{k=1}^{d} \binom{d-1}{k-1} x_k = a \\ & \sum_{k=0}^{d} \binom{d}{k} x_k = 1 \\ & x_k \geq 0 \end{aligned}$$

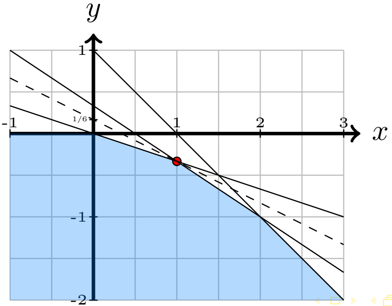
THE UNIVERSITY OF IOWA

Example 1: Solution

- The dual problem:

$$\begin{array}{ll} \text{maximize} & ax + y \\ \text{subject to} & y \le 0; \quad x + d \cdot y \le 0 \\ & \binom{d-1}{k-1} x + \binom{d}{k} y \le \binom{d-2}{k-2}, \text{ for } k = 2, \ldots, d. \\ \textbf{Solution:} & \dfrac{(2ad - \lfloor ad \rfloor - 1)\lfloor ad \rfloor}{d(d-1)} \end{array}$$

- Case $d = 3$:



BCM and TDM - The connection

Theorem (Embrechts et al. (2016))

Matrix T is a TDM iff αT is a BCM, with identical diagonal elements, for some $\alpha > 0$.

Proposition

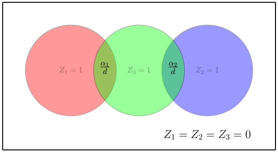
Matrix T is a TDM iff T/d is a BCM with diagonal elements equal to $1/d$.

Illustrative Example 2

- Example from Embrechts et al. (2016): TDM iff $\sum_{i=1}^{d-1} \alpha_i \le 1$.

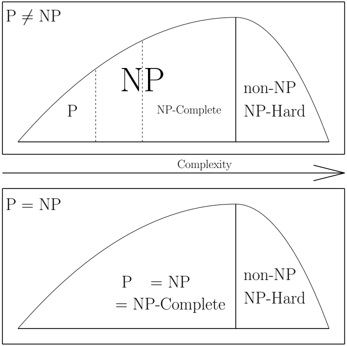
$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \alpha_1 \\ 0 & 1 & \cdots & 0 & \alpha_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_{d-1} \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{d-1} & 1 \end{pmatrix},$$

- $\Pr\left(\bigoplus_{i=1}^{d-1} A_i A_d\right) = \sum_{i=1}^{d-1} \Pr(A_i A_d) \le \Pr(A_d) \iff \sum_{i=1}^{d-1} \alpha_i \le 1$.

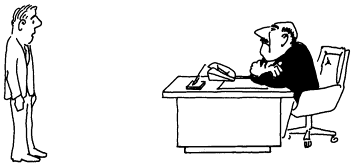


Complexity of Decision Problems

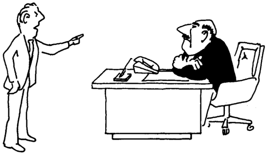
- Worst case Complexity
- Polynomial complexity - Good
- NP Complete
  - NP
  - NP Hard
- To show Γ is NP-Complete
  - Show Γ is NP
  - Select a known NP-complete Γ′
  - Transform Γ′ to Γ in polytime



Essence of NP Completeness - Garey and Johnson (1979)



"I can't find an efficient algorithm, I guess I'm just too dumb."



"I can't find an efficient algorithm, because no such algorithm is possible!"

"I can't find an efficient algorithm, but neither can all these famous people."

A Connection with Max Cut Problem

- Farkas Lemma transforms problem
  - Non-emptiness of a bounded convex set
- Issue - Exponential number of constraints
- Ellipsoid Method - Shor (1977) and Khachian (1979)
  - Shrinking containing ellipsoids
  - Separation Oracle being polytime is key
  - Maximum Weighted Matching problem
    - Weighted min-cut Problem
- Separation Oracle in the TDM case
  - Transforms to a max-cut problem (NP complete)

Realization Problem for TDM is NP-Complete

- COR_d - correlation polytope

  $\{(\mathbb{P}(A_1), \ldots, \mathbb{P}(A_d), \mathbb{P}(A_1 A_2), \ldots, \mathbb{P}(A_{d-1} A_d)) : \mathbb{P} \text{ a prob. measure } ; A_1, \ldots A_d \text{ events}\}$

  - 2^d Extreme points
- Determining membership in COR_d is NP-Complete
  - Pitowsky (1991)
  - 1-in-3 SAT problem
  - 3-colorability of graph

Theorem

- *Realization problem for TDM is NP*
- *TDM realization is NP hard*
  - *Transformation reduces COR_d realization to TDM realization*

Algorithm

- Step 1: Matrix T is a TDM iff T/d is a BCM with diagonal elements equal to $1/d$.
  - Convert T/d to the corresponding $\tilde{p}$ probability vector
  - Resultant Problem: Existence of $\tilde{q} >= 0$ such that $C_d \cdot \tilde{q} = \tilde{p}$
- Step 2: Apply **Farkas Lemma**:

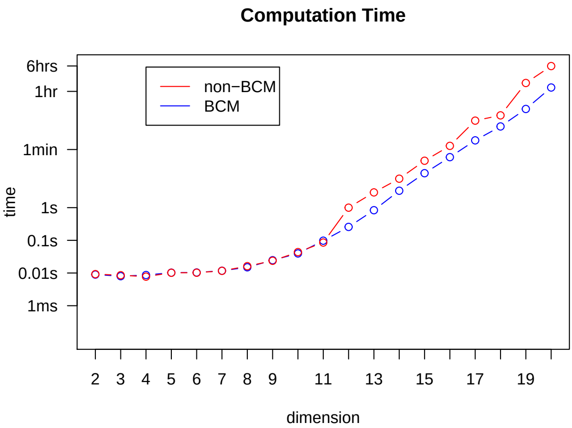
$$\begin{aligned} & \exists \tilde{q} \in \mathbb{R}_+^{2^d} \text{ such that } C_d \cdot \tilde{q} = \tilde{p} \\ \Longleftrightarrow \quad & \min_{\mathbf{y} \in \mathbb{R}^{d(d+1)/2+1} | C_d^\top \mathbf{y} \geq 0} \tilde{p}^\top \mathbf{y} \geq 0 \end{aligned}$$

  - Solve the resultant Linear Programming problem

Performance of the Algorithm

- Since the algorithm ultimately works with BCMs
  - We skipped Step 1 (low compute time)
  - Chose BCMs directly for Step 2
- Considered parametric BCMs for test cases with $d = 2, \ldots, 20$
- For positive test cases used independent BCMs
- For negative test cases used Example 1 with $a = .5$, $b = b_{0.5} - .001$
- Used `linprog` (MATLAB®) for LP solver
  - Mac-book Pro 2.7 GHz Intel Core i7 with 16GB RAM

LP method using Matlab



Any point in the convex cone of vertices (dim 2–6)

- Simulated 100 TDMs for each dimension d from 3 to 6
    - Fiebig et al. (2017):

d	3	4	5	6
# of vertices (N_d)	5	15	214	28895

    - K - uniformly distributed on $\{1, \ldots, N_d\}$
    - Randomly chose K vertices from among the N_d vertices
    - Generated a probability vector from K-simplex
        - Used *Stick Breaking* with $\mathrm{Beta}(1/K, 1 - 1/K)$ variables
- Used the algorithm and `linprog` (MATLAB®) on the simulated TDM ten times
- Record the minimum among the ten compute times

Variation in Compute Time

- Time in units of millisecond.

d	3	4	5	6
Mean	9	9	9	13
(Min,Max)	(9,10)	(8,10)	(8,10)	(10,16)

- Conclusion: Compute time does not vary significantly w.r.t. input matrix

Embrechts, P., Hofert, M., and Wang, R. (2016). Bernoulli and tail-dependence compatibility. *Ann. Appl. Probab.*, 26(3):1636–1658.

Fiebig, U.-R., Strokorb, K., and Schlather, M. (2017). The realization problem for tail correlation functions. *Extremes*, 20(1):121–168.

Garey, M. R. and Johnson, D. S. (1979). *Computers and intractability*. W. H. Freeman and Co., San Francisco, Calif. A guide to the theory of NP-completeness, A Series of Books in the Mathematical Sciences.

Hogg, R. V. and Craig, A. T. (1978). *Introduction to mathematical statistics*. Macmillan Publishing Co., Inc., New York; Collier Macmillan Publishers, London, fourth edition.

McNeil, A. J., Frey, R., and Embrechts, P. (2015). *Quantitative risk management*. Princeton Series in Finance. Princeton University Press, Princeton, NJ, revised edition. Concepts, techniques and tools.

Pitowsky, I. (1991). Correlation polytopes: their geometry and complexity. *Math. Programming*, 50(3, (Ser. A)):395–414.