Simulation

1.1 Inverse Transform Method

The inverse transform method is identical in both the continuous and discrete cases, albeit the presentation is different in the text and there is a difference in their implementation. Essentially, the method is about generating a uniform random variable on the unit interval $(0, 1)$ and using $F^{-1}(U)$ as the simulated random variable from the distribution function $F$. As mentioned in class, one has to suitably define $F^{-1}$ in the discrete case but this re-definition is not important - what is important is to note that the properties in both the discrete and continuous case remain the same.

Some remarks on the method itself:

i. Always applicable, whatever $F$ be - discrete, continuous or even mixed.
ii. Not always efficient - Alternate methods might be faster.
iii. One uniform random variable needed for one simulated random variable from $F$.
iv. Small values of $U$ will always correspond to small values of the simulated variable and vice versa.

Some remarks on variations:

i. Sometimes, like in the case of exponential, the formula for $F^{-1}(U)$ involves finding $1 - U$ - this is one minus operation which adds to the simulation time, especially significant when one is simulating a billion random variables. And moreover the operation is redundant as $1 - U$ has the same distribution as $U$. But the change of $1 - U$ to $U$ makes small values of $U$ correspond to large values of the simulated variable. Problems based on this removal of redundancy have appeared quite a few times in the SOA exams.
ii. On occasions you want to generate $Z = \psi(X)$, for some function $\psi(\cdot)$. For example, $Z$ could be the present value random variable corresponding to an insurance product or the loss random variable and $X$ would then be the future life time random variable. There are two ways of using the inverse transform method. First, use it to generate $X$ and use the simulated value of $X$ to get a simulated value of $Z$ by calculating $\psi(X)$. Second, find the distribution of $Z$ and apply the inverse transform method directly to simulate $Z$. Some comments:
a. If $\psi(\cdot)$ is monotone non-decreasing then both the methods are one and the same. An example, will be the present value random variable from a life annuity product.

b. If $\psi(\cdot)$ is monotone non-increasing then under the first method small values of $U$ correspond to large values of the simulated value of $Z$. An example, will be the loss random variable from a whole life insurance product.

c. If $\psi(\cdot)$ is not monotone then the two methods will be quite distinct. An example, will be the loss random variable from a pure endowment product.

The algorithm for the discrete case: Let $X$ take the values $x_1 < x_2 < \ldots$ with probabilities $p_1, p_2, \ldots$, respectively. Then

$$X = \begin{cases} x_1, & u < p_1; \\ x_j, & \text{if for some } j \geq 2, \sum_{k=1}^{j-1} p_k \leq u < \sum_{k=1}^{j} p_k; \end{cases}$$

On a Variation for the Discrete Case The above algorithm can be made more efficient if the $x'_i$s are ordered in decreasing order of $p'_i$s, i.e. if $x_1$ is the mode, $x_2$ is the value which $X$ assumes with second highest probability and so on...

Acceptance-Rejection Method

This method is about using an auxiliary random variable $Y$ to generate $X$. The method is not always applicable:

i. Discrete case:

a. Applicable if

$$\Pr(X = x) > 0 \Rightarrow \Pr(Y = x) > 0, \forall x \quad \text{and moreover} \quad \max \frac{\Pr(X = x)}{\Pr(Y = x)} < \infty$$

b. The value $c$ is such that

$$\max \frac{\Pr(X = x)}{\Pr(Y = x)} \leq c \quad \text{and the optimum value satisfies} \quad \max \frac{\Pr(X = x)}{\Pr(Y = x)} = c^*$$

c. $c \geq 1$ and equal to one if and only if $X \overset{d}{=} Y$. Small values of $c$ are preferred to large values.

d. The algorithm:

1. Simulate $Y$ and let the simulated value be $y$.
2. Generate $U$ from $U(0, c\Pr(Y = y)) = c\Pr(Y = y) U(0, 1)$
3. If $U < \Pr(X = y)$, then define the simulated value of $X$ as $y$. Else return to step 1 (i.e. start again).

e. The probability of rejecting a generated value of $Y$ is $1 - \frac{1}{c}$. The expected value of the number of $Y'$s needed to generate a single value of $X$ is $c$ and the distribution of the number of $Y'$s needed is one plus a geometric with $\beta = c - 1$.

ii. Continuous Case:
a. Same as the discrete case except replace $\Pr(X = x)$ by $f(x)$ and $\Pr(Y = y)$ by $g(y)$, where $f(\cdot)$ and $g(\cdot)$ are densities of $X$ and $Y$ respectively.