Finding the distribution of $X+Y$ when $X$ and $Y$ are iid $U(0,1)$

**Method 1: Geometric**

\[
\begin{align*}
\text{Area will be equal to area of the unit square minus the area of the dotted triangle:} \\
\text{area} = \frac{1}{2} s^2 \quad \text{when} \quad 0 \leq s \leq 2 \\
= 1 - \frac{1}{2} (2-s)^2
\end{align*}
\]

Hence the distribution function $F(s)$ of $X+Y$ is given by

\[
F(s) = \Pr(X+Y \leq s) = \begin{cases} 
0 & s \leq 0 \\
\frac{s^2}{2} & 0 < s < 1 \\
1 - \frac{1}{2} (2-s)^2 & 1 \leq s < 2 \\
1 & s \geq 2
\end{cases}
\]

Check: The resulting function should be continuous, and it is as can be seen by the fact points $s=1$ and 2.
The density then is the derivative of $F_{X,Y}$ which is given by

$$f_{X,Y}(s) = \begin{cases} \frac{s}{2} & 0 < s < 1 \\ 2 - s & 1 \leq s < 2 \\ 0 & \text{otherwise} \end{cases}$$

i.e.

One can directly argue from figure 0 to figure 2. Can you?

**Method 2: Brute Force**

$$f_{X,Y}(s) = \int f_X(t) f_Y(s-t) \, dt, \quad \forall s \in [0,2]$$

the limits are 0 to 1 as $f_X(t)$ is positive only between these limits.

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But that would be the same with \( f_y \) too. Hence
\[ 0 < S - t < 1 \]
\[ \Leftrightarrow \quad S - 1 < t < S. \]

Hence combining the limits on \( t \), we have
\[ (S-1)V_0 < t < S V \]
\[ V = \text{maximum} \quad \lambda = \text{minimum}. \]

Hence
\[ f_{x+1}(s) = \int_{(S-1)V_0}^{S V} f_x(t) f_y(s+t) \, dt. \]

But as the densities are equal to \( \lambda \) when positive we have
\[ f_{x+1}(s) = \int_{(S-1)V_0}^{S V} dt = [S V - (S-1)V_0] \]

\[ = \begin{cases} 
0 & S < 0 \text{ or } S > 2, \\
S - 0 = S & 0 \leq S < 1, \\
1 - (S-1) = 2-S & 1 \leq S < 2. 
\end{cases} \]