1. \( u_\alpha(w) = -\exp\{-\alpha w\} \) \( \forall w \)
\[
\lim_{\alpha \to 0} \alpha u_\alpha = E(X)
\]

(ii) \( u^*_\alpha(w) = \frac{1 - \exp\{-\alpha w\}}{\alpha} \) \( \forall w \)
\[
\text{show} \quad C^*_\alpha(w) = u^*_\alpha(w)
\]
\[
\Rightarrow M^*_\alpha(w - C^*_\alpha) = E[ u^*_\alpha(w_0 - X) ]
\]
\[
= \frac{1 - \exp[-\alpha(w_0 - C^*_\alpha)]}{\alpha} = E\left[ \frac{1}{\alpha} \left( 1 - \exp(-\alpha(w_0 - X)) \right) \right]
\]
\[
\Rightarrow \exp(\alpha C^*_\alpha) = E[\exp(\alpha X)] = M_X(\alpha)
\]
\[
\Rightarrow C^*_\alpha = \ln M_X(\alpha)/\alpha
\]

when \( \alpha \to 0 \),
\[
\lim_{\alpha \to 0} \frac{\ln M_X(\alpha)}{\alpha} = \frac{d \ln M_X(w)}{dw} = \frac{M'_X(w)}{M_X(w)} = E(X)
\]
\[
\therefore C^*_\alpha = C_0
\]

(iii) \( \tilde{u}(w) = a + bw \quad b > 0 \)
\[
\tilde{u}(w_0 - C) = E[\tilde{u}(w_0 - X)]
\]
\[
a + b(w_0 - C) = E[a + b(w_0 - X)]
\]
\[
a + b(w_0 - C) = a + bE(X) - bE(C)
\]
\[
\therefore C = E(X)
\]

it is a linear utility function
1.2 \[ X = 2^N \]  \[ f(N) = \left(\frac{1}{2}\right)^N \quad N=1,2,3, \ldots \]

\[ E [ U(W) ] = E [ \log 2^N ] = \log 2 \times E(N) \]

\[ E(N) = \sum_{k=0}^{\infty} \frac{k}{2^k} \left(\frac{1}{2}\right)^k + \sum_{k=0}^{\infty} \frac{k}{2^k} \left(\frac{1}{2}\right)^k + \ldots \]

\[ = 1 + \frac{1}{2} + \frac{1}{2} + \ldots \]

\[ = 2 \]

\[ \therefore \quad E( U(x) ) = 2 \log 2 \]

1.3 \[ W_0 = 100 \quad U(W) = \log W \]

\[ P_r (X = 0) = P_r (X = 51) = \frac{1}{2} \]

a. \[ \begin{align*}
    U(W_0 - G) & \geq E [ U(W_0 - X) ] \\
    \log (100 - G) & \geq E [ \log (100 - X) ] \\
    E [ \log (100 - G) ] & = \frac{1}{2} \log 100 + \frac{1}{2} \log (100 - 51) \\
    & = \frac{1}{2} \log 4900 \\
    \Rightarrow 100 - G & \geq 70 \\
    \Rightarrow G & \leq 30
\end{align*} \]

b. \[ \begin{align*}
    U(W_1) & = E [ U(W_1 + H - X ) ] \\
    \text{where} \quad U(W_1) & = 650 \quad \text{and} \quad U(W) = \log W \\
    \log 650 & = E [ \log (650 + H - X) ] \\
    & = \frac{1}{2} \log (650 + H) + \frac{1}{2} \log (599 + H) \\
    & = \frac{1}{2} \log [(650 + H)(599 + H)] \\
    \Rightarrow 650^2 & = (650 + H)(599 + H) \\
    \therefore \quad H & = 26
\end{align*} \]
1.14 Complete insurance  \( G = 40 \)
\[
U(W - G) = U(W - 40) = -e^{-0.005(W - 40)} = -1.2276 e^{-0.005W}
\]

Partial insurance  \( G = 25 \)
\[
E\left[ U(W - G - X + \frac{X}{2}) \right] = 0.75 U(W - 25) + 0.25 \int_0^\infty U(W - 25 - \frac{X}{2}) f(X) dX
\]
\[
= 0.75 e^{-0.005(W - 25)} + 0.25 \int_0^\infty e^{-0.005(W - 25 - \frac{X}{2})} \cdot 0.01 e^{-0.1X} dX
\]
\[
= -1.2276 e^{-0.005W}
\]

\[\therefore\] complete insurance maximized the expected utility.

1.18 \( f(x) = 0.1e^{-0.1x}, x > 0 \)

a. \( X \sim \text{exponential} \quad (\lambda = 0.1) \)
\[
E(X) = \frac{1}{\lambda} = 10
\]
\[
\text{Var}(X) = \frac{1}{\lambda^2} = 100
\]

b. pure premium = 5
\[
E(I(x)) = \beta = 5 = \frac{1}{2} E(X)
\]
\[
I(x) = \begin{cases} 
0 & \text{if } x < d \\
 x - d & \text{if } x \geq d 
\end{cases}
\]
\[
\beta = \int_0^d \left[ 1 - F(x) \right] dx
\]
\[
5 = \int_0^d e^{-0.1x} dx = 10x e^{-0.1d}
\]
\[
d = 10 \log 2
\]
1.19 \( f(x) = \frac{1}{100} \quad 0 < x < 100 \)

a. \( X \sim \text{Uniform}(0, 100) \)
\( E(X) = 50 \quad \text{Var}(X) = \frac{100^2}{12} \)

b. \( E(I(x)) = \beta = P = 12.5 \)
\( E(kx) = 12.5 \)
\( K = \frac{12.5}{E(X)} = 0.25 \quad \text{For proportional policy} \)

For the stop-loss policy, we have
\( \beta = \int_0^{12.5} (1 - F(x)) \, dx \)
\( 12.5 = \int_0^{100} (1 - \frac{x}{100}) \, dx \)
\( = \int_0^{100} (1 - \frac{x}{100}) \, dx \)
\( \Rightarrow d = 50 \)

c. \( \text{Var}[X - I(X)] = \text{Var}[X - 0.25X] \)
\( = 0.75^2 \text{Var}(X) \)
\( = 468.75 \)

\( \text{Var}[X - Id(X)] = E[(X - Id(X))^2] - [E(X - Id(X))]^2 \)
\( = \int_0^d x^2 \frac{1}{100} \, dx + \int_0^{100} \frac{d^2}{100} \, dx \)
\( - \int_0^d x \frac{1}{100} \, dx - \int_0^{100} \frac{d}{100} \, dx \)
\( = 260 \)

\( \text{Var}[X - I(X)] > \text{Var}[X - Id(X)] \)
\[ \text{(1.21) } E \left[ (X - I(X))^2 \right] = E \left[ (X - I(X))^2 \right] - 2(\mu - \beta)(X - I(X)) + (\mu - \beta)^2 \]

\[ = E \left[ (X - I(X))^2 \right] - 2(\mu - \beta)^2 + (\mu - \beta)^2 \]

\[ = E \left[ (X - I(X))^2 \right] - \left( E(X) - I(X) \right)^2 \]

\[ = \text{Var}(X - I(X)) \quad (\text{here } E(X) = \mu, E(I(X)) = \beta) \]

\[ \text{Var}(X - I(X)) \text{ is minimized when } I(X) = I^*(X) \]

\[ (x^* + 2z^*) \geq 2xz \quad \Rightarrow \quad \lambda^* - 2z^* \geq 2(x - z)z \]

\[ \Rightarrow \left[ X - I(X) \right]^2 - \left[ X - I^*(X) \right]^2 \geq [2x - 2I^*(X)][I^*(X) - I(X)] \]

\[ \text{here if } I^*(X) = I(X) \text{ left side = right side = 0} \]

\[ \text{if } I^*(X) > I(X) \quad I^*(X) > 0 \]

\[ \Rightarrow \quad X - I^*(X) = d^* \]

\[ \text{if } I^*(X) < I(X) \]

\[ (I(X) - I^*(X)) \geq 0 \quad \text{and } I^*(X) - I(X) \geq d^* \]

\[ 2x \cdot [2I^*(X)] [I^*(X) - I(X)] \geq 2[I^*(X) - I(X)]d^* \]

\[ \Rightarrow \quad E \left[ (X - I(X))^2 \right] - E \left[ (X - I^*(X))^2 \right] \geq 2E \left[ I^*(X) - I(X) \right]^2 \]

\[ \Rightarrow \quad \forall \left[ (X - I(X))^2 \right] \geq \forall \left[ (X - I^*(X))^2 \right] \]
From lemma rule
\[ u(w) - u(z) \leq (w-z)u''(z) \]
for \( u''(w) > 0 \), \( w, z \in [a, b] \)

Thus
\[ u(w-x+I_0(x)-p) - u(w-x+I_0(x)-E(x)) \leq [I(x) - p - I_0(x) + E(x)] u'(w-x+I_0(x)-E(x)) \]
\[ = (I(x) - p + E(x) - x) u'(w-E(x)) \]
\[ \because I_0(x) = x \]

Take expectation
\[ E[u(w-x+I_0(x)-p) - u(w-x+I_0(x)-E(x))] \leq E[I(x) - p + E(x) - x] u'(w-E(x)) = 0 \]
\[ \therefore E[u(w-x+I_0(x)-p)] \leq E[u(w-x+I_0(x)-E(x))] = E[u(w-E(x))] = u(w-E(x)) \]

\[ \because \text{Full coverage insurance is optimal} \]

1.23 a. \[ Var[I(x)] = Var[I(x) - x + x] \]
\[ = Var(X-I(x)) + Var(x) - 2 \text{COV}(X-I(x), X) = V + Var(X) - 2 \text{COV}(X-I(x), X) \]

b. \[ V + Var(X) \geq 2 (U \text{Var}(X))^{1/2} \]
\[ \therefore Var[I(x)] \geq 2 (U \text{Var}(X))^{1/2} - 2 \text{COV}(X-I(x), X) \]
\[ \therefore \text{Var}[I(x)] \geq 0 \]

Take equal sign
\[ \rho_X, X-I(x) = \frac{\text{COV}(X, X-I(x))}{(U \text{Var}(X))^{1/2}} = 1 \]
\[ d. \quad \rho_{X, X-I(x)} = \frac{\text{cov}(X, X-I(x))}{(\sqrt{\text{Var}(X)})^2} = \frac{\text{cov}(X, ax)}{(\sqrt{\text{Var}(X)})^2} \]
\[ = \frac{a \text{ Var}(X)}{(\sqrt{\text{Var}(X)})^2} = 1 \]

\[ a = \sqrt{\text{Var}(X)} \]

\[ \therefore \quad I(X) = \left[ 1 - \frac{1}{\sqrt{\text{Var}(X)}} \right] X \]