NOTES FOR 02/16 CLASS (#3, §12.1)

Definition. For any domain \( R \) the rank of an \( R \)-module \( M \) is the maximum number of \( R \)-linearly independent elements of \( M \).

Problem. Let \( A \), and \( B \) be \( R \)-modules of ranks \( m \) and \( n \), respectively. Prove that the rank of \( A \oplus B \) is \( m + n \).

Proof. Let \( \mathcal{L}_A = \{x_1, x_2, \ldots, x_m\} \) be a maximal set of \( R \)-linearly independent elements of \( A \), and let \( \mathcal{L}_B = \{y_1, y_2, \ldots, y_n\} \) be a maximal set of \( R \)-linearly independent elements of \( B \). Let \( \mathcal{L}_A' = \{(x_1,0),(x_2,0),\ldots,(x_m,0)\} \), and \( \mathcal{L}_B' = \{(y_1,0),(y_2,0),\ldots,(y_n,0)\} \). Then \( \mathcal{L}_A' \cup \mathcal{L}_B' \) is a linearly independent subset of \( A \oplus B \). Let \( N \) be the submodule generated by them. We claim that \( A \oplus B/N \) is a torsion \( R \)-module. Let \((a,b)\) be any element in \( A \oplus B \). By #2, (a) in Section 12.1, we know that there exists a nonzero \( r_1 \in R \) such that \( r_1a \) belongs to the submodule generated by \( \mathcal{L}_A \), and there exists a nonzero \( r_2 \in R \) such that \( r_2b \) belongs to the submodule generated by \( \mathcal{L}_B \). Consequently, if \( r = r_1r_2 \), then \( r \) is nonzero and \( r(a,b) \in N \). This proves the claim, and according to #2, (b), it implies that the rank of \( A \oplus B \) is \( m + n \). \( \square \)