Chapter 14

The following exercise computes the Galois group of \( spl(x^4 - 2x^2 - 2) \) over \( \mathbb{Q} \) in several steps. We keep the notation in the text. (See page 582.)

16. a. Let \( f(x) = x^4 - 2x^2 - 2 \). By the Eisenstein’s criterion, \( f(x) \) is irreducible in \( \mathbb{Z}[x] \), and by Gauss’ Lemma (see Corollary 6, pg 304) \( f(x) \) is irreducible over \( \mathbb{Q} \).

b. This is straightforward to check.

c. Note that \( K_1 \subset \mathbb{R} \). If \( K_1 = K_2 \), then \( \alpha_2 \in K_1 \), and this implies that \( \alpha_1 \alpha_2 \in K_1 \). However, \( \alpha_1 \alpha_2 \) \((= 2\sqrt{-1})\) is a complex number. Thus \( K_1 \neq K_2 \). It is clear that \( \mathbb{Q}(\sqrt{3}) \subset K_1 \cap K_2 \) since \( \frac{\alpha_1^2 - \alpha_2^2}{2} = \sqrt{3} \). Hence \( [K_1 \cap K_2 : \mathbb{Q}] \geq 2 \). On the other hand \( [K_1 : \mathbb{Q}] = 4 \), consequently \( [K_1 \cap K_2 : \mathbb{Q}] = 2, 4 \). Since \( K_1 \neq K_2 \), we see that \( K_1 \cap K_2 = \mathbb{Q}(\sqrt{3}) \) as they are both of degree 2 over \( \mathbb{Q} \), and \( \mathbb{Q}(\sqrt{3}) \subset K_1 \cap K_2 \). Since the splitting field of a polynomial is generated by all the roots, it is also clear that \( K_1 K_2 = \mathbb{Q}(\sqrt{3}) \).

d. Let \( F = \mathbb{Q}(\sqrt{3}) = K_1 \cap K_2 \). Since \( K_1/F \) and \( K_2/F \) are quadratic extensions, and \( \text{char}(F) \neq 2 \), it follows that they are both Galois extensions, and hence the compositum \( K_1 K_2 \) is also Galois over \( F \). It is in fact clear that \( K_1 = F(\alpha_1), K_2 = F(\alpha_2) \), and \( K_1 K_2 = F(\alpha_1, \alpha_2) \). Now let \( G = G(K_1 K_2/F) \). For any \( \rho \in G \), since \( \alpha_1^2, \alpha_2^2 \in F \), we see that \( \rho(\alpha_1) = \pm \alpha_1 \) and \( \rho(\alpha_2) = \pm \alpha_2 \). Moreover, any such \( \rho \) is completely determined by its action on \( \{\alpha_1, \alpha_2\} \). Let \( \sigma \in G \) be the element which maps \( \alpha_1 \mapsto -\alpha_1 \) and fixes \( \alpha_2 \); let \( \tau \in G \) be the element which maps \( \alpha_2 \mapsto -\alpha_2 \) and fixes \( \alpha_1 \). Then \( \sigma^2 = \tau^2 = 1 \); and \( \sigma \tau = \tau \sigma \). Thus \( G \) is the Klein 4-group

\[ G = \{1, \sigma, \tau, \sigma \tau : \sigma^2 = \tau^2 = 1; \sigma \tau = \tau \sigma\} \]

In particular \([K_1 K_2 : F] = 4 \). All the subgroups of \( G \) are given by

\[ \{1\}, H_1 = \{1, \sigma\}, h_2 = \{1, \tau\}, H_3 = \{1, \sigma \tau\}, G \]

Using the fact that any element in \( K_1 K_2 \) can be written uniquely in the form

\[ a + b\alpha_1 + c\alpha_2 + d\alpha_1 \alpha_2, a, b, c, d \in F. \]
we can determine all the fixed subfields of $K_1K_2$ containing $F$:

\[
(K_1K_2)^{\{1\}} = K_1K_2 \\
(K_1K_2)^{H_1} = K_2 \\
(K_1K_2)^{H_2} = K_1 \\
(K_1K_2)^{H_3} = F(\alpha_1\alpha_2) = F(\sqrt{-2}) \\
(K_1K_2)^G = F
\]

e. We have already observed that $K_1K_2$ is the splitting field of $f(x)$ over $\mathbb{Q}$. Since $[K_1K_2 : \mathbb{Q}] = [K_1K_2 : F][F : \mathbb{Q}]$, we see that $[K_1K_2 : \mathbb{Q}] = 8$. Now let $r$ be the automorphism of $K_1K_2$ over $\mathbb{Q}$ which maps $\alpha_1 \mapsto \alpha_2 \mapsto -\alpha_1 \mapsto -\alpha_2$. It is easily verified that $r$ is of order 4 (check this!). Let $s$ be the automorphism of $K_1K_2$ over $\mathbb{Q}$ that permutes $\{\alpha_1, \alpha_2\}$; in particular $s$ is of order 2. Further $rs = sr^{-1}$ since they agree on the generators $\{\alpha_1, \alpha_2\}$. Hence $G(K_1K_2/\mathbb{Q})$ is the dihedral group $D_8$. 