TWISTED EXTERIOR SQUARE LIFT FROM $GU(2,2)_{E/F}$ TO GL_6/F

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Abstract. Suppose E/F is a quadratic extension of number fields and $GU(2,2)$ is the quasi-split unitary similitude group attached to E/F. We prove functoriality from globally generic cuspidal representations of $GU(2,2)(\mathbb{A}_F)$ to automorphic representations of $GL_6(\mathbb{A}_F)$ corresponding to the twisted exterior square map on the dual side. For a split quadratic algebra E/F, the twisted exterior square map reduces to the usual exterior square map from $GL_4(\mathbb{C})$ to $GL_6(\mathbb{C})$. We also discuss functoriality for the twisted symmetric fourth power map and discuss its relation with functoriality of the twisted exterior square.

1. Introduction.

Let E/F be a quadratic extension of number fields, and $GU(2,2)_{E/F}$ be the quasi-split unitary similitude group defined with respect to E/F. We write $\delta_{E/F}$ for the idele class character of F associated to E/F via class field theory. To ease our notation, we sometimes suppress the symbol "E/F" while denoting unitary or unitary similitude groups which are defined with respect to E/F.

The purpose of this paper is to prove Langlands functoriality from globally generic cuspidal representations of $GU(2,2)_{E/F}(\mathbb{A}_F)$ to automorphic representations of $GL_6(\mathbb{A}_F)$, corresponding to the twisted exterior square map (see Section 2). If E/F is a split quadratic algebra, then our case of functoriality is same as the one established in [Ki1], namely, the exterior square lift from cuspidal representations of $GL_4(\mathbb{A}_F)$ to automorphic representations of $GL_6(\mathbb{A}_F)$.

We note that there is no twisted exterior square lift from cuspidal representations of $U(2,2)(\mathbb{A}_F)$ to automorphic representations of $GL_6(\mathbb{A}_F)$ due to the following reason: The usual exterior square map from $GL_4(\mathbb{C})$ to $GL_6(\mathbb{C})$ doesn't extend to yield an L-group homomorphism from $GL_4(\mathbb{C}) \rtimes Gal(E/F)$, the L-group of $U(2,2)_{E/F}$, to $GL_6(\mathbb{C})$, the L-group of GL_6/F. We need to use the group $GU(2,2)_{E/F}$ in which case there is an L-group homomorphism $(GL_4(\mathbb{C}) \times GL_1(\mathbb{C})) \rtimes Gal(E/F) \longrightarrow GL_6(\mathbb{C})$ extending the usual exterior square map $GL_4(\mathbb{C}) \longrightarrow GL_6(\mathbb{C})$. We refer to this L-group homomorphism as the twisted exterior square map. This situation is similar to the classical situation of Sp_4

[1] Partially supported by an NSERC grant and an AMS Centennial Fellowship.

versus GSp_4. There is an L-group homomorphism from $GSp_4(\mathbb{C})$, the L-group of GSp_4, to $GL_4(\mathbb{C})$. However, there is no L-group homomorphism from $SO_5(\mathbb{C})$, the L-group of Sp_4, to $GL_4(\mathbb{C})$.

Our goal is to prove Langlands functoriality for the twisted exterior square map $\wedge_t^2$. (We refer the reader to the main body of the paper for any unexplained notation.) To be precise, we prove (Theorem 6.2):

Theorem 1.1. *Let π be a globally generic cuspidal representation of $GU(2,2)(\mathbb{A}_F)$, and let T be the set of finite places of F defined by $T = \{v < \infty |\ v$ does not split in E and π_v is supercuspidal $\}$. Then there exists an automorphic representation Π of $GL_6(\mathbb{A})$ such that $\Pi_v \simeq \wedge_t^2(\pi_v)$ for $v \notin T$. It is of the form*

$$\Pi = \sigma_1 \boxplus \cdots \boxplus \sigma_k,$$

where σ_i's are (unitary) cuspidal representations of $GL_{n_i}(\mathbb{A}_F)$. Moreover, we have $\Pi_E \simeq \wedge^2(BC(\pi))$, where Π_E is the base change lift of Π to an automorphic representation of $GL_6(\mathbb{A}_E)$.

What we prove is weaker than the automorphy of $\wedge_t^2(\pi)$ since the local Langlands correspondence is not available for supercuspidal representations of $GU(2,2)(F_v)$.

Remark. Given a generic cuspidal representation π of $GU(2,2)(\mathbb{A}_F)$, let $BC(\pi)$ be the stable base change lift to $GL_4(\mathbb{A}_E) \times GL_1(\mathbb{A}_E)$ (cf. Section 5). Let $\wedge^2(BC(\pi))$ be the exterior square lift of $BC(\pi)$ to $GL_6(\mathbb{A}_E) \times GL_1(\mathbb{A}_E)$. Since $\wedge^2(BC(\pi))$ is θ-invariant, by the result of Arthur-Clozel [A-C], there exists an automorphic representation Π of $GL_6(\mathbb{A}_F) \times GL_1(\mathbb{A}_F)$ whose base change to $GL_6(\mathbb{A}_E) \times GL_1(\mathbb{A}_E)$, is $\wedge^2(BC(\pi))$. However, this Π is unique only up to twisting by δ, the quadratic character associated to E/F. We cannot conclude which one, Π or $\Pi \otimes \delta$, corresponds to the twisted exterior square L-group homomorphism. Hence it is necessary to use the converse theorem method below.

Our methodology is to use the converse theorem of Cogdell and Piatetski-Shapiro [Co-PS] and the Langlands-Shahidi method [Sh1,Sh2] of studying L-functions. We will now give a brief outline of the paper. The necessary L-functions and their analytic properties are discussed in Section 3. There are two ways one can go about applying the converse theorem to prove new cases of functoriality. One way is to use the stability of the Langlands-Shahidi local factors (see for example [CKPSS, Ki-Kr]), and the other way is to use Ramakrishnan's descent criterion (see for example [Ki1,Kr]). The problem of stabilizing the Langlands-Shahidi local factors is very difficult. Recently, this stability problem has been resolved for a wide class of groups including the ones considered in this paper (cf. [CPSS]). However, since [CPSS] was not available when we were writing this paper, we use Ramakrishnan's descent criterion in order to prove the above mentioned theorem. We carry out the main steps in Sections 4, 5, and 6. Some of the intermediate results that we obtain in order to use the descent criterion seem to be of independent interest. (See Section 5.) Our proofs in Section 5 (and also Section 9) rely on some results from the powerful theory of descent developed by Ginzburg, Rallis, and Soudry [So].

Given a generic cuspidal representation π of $PGSp_4(\mathbb{A}_F)$, in order to obtain the lift to $GL_4(\mathbb{A}_F)$ using the converse theorem, we need to identify $PGSp_4$ with SO_5, and apply the converse theorem to the group SO_{2n+1}. Likewise, when we have a generic cuspidal representation of $PGU(2,2)(\mathbb{A}_F) \simeq PU(2,2)(\mathbb{A}_F)$, we may obtain the twisted exterior square lift to $GL_6(\mathbb{A}_F)$ in another way as follows: Let SO_6^- be the quasi-split orthogonal group, and consider the map $SO_6^- \longrightarrow PU(2,2)$ (see Section 7). Given a generic cuspidal representation π of $PGU(2,2)(\mathbb{A}_F)$, we can consider π as a cuspidal representation of $SO_6^-(\mathbb{A}_F)$, and then apply the converse theorem to the group SO_{2n}^-. We give a short description of the proof in Section 7.

In Section 8, we define the twisted analogue of the symmetric mth power map and deduce functoriality of the twisted symmetric mth power of cuspidal representations of $GU(1,1)(\mathbb{A}_F)$ for $m = 2,3,4$. Since the derived group of $GU(1,1)$ is SL_2, this essentially follows from the functoriality of the usual symmetric mth power, $m = 2,3,4$. (See [Ge-J,Ki1,Ki-Sh].) Finally, in Section 9, we discuss the relation between the twisted exterior square lift and the twisted symmetric fourth power lift.

Acknowledgments. We would like to thank S. Kudla for helping us to understand unitary similitude groups. We would like to thank F. Shahidi and J. Cogdell for helpful discussions. Thanks are due to J.K. Yu for suggesting the proof of Lemma 2.1. The first author started this project when he was at the Institute for Advanced Study in the special year 1999-2000 in the Theory of Automorphic Forms and L-functions. He would like to thank the organizers, E. Bombieri, H. Iwaniec, R.P. Langlands, and P. Sarnak for inviting him to participate in the special year. The second author is grateful to the Horace H. Rackham School of Graduate Studies at the University of Michigan for their support in the summer of 2005 when part of this project was completed. Finally, we thank the referee for many helpful suggestions.

2. Unitary similitude groups and twisted exterior square.

Throughout this paper we fix E/F a quadratic extension of number fields and let $Gal(E/F) = \{1, \theta\}$. For any positive integer l, let J_l be the $l \times l$ matrix given by

$$J_l = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & \end{pmatrix}.$$

Let $J'_{2l} = \begin{pmatrix} & J_l \\ -J_l & \end{pmatrix}$. Let $\mathbf{G}_l = GU(l,l)_{E/F}$ be the quasi-split unitary similitude group defined with respect to the form J'_{2l}. It is the F-form of $GL_{2l} \times GL_1$ defined by the following Galois action:

$$\sigma(g\times\lambda)=\begin{cases} g\times\lambda, & \text{if } \sigma\in \mathrm{Gal}(\bar F/E)\\ \bar\lambda {J'_{2l}}^{-1}{}^t g^{-1}J'_{2l}\times\bar\lambda, & \text{otherwise.}\end{cases}$$

If A is any F-algebra, then

$$\mathbf{G}_l(A)=\{g\in GL_{2l}(E\otimes_F A): {}^t\bar g J'_{2l} g=\lambda J'_{2l},\ \text{ for some } \lambda\in A^\times\}.$$

(Here $g\mapsto \bar g$ is the action induced by θ; the action of θ on $E\otimes_F A$ is given by $b\otimes a\mapsto\theta(b)\otimes a$.) It is worth noting that λ depends on g. In particular, the adelic group $\mathbf{G}_l(\mathbb{A}_F)$ is given by

$$\mathbf{G}_l(\mathbb{A}_F)=\{g\in GL_{2l}(\mathbb{A}_E)|\, {}^t\bar g J'_{2l} g=\lambda J'_{2l},\ \text{for some } \lambda\in\mathbb{A}_F^\times\}.$$

Let $\mathbf{B}=\mathbf{TU}$ be the Borel F-subgroup of $\mathbf{G}_l$ corresponding to the upper triangular matrices. We write the L-group of $\mathbf{G}_l$ as ${}^L\mathbf{G}_l=(GL_{2l}(\mathbb{C})\times GL_1(\mathbb{C}))\rtimes Gal(E/F)$; here the action of θ is given by $\theta(g,\lambda)=({J'_{2l}}^{-1}{}^t g^{-1}J'_{2l},\lambda det(g))$ (cf. [R1]). We further set $\theta(g)={J'_{2l}}^{-1}{}^t g^{-1}J'_{2l}$, $g\in GL_{2l}(\mathbb{C})$. Note that the center of $\mathbf{G}_l(\mathbb{A}_F)$ is isomorphic to $\mathbb{A}_E^*$, therefore for any automorphic representation π of $\mathbf{G}_l(\mathbb{A}_F)$, its central character ω_π is an idele class character of E.

Let us suppose v is a place of F which remains inert in E. Then the group $\mathbf{G}_l(F_v)$ is given by

$$\mathbf{G}_l(F_v)=\left\{g\in GL_{2l}(E_v)\middle|\ {}^t\bar g J'_{2l} g=\lambda J'_{2l},\ \text{ for some } \lambda\in F_v^\times\right\};$$

and a typical element in $\mathbf{T}(F_v)$ is given by $t=\mathrm{diag}(t_1,...,t_l,a\bar t_l^{-1},...,a\bar t_1^{-1})$, $t_i\in E_v^\times, a\in F_v^\times$. Hence any character of $\mathbf{T}(F_v)$ is of the form $\nu_1\otimes\cdots\otimes\nu_l\otimes\nu_0$, where ν_i's are characters of $E_v^\times$ and ν_0 is a character of $F_v^\times$; it is given by $t\longmapsto \nu_1(t_1)\cdots\nu_l(t_l)\nu_0(a)$. Now, if π_v is any spherical representation of $\mathbf{G}_l(F_v)$ induced by a character $\chi=\nu_1\otimes\cdots\otimes\nu_l\otimes\nu_0$ of $\mathbf{T}(F_v)$ with $\nu_i, 1\le i\le l$, and ν_0 unramified, then the semi-simple conjugacy class of π_v is represented by the element

$$(\mathrm{diag}(\nu_1(\varpi),...,\nu_l(\varpi),1,...,1),\nu_0(\varpi),\theta)\in(GL_{2l}(\mathbb{C})\times GL_1(\mathbb{C}))\rtimes Gal(E/F);$$

the central character of π_v is given by $\omega_{\pi_v}=\nu_1\cdots\nu_l(\nu_0\circ N_{E_v/F_v})$ and its restriction to $F_v^\times$ is given by $\omega_{\pi_v}|_{F_v^\times}=\nu_1(\varpi)\cdots\nu_l(\varpi)\nu_0(\varpi)^2$.

On the other hand, if v splits as (w_1,w_2) in E, since $E\otimes_F F_v\simeq E_{w_1}\times E_{w_2}$, we have $GL_{2l}(E\otimes_F F_v)\simeq GL_{2l}(E_{w_1})\times GL_{2l}(E_{w_2})$; any $g\in GL_{2l}(E\otimes_F F_v)$ can be identified with $(g_1,g_2)\in GL_{2l}(E_{w_1})\times GL_{2l}(E_{w_2})$, and under this identification, $\bar g$ is given by (g_2,g_1). Consequently, the group $\mathbf{G}_l(F_v)$ is given by

$$\begin{aligned}\mathbf{G}_l(F_v)&=\{g\in GL_{2l}(E\otimes_F F_v): {}^t(g_2,g_1)\,J'_{2l}\,(g_1,g_2)=\lambda\,J'_{2l},\ \text{for some } \lambda\in F_v^\times\}\\ &=\{(g_1,\lambda\,{J'_{2l}}^{-1}{}^t g_1^{-1}J'_{2l})\subset GL_{2l}(E_{w_1})\times GL_{2l}(E_{w_2}),\ \lambda\in F_v^\times\}.\end{aligned}$$

Here the dependency of λ on (g_1, g_2) is through the relation $g_2 = \lambda {J'_{2l}}^{-1} {}^t g_1^{-1} J'_{2l}$. Thus, by fixing an embedding $E \hookrightarrow \bar{F}_v$, and hence an isomorphism $E_{w_1} \simeq F_v$, we see that

$$\mathbf{G}_l(F_v) \simeq GL_{2l}(F_v) \times GL_1(F_v),$$

where the isomorphism is given by the map $(g_1, \lambda {J'_{2l}}^{-1} {}^t g_1^{-1} J'_{2l}) \longmapsto (g_1, \lambda)$. Let us determine where the center of $\mathbf{G}_l(F_v)$ goes under this isomorphism: The center of $\mathbf{G}_l(F_v)$ is $\{(a_1 I_{2l}, a_2 I_{2l}) | \, a_1, a_2 \in F_v^*\}$, with $\lambda = a_1 a_2$; it corresponds to the center of $GL_{2l}(F_v) \times GL_1(F_v)$ through the identification $(a_1 I_{2l}, a_2 I_{2l}) \longmapsto (a_1 I_{2l}, a_1 a_2)$. Now, for any automorphic representation π of $\mathbf{G}_l(\mathbb{A}_F)$, let us look at its local component π_v at v; it can be identified with a representation of the form $\pi_v = \tau_v \otimes \chi_v$, where τ_v is a representation of $GL_{2l}(F_v)$ and χ_v is a character of $F_v^\times$. If ω_π denotes the central character of π, then its local component at w_1 is given by $\omega_{\tau_v} \chi_v$ and its component at w_2 is given by χ_v; hence the local component of $\omega_\pi|_{\mathbb{A}_F^\times}$ at v is given by $(\omega_\pi|_{\mathbb{A}_F^\times})_v = \omega_{\tau_v} \chi_v^2$. Further, if π_v is spherical, its semi-simple conjugacy class is represented by

$$(\mathrm{diag}(\alpha_1, ..., \alpha_{2l}), \lambda, 1) \in (GL_{2l}(\mathbb{C}) \times GL_1(\mathbb{C})) \rtimes Gal(E/F),$$

where $\mathrm{diag}(\alpha_1, ..., \alpha_{2l})$ represents the semi-simple conjugacy class of τ_v, and $\lambda = \chi_v(\varpi)$. In the event π_v is not spherical, thanks to the work of Harris and Taylor [H-T], and Henniart [Hen], the local Langlands correspondence is known in this case.

Next, we define the twisted exterior square map. Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{C}^4$ and let $\{f_1, f_2, \cdots, f_6\}$ denote the basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$ of $\wedge^2 \mathbb{C}^4$. The standard action of $g \in GL_4(\mathbb{C})$ on $\mathbb{C}^4$ induces the action $\wedge^2 g : \wedge^2 \mathbb{C}^4 \longrightarrow \wedge^2 \mathbb{C}^4$, defined by $(\wedge^2 g)(e_i \wedge e_j) = g e_i \wedge g e_j$; we let $\wedge^2 g \in GL_6(\mathbb{C})$ denote the matrix of this transformation with respect to the basis $\{f_1, f_2, \cdots, f_6\}$. Let us write $GU(2,2)$ to denote the group $\mathbf{G}_2$; then the connected component of ${}^L GU(2,2)$ is $GL_4(\mathbb{C}) \times GL_1(\mathbb{C})$; and we consider the six dimensional representation

$$\wedge^2 : GL_4(\mathbb{C}) \times GL_1(\mathbb{C}) \longrightarrow GL_6(\mathbb{C})$$

given by $(g, \lambda) \longmapsto (\wedge^2 g) \lambda$.

Lemma 2.1. *The representation $\wedge^2 : GL_4(\mathbb{C}) \times GL_1(\mathbb{C}) \longrightarrow GL_6(\mathbb{C})$ extends to a representation of ${}^L GU(2,2) = GL_4(\mathbb{C}) \times GL_1(\mathbb{C}) \rtimes Gal(E/F)$.*

Proof. The highest weight associated to the irreducible representation $\wedge^2$ is $e_1 + e_2$. The representation $\wedge^2 \circ \theta$ is also irreducible with highest weight $e_1 + e_2$. By the theory of highest weights, the representations $\wedge^2$ and $\wedge^2 \circ \theta$ are equivalent. Hence there is a matrix $A \in GL_6(\mathbb{C})$ such that

$$\wedge^2(\theta(g, \lambda)) = A^{-1} \wedge^2 (g, \lambda) A, \forall g \in GL_4(\mathbb{C}).$$

Now, we can extend $\wedge^2$ to ${}^L GU(2,2)$ by mapping

$$(g,\lambda,1) \longmapsto \wedge^2(g)\lambda, \text{ and } (1,1,\theta) \longmapsto A.$$

One checks that this yields a well defined representation of ${}^L GU(2,2)$. $\square$

Let us write $\wedge_t^2 : {}^L GU(2,2) \longrightarrow GL_6(\mathbb{C})$ to denote the representation obtained in Lemma 2.1 and we call it the twisted exterior square map.

Remark. We note that there is no twisted exterior square map from ${}^L U(2,2)$ to $GL_6(\mathbb{C})$. In other words, the highest weight associated to the irreducible representation $\wedge^2$ of $GL_4(\mathbb{C})$ is $e_1 + e_2$ where as the highest weight associated to $\wedge^2 \circ \theta : GL_4(\mathbb{C}) \longrightarrow GL_6(\mathbb{C})$ is $-e_3 - e_4$. Hence the representations $\wedge^2$ and $\wedge^2 \circ \theta$ are not equivalent. Consequently, $\wedge^2$ doesn't extend to a representation of ${}^L U(2,2)$. The situation is similar to Sp_4 versus GSp_4. There is a map from $GSp_4(\mathbb{C})$, the L-group of GSp_4, to $GL_4(\mathbb{C})$. However, there is no map from $SO_5(\mathbb{C})$, the L-group of Sp_4, to $GL_4(\mathbb{C})$. For the sake of reference, we record the irreducible representations of $SO_5(\mathbb{C}) \simeq PSp_4(\mathbb{C})$. Let ρ be the standard representation of $Sp_4(\mathbb{C})$. Then $\wedge^2\rho = \tau \oplus 1$, where $\tau : Sp_4(\mathbb{C}) \longrightarrow GL_5(\mathbb{C})$. The irreducible representations of $SO_5(\mathbb{C})$ are of the form $Sym^k(\rho) \otimes Sym^l(\tau)$, where $k, l \geq 0$ and k even. On the other hand, the irreducible representations of $Sp_4(\mathbb{C})$ are $Sym^k(\rho) \otimes Sym^l(\tau)$ without the condition on k.

The choice of A in the proof of Lemma 2.1 can be made explicit. Define a symmetric bilinear form Q by

$$e_i \wedge e_j \wedge e_k \wedge e_l = Q(e_i \wedge e_j, e_k \wedge e_l) e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

and extend it to $\wedge^2\mathbb{C}^4$ by linearity. Note that

$$ge_i \wedge ge_j \wedge ge_k \wedge ge_l = det(g) e_i \wedge e_j \wedge e_k \wedge e_l = Q(ge_i \wedge ge_j, ge_k \wedge ge_l) e_1 \wedge e_2 \wedge e_3 \wedge e_4;$$

hence $Q(ge_i \wedge ge_j, ge_k \wedge ge_l) = det(g) Q(e_i \wedge e_j, e_k \wedge e_l)$. Let S be the matrix associated to Q with respect to the basis $\{f_1, f_2, \cdots, f_6\}$. Then

$$S = \begin{pmatrix} & & & & & 1 \\ & & & & -1 & \\ & & & 1 & & \\ & & 1 & & & \\ & -1 & & & & \\ 1 & & & & & \end{pmatrix};$$

and we have shown that $\wedge^2 g \in GO_6(\mathbb{C})$, where

$$GO_6(\mathbb{C}) = \{g \in GL_6(\mathbb{C}) | \, {}^t g S g = \lambda(g) S, \lambda(g) \in \mathbb{C}^\times\}.$$

In fact, it is easy to see that $\wedge^2 g \in GSO_6(\mathbb{C}) = \{h \in GO_6(\mathbb{C}) |\, det(h)\lambda(h)^{-3} = 1\}$: Since $(\wedge^2 g)f_1\wedge\cdots\wedge(\wedge^2 g)f_6 = det(\wedge^2 g)(f_1\wedge\cdots\wedge f_6)$ and $(\wedge^2 g)f_i\wedge(\wedge^2 g)f_j = det(g)(f_i\wedge f_j)$, $i \neq j$, it follows that $det(\wedge^2 g) = det(g)^3$, and consequently $\wedge^2 g \in GSO_6(\mathbb{C})$. Now, we can take $A = S(\wedge^2(J_4'))$ in the proof of Lemma 2.1. Note that $A^2 = I$ and $det(A) = -1$. Hence $A \in GO_6(\mathbb{C}) - GSO_6(\mathbb{C})$ and one can verify that $\wedge^2(\theta(g,\lambda)) = A^{-1}\theta(g,\lambda)A$.

Now, we introduce the relevant base change maps on the dual group side. For any reductive group $\mathbf{G}$ defined over F, we have the usual base change map

$$^L\mathbf{G} \longrightarrow {}^L(R_{E/F}\mathbf{G})$$

given by the diagonal embedding $(g,\gamma) \longmapsto (g,g,\gamma)$, $\gamma \in \Gamma_F$. For $\mathbf{G} = GU(2,2)$, which splits over E, we have

$$^L(R_{E/F}GU(2,2)) = GL_4(\mathbb{C}) \times GL_1(\mathbb{C}) \times GL_4(\mathbb{C}) \times GL_1(\mathbb{C}) \rtimes Gal(E/F),$$

where $\tau \in Gal(E/F)$ acts by $((g,\lambda),(g',\lambda')) \mapsto (\tau(g',\lambda'),\tau(g,\lambda))$. Now, let BC denote the composition

$$BC : {}^LGU(2,2) \longrightarrow {}^L(R_{E/F}GU(2,2)) \longrightarrow {}^L(R_{E/F}(GL_4 \times GL_1)),$$

where the first arrow is the base change map mentioned above, and the second arrow is the isomorphism given by

$$((g_1,\lambda_1),(g_2,\lambda_2),\tau) \longmapsto ((g_1,\lambda_1),(\theta(g_2,\lambda_2),\tau),\ \tau \in Gal(E/F).$$

In other words, the map BC is given by $(g,\lambda,\tau) \longmapsto (g,\lambda,\theta(g),\lambda det(g),\tau)$; we call it the stable base change map. (See [Ki-Kr] for the definition of the stable base change map for unitary groups.) We discuss functoriality for the base change map BC in Section 5.

Let $B : GL_6(\mathbb{C}) \times Gal(E/F) \longrightarrow (GL_6(\mathbb{C}) \times GL_6(\mathbb{C})) \rtimes Gal(E/F)$ be the base change map $(g,\tau) \longmapsto (g,g,\tau)$, and consider the diagram

$$\begin{array}{ccc} {}^LGU(2,2) & \xrightarrow{\wedge_t^2} & {}^LGL_6 \\ {\scriptstyle BC}\big\downarrow & & {\scriptstyle B}\big\downarrow \\ {}^LR_{E/F}(GL_4 \times GL_1) & \xrightarrow{\wedge^2} & {}^LR_{E/F}(GL_6), \end{array}$$

where the exterior square map in the bottom horizontal arrow is defined as

$$\wedge^2(g_1,\lambda_1,g_2,\lambda_2,\tau) = ((\wedge^2 g_1)\lambda_1,(\wedge^2 g_2)\lambda_2,\tau),\ \tau \in Gal(E/F).$$

We claim that the above diagram is commutative up to conjugacy, i.e., $B \circ \wedge_t^2 = (I,A,1)(\wedge^2 \circ BC)(I,A,1)$: note that

$$B \circ \wedge_t^2(g,\lambda,\tau) = \begin{cases} ((\wedge^2 g)\lambda,(\wedge^2 g)\lambda,1), & \text{if } \tau = 1 \\ ((\wedge^2 g)\lambda A,(\wedge^2 g)\lambda A,\theta), & \text{if } \tau = \theta \end{cases};$$

and

$$\wedge^2 \circ BC(g,\lambda,\tau) = \begin{cases} ((\wedge^2 g)\lambda,(\wedge^2(\theta(g)))\lambda det(g),1), & \text{if } \tau = 1 \\ ((\wedge^2 g)\lambda,(\wedge^2(\theta(g)))\lambda det(g),\theta), & \text{if } \tau = \theta \end{cases};$$

now our claim follows since $\wedge^2(\theta(g,\lambda)) = A(\wedge^2(g,\lambda))A$.

3. Properties of relevant L-functions.

Let $\mathbf{G} = Spin_{2n}^-$ be a quasi-split spin group over a quadratic extension E/F. It is a simply connected 2-fold covering group of the quasi-split orthogonal group SO_{2n}^- corresponding to a quadratic form of index $n-1$ relative to F but index n relative to E. Namely, it is given by the quadratic form $x_1x_n + x_2x_{n+1} + \cdots + x_{n-1}x_{2n-2} + N$, where $N : E \longrightarrow F$ is the norm and we consider E as a two-dimensional vector space over F. Let $\{\alpha_1 = e_1 - e_2, ..., \alpha_{n-2} = e_{n-2} - e_{n-1}, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\}$ be the non-restricted simple roots (see [B2]). Let $\beta_1 = \alpha_1 = e_1 - e_2, ..., \beta_{n-2} = \alpha_{n-2} = e_{n-2} - e_{n-1}, \beta_{n-1} = \frac{1}{2}(\alpha_{n-1} + \alpha_n) = e_{n-1}$. Then $\Delta = \{\beta_1, ..., \beta_{n-1}\}$ forms a set of simple roots of type B_{n-1}. For each α_i, let H_{α_i} be the corresponding coroot. Then an element in the maximal torus of $Spin_{2n}^-$ can be written as

$$H_{\alpha_1}(t_1) \cdots H_{\alpha_{n-2}}(t_{n-2}) H_{\alpha_{n-1}}(t_{n-1}) H_{\alpha_n}(\bar{t}_{n-1}),$$

where $t_i \in F^*$ for $i = 1, ..., n-2$ and $t_{n-1} \in E^*$. On the other hand an element in the maximal split torus can be written as $H_{\alpha_1}(t_1) \cdots H_{\alpha_{n-2}}(t_{n-2}) H_{\alpha_{n-1}}(t_{n-1}) H_{\alpha_n}(t_{n-1})$, where $t_i \in F^*$ for all i.

Now we consider a special case of ${}^2D_n - 1$ case in [Sh1], namely, the case attached to $\theta = \Delta - \{\beta_{n-3}\}$. In general, the case ${}^2D_n - 1$ would give the analytic properties of L-functions required to establish functorial transfers (via the converse theorem) from $GSpin_{2k}^-(\mathbb{A}_F) \longrightarrow GL_{2k}(\mathbb{A}_F)$ when $k \geq 3$. In particular, one can obtain functorial transfers from globally generic cuspidal representations of $SO_{2k}^-(\mathbb{A}_F)$ to automorphic representations of $GL_{2k}(\mathbb{A}_F)$ when $k \geq 4$. When $k = 3$, due to the accidental isomorphism $Spin_6^- \simeq SU(2,2)$, we obtain a twisted exterior square transfer (also referred to as twisted exterior square lift) from globally generic cuspidal representations of $GU(2,2)_{E/F}(\mathbb{A}_F)$ to automorphic representations of $GL_6(\mathbb{A}_F)$, corresponding to the map

$$\wedge_t^2 : (GL_4(\mathbb{C}) \times GL_1(\mathbb{C})) \rtimes Gal(E/F) \longrightarrow GL_6(\mathbb{C}).$$

Our focus in this paper is the case $k = 3$. Let $\mathbf{P} = P_\theta = \mathbf{MN}$, and $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$. Then $\mathbf{A}(\overline{F}) = \{a(t) |\, t \in \overline{F}\}$;

$$a(t) = \begin{cases} H_{\alpha_1}(t) \cdots H_{\alpha_{n-3}}(t^{n-3}) H_{\alpha_{n-2}}(t^{n-3}) H_{\alpha_{n-1}}(t^{\frac{n-3}{2}}) H_{\alpha_n}(t^{\frac{n-3}{2}}), & \text{if } n \text{ odd} \\ H_{\alpha_1}(t^2) H_{\alpha_2}(t^4) \cdots H_{\alpha_{n-3}}(t^{2(n-3)}) H_{\alpha_{n-2}}(t^{2(n-3)}) H_{\alpha_{n-1}}(t^{n-3}) H_{\alpha_n}(t^{n-3}), & \text{if } n \text{ even} \end{cases}$$

We note that $\mathbf{A}$ is a 1-dimensional torus that splits over F. Since $\mathbf{G}$ is simply connected, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence

$$\mathbf{M}_D = SL_{n-3} \times SU(2,2).$$

Note that $Spin_6^- \simeq SU(2,2)$. Further

$$\mathbf{A} \cap \mathbf{M}_D(\overline{F}) = \begin{cases} \{H_{\alpha_1}(t) \cdots H_{\alpha_{n-4}}(t^{n-4}) H_{\alpha_{n-1}}(t^{\frac{n-3}{2}}) H_{\alpha_n}(t^{\frac{n-3}{2}}) |\ t^{n-3} = 1\}, & \text{if } n \text{ odd} \\ \{H_{\alpha_1}(t^2) \cdots H_{\alpha_{n-4}}(t^{2(n-4)}) H_{\alpha_{n-1}}(t^{n-3}) H_{\alpha_n}(t^{n-3}) |\, t^{2(n-3)} = 1\}, & \text{if } n \text{ even} \end{cases}$$

Let us recall that there are isomorphisms $f : SU(2,2) \longrightarrow SL_4$ and $q_1 : R_{E/F}GL_1 \longrightarrow GL_1 \times GL_1$, both defined over E. We now define $f' : \mathbf{A} \times \mathbf{M} \longrightarrow GL_{n-3} \times GU(2,2)$ by

$$\begin{array}{ccc}
\mathbf{A} \times \mathbf{M}_D & \xrightarrow{\phi_1} & \mathbf{A} \times SL_{n-3} \times SL_4 \\
 & & \downarrow p \\
f' \downarrow & & GL_1 \times GL_1 \times GL_1 \times SL_{n-3} \times SL_4 \\
 & & \downarrow q \\
GL_{n-3} \times GU(2,2) & \underset{\phi_2}{\longleftarrow} & GL_1 \times R_{E/F}GL_1 \times SL_{n-3} \times SU(2,2)
\end{array}$$

where $\phi_1(a(t), x, u) = (a(t), x, f(u)), x \in SL_{n-3}, u \in SU(2,2)$;

$$p(a(t), x, y) = \begin{cases} \{(t, t^{\frac{n-3}{2}}, t^{\frac{n-3}{2}}, x, y)\}, & \text{if } n \text{ odd} \\ \{(t^2, t^{n-3}, t^{n-3}, x, y)\}, & \text{if } n \text{ even} \end{cases}, \quad x \in SL_{n-3}, y \in SL_4;$$

$q(t_1, t_2, t_3, x, y) = (t_1, q_1^{-1}(t_2, t_3), x, f^{-1}(y))$; and ϕ_2 is the obvious map induced by the homomorphisms $R_{E/F}GL_1 \times SU(2,2) \longrightarrow GU(2,2)$ and $GL_1 \times SL_{n-3} \to GL_{n-3}$, respectively. It is a routine matter to check that f' is F-rational and that it factors through $\mathbf{A} \cap \mathbf{M}_D$. Therefore f' yields an injection $f : \mathbf{M} \longrightarrow GL_{n-3} \times GU(2,2)$ which is the identity map when restricted to the derived group $\mathbf{M}_D$.

For the remainder of the paper, we fix a non-trivial additive character $\psi = \otimes \psi_v$ of $F \backslash \mathbb{A}_F$. Let σ be a cuspidal representation of $GL_{n-3}(\mathbb{A}_F)$ and let π be a globally ψ-generic representation of $GU(2,2)(\mathbb{A}_F)$. We recall that "globally ψ-generic" means that there is a vector in the space of π such that its ψ-Whittaker coefficient is non-zero. Let ω_σ and ω_π denote the central characters of σ and π, respectively. Let Σ be a cuspidal representation of $\mathbf{M}(\mathbb{A}_F)$, determined by the map f, σ, and π. To be precise, if $\mathbb{A}_F^\times$ is embedded as the center, of say, GL_{n-3}, then $\mathbf{M}(\mathbb{A}_F)\mathbb{A}_F^\times$ is co-compact in $GL_{n-3} \times GU(2,2)(\mathbb{A}_F)$. Consequently, $\sigma \otimes \pi|_{f(\mathbf{M}(\mathbb{A}_F))}$ decomposes into a direct sum of irreducible cuspidal representations of $\mathbf{M}(\mathbb{A}_F)$. We take Σ to be any irreducible constituent of this direct sum, the choice of the constituent does not matter since they all give rise to the same L-functions. The central character of Σ is given by

$$\omega_\Sigma = \begin{cases} \omega_\sigma(\omega_\pi|_{\mathbb{A}_F^*}^{\frac{n-3}{2}}), & \text{if } n \text{ odd}, \\ \omega_\sigma^2(\omega_\pi|_{\mathbb{A}_F^*}^{n-3}), & \text{if } n \text{ even}. \end{cases}$$

In what follows, we write ρ_n to denote the standard representation of $GL_n(\mathbb{C})$.

Let us write $\Sigma = \otimes_v \Sigma_v$. The Langlands-Shahidi method [Sh1] attaches to the data $(\mathbf{G}, \mathbf{M}, \Sigma)$, two L-functions $L(s, \Sigma, r_1), L(s, \Sigma, r_2)$. Here $r = r_1 \oplus r_2$ is the adjoint action of ${}^L M$ on ${}^L\mathfrak{n}$, where $\mathfrak{n}$ is the Lie algebra of $\mathbf{N}$. We will see that $L(s,\Sigma,r_1)$ is exactly the L-function we need to apply the converse theorem in order to obtain the twisted exterior square lift. When v is unramified and Σ_v is spherical, we can explicitly compute $L(s,\Sigma_v,r_1)$ and $L(s,\Sigma_v,r_2)$: If v splits as (w_1,w_2) in E, then π_v is an unramified representation of $GL_4(F_v)\times GL_1(F_v)$. Write $\pi_v=\tau_v\otimes\chi_v$. In this case,

$$L(s,\Sigma_v,r_{1,v})=L(s,\sigma_v\otimes\tau_v,\rho_{n-3}\otimes\wedge^2\rho_4\otimes\chi_v)=L(s,\sigma_v\times\wedge_t^2(\pi_v))$$
$$L(s,\Sigma_v,r_{2,v})=L(s,\sigma_v,\wedge^2\otimes(\omega_{\tau_v}\chi_v^2)).$$

Here note that $\omega_{\tau_v}\chi_v^2=(\omega_\pi|_{\mathbb{A}_F^*})_v=(\omega_\pi)_{w_1}(\omega_\pi)_{w_2}$.

Suppose v is inert. Let $\sigma_v=\pi(\mu_1,...,\mu_{n-3})$, and $\pi_v=\pi(\nu_1,\nu_2,\nu_0)$, where $\mu_1,...,\mu_{n-3},\nu_0$ are characters of $F_v^\times$ and ν_1,ν_2 are characters of $E_v^\times$. Here the character $\nu_1\otimes\nu_2\otimes\nu_0$ acts on $\mathrm{diag}(a,b,\lambda\bar{b}^{-1},\lambda\bar{a}^{-1})$ by $\nu_1(a)\nu_2(b)\nu_0(\lambda)$. So $\omega_{\pi_v}=\nu_1\nu_2(\nu_0\circ N_{E_v/F_v})$ and $\omega_{\pi_v}|_{F_v^\times}=(\nu_1\nu_2|_{F_v^\times})\nu_0^2$. We compute the local L-function for n odd. The "even" case is similar. Let Σ_v be induced from a character χ. Then

$$\chi\circ H_{\alpha_1}=\mu_1\mu_2^{-1},\cdots,\chi\circ H_{\alpha_{n-4}}=\mu_{n-4}\mu_{n-3}^{-1},\ \chi\circ H_{\alpha_{n-2}}=\nu_2|_{F_v^\times},$$
$$\chi\circ H_{\alpha_{n-1}}(\varpi)H_{\alpha_n}(\varpi)=\nu_1\nu_2^{-1}(\varpi),\quad \chi(a(\varpi))=\omega_{\sigma_v}(\varpi)\omega_{\pi_v}(\varpi)^{\frac{n-3}{2}},$$

where ϖ is a uniformizing element in $F_v^\times$. From this, we see that $\chi\circ H_{\alpha_{n-3}}=\mu_{n-3}\nu_0$, and

$$L(s,\Sigma_v,r_{1,v})=L(s,\sigma_v\otimes\pi_v,\rho_{n-3}\otimes\wedge_t^2)=L(s,\sigma_v\times\wedge_t^2(\pi_v))$$
$$L(s,\Sigma_v,r_{2,v})=L(s,\sigma_v,\wedge^2\otimes(\omega_{\pi_v}|_{F_v^\times})),$$

where $\wedge_t^2(\pi_v)$ is the unramified representation of $GL_6(F_v)$ given by

$$\wedge_t^2(\pi_v)=\pi(\nu_0(\nu_1\nu_2|_{F_v^\times})^{\frac12},\nu_0\omega_{E_v/F_v}(\nu_1\nu_2|_{F_v^\times})^{\frac12},\nu_0(\nu_1|_{F_v^\times}),\nu_0(\nu_2|_{F_v^\times}),\nu_0(\nu_1\nu_2|_{F_v^\times}),\nu_0).$$

For the ramified places v, we let $L(s,\Sigma_v,r_{1,v})$ and $L(s,\Sigma_v,r_{2,v})$ be the ones defined in [Sh2, Section 7]. In particular, if $v|\infty$, then $L(s,\Sigma_v,r_{i,v})$, $i=1,2$, is the Artin L-function defined through Langlands parameterization in each of the cases. Also, for each place v of F, let $I(s,\Sigma_v)$ be the induced representation and let $N(s,\Sigma_v,w_0)$ be the normalized local intertwining operator given by

$$A(s,\Sigma_v,w_0)=\frac{L(s,\Sigma_v,r_{1,v})L(2s,\Sigma_v,r_{2,v})}{L(1+s,\Sigma_v,r_{1,v})L(1+2s,\Sigma_v,r_{2,v})}\frac{N(s,\Sigma_v,w_0)}{\epsilon(s,\Sigma_v,r_{1,v},\psi_v)\epsilon(2s,\Sigma_v,r_{2,v},\psi_v)},$$

where $A(s,\Sigma_v,w_0)$ is the unnormalized local intertwining operator.

Proposition 3.1. *The normalized local intertwining operators $N(s, \Sigma_v, w_0)$ are holomorphic and non-zero for $Re(s) \geq \frac{1}{2}$ for all v.*

Proof. If v splits, π_v is a representation of $GL_4(F_v) \times GL_1(F_v)$ and our assertion follows from [Ki1, Proposition 3.1]. If v is inert or ramified, we proceed as in [Ki-Kr, Proposition 5.2]; we need two ingredients: Conjecture 7.1 of [Sh2] on the holomorphy of the local L-functions, and standard module conjecture. Conjecture 7.1 has been proved in our case in [Ca-Sh]. The standard module conjecture is proved in [Mu]. If Σ_v is tempered, then by Conjecture 7.1, $L(s, \Sigma_v, r_{1,v})L(2s, \Sigma_v, r_{2,v})$ is holomorphic for $Re(s) > 0$. Hence our result follows. If Σ_v is non-tempered, then we write it as a full induced representation and write $N(s, \Sigma_v, w_0)$ as a product of rank-one intertwining operators, and then show that each rank-one operators are holomorphic. The non-vanishing part follows from the holomorphy of $N(s, \Sigma_v, w_0)$. (See [Zh] for example.) □

Let χ be an idele class character of F which is highly ramified for some finite place. Let Σ_χ be the corresponding cuspidal representation of $\mathbf{M}(\mathbb{A})$ determined by π and $\sigma \otimes \chi$. The analytic properties of $L(s, \Sigma_\chi, r_1)$ are well known due to the works of Stephen Gelbart, Henry Kim and Freydoon Shahidi. Let us gather the known analytic properties in the following:

Proposition 3.2. *The L-function $L(s, \Sigma_\chi, r_1)$ is "nice", i.e., entire, bounded in vertical strips, and satisfies the functional equation*

$$L(s, \Sigma_\chi, r_1) = \epsilon(s, \Sigma_\chi, r_1)L(1 - s, \Sigma_\chi, \tilde{r}_1).$$

Proof. The meromorphic continuation and the functional equation of $L(s, \Sigma_\chi, r_1)$ is part of a general result due to Shahidi [Sh2]. The fact that the L-functions under discussion is entire follows from [Ki-Sh, Proposition 2.1]. Finally, the boundedness in vertical strips follows from [Ge-Sh]. We note that in verifying the analyticity and the boundedness in vertical strips, one needs to have Proposition 3.1 in hand. □

4. Weak lift: The "good" case.

With notations as in Section 2, let $\pi = \otimes_v \pi_v$ be a globally generic cuspidal representation of $GU(2, 2)(\mathbb{A}_F)$. Let $T = \{v < \infty | \ v \text{ does not split in } E \text{ and } \pi_v \text{ is supercuspidal}\}$. Then if $v \notin T$, any π_v can be parametrized by $\phi_v : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow (GL_4(\mathbb{C}) \times GL_1(\mathbb{C})) \rtimes Gal(E/F)$. For split places, it is the result of Harris-Taylor, Henniart [H-T, Hen]. For real places, it is the result of Langlands [La]. For $U(2, 2)$ over p-adic places, the parametrization is explicitly written in [Ko-Ko, §3.3], based on the calculations of [Ko]. By [B1, page 43], we can lift the parameters to those of $GU(2, 2)$ over p-adic places. We refer the reader to [Ta1, Section 2] where the relationship between representations of $GU(2, 2)$ and $U(2, 2)$ is discussed in a more general setting; Tadic [Ta1] generalized the case of $SL_n \subset GL_n$. We discuss the global version of this relationship in Section 5 (see Theorem 5.3). Then we have $\wedge_t^2 \circ \phi_v : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow GL_6(\mathbb{C}) \times Gal(E/F)$ which in turn gives rise to the irreducible admissible representation $\wedge_t^2(\pi_v)$ of $GL_6(F_v)$ by the local Langlands correspondence (cf. [H-T, Hen]).

More explicitly, if v splits, π_v is an irreducible representation of $GL_4(F_v) \times GL_1(F_v)$ of the form $\tau_v \otimes \chi_v$, where τ_v is an irreducible representation $GL_4(F_v)$. Then $\wedge_t^2(\pi_v)$ is the representation $\wedge^2 \tau_v \otimes \chi_v$.

If v is inert and π_v is spherical, the semi-simple conjugacy class of π_v is given by $(\text{diag}(a, b, 1, 1), \lambda, \theta) \in (GL_4(\mathbb{C}) \times GL_1(\mathbb{C})) \rtimes Gal(E/F)$. We can see easily that

$$\wedge_t^2(\text{diag}(a, b, 1, 1), \lambda, \theta) = \lambda \text{diag}(\begin{pmatrix} ab & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}),$$

so that the eigenvalues are $\lambda, \lambda a, \lambda b, \lambda ab$ and the roots of $X^2 - ab\lambda^2$. Then the semi-simple conjugacy class of $\wedge_t^2(\pi_v)$ is given by $\text{diag}(\lambda, \lambda a, \lambda b, \lambda ab, \lambda\alpha, \lambda\beta) \in GL_6(\mathbb{C})$, where α, β are roots of $X^2 - ab$.

Recall that we have $\gamma(s, \Sigma_v, r_{1,v}, \psi_v), L(s, \Sigma_v, r_{1,v})$, defined by Langlands-Shahidi method. They are defined as normalizing factors of intertwining operators. On the other hand, we can define $\gamma(s, \sigma_v \times \wedge_t^2(\pi_v), \psi_v), L(s, \sigma_v \times \wedge_t^2(\pi_v))$ by the Rankin-Selberg method which uses the theory of integral representations. By the local Langlands correspondence, the local γ and L-factors defined via the Rankin-Selberg method are known to be the γ and L-factors attached to the corresponding finite dimensional complex representations of the Weil-Deligne group. We now prove that for $v \notin T$ these two different definitions of local factors are indeed the same.

Proposition 4.1. *If $v \notin T$, we have*

$$\gamma(s, \Sigma_v, r_{1,v}, \psi_v) = \gamma(s, \sigma_v \times \wedge_t^2(\pi_v), \psi_v), \quad L(s, \Sigma_v, r_{1,v}) = L(s, \sigma_v \times \wedge_t^2(\pi_v)). \tag{4.1}$$

for all generic irreducible admissible representation σ_v of $GL_m(F_v)$, and all m.

Proof. Observe that (4.1) is true if $v|\infty$ by [Sh5] or if $v < \infty$ and all the local data are unramified by [Sh2]. In general for $v \notin T$, we only give the idea of the proof. We refer the reader to [Ki1, Proposition 4.2] for details. By multiplicativity of γ and L-factors, we can write both sides as products of rank-one γ and L-factors, which are γ and L-factors for $GL_k \times GL_l$. In this case, Shahidi [Sh4] proved the equality of the Langlands-Shahidi γ and L-functions with those defined via the Rankin-Selberg method. For $GL_k \times GL_l$, as mentioned above, the Rankin-Selberg local factors are same as those of Artin by the local Langlands correspondence. Hence our result follows. □

Since we do not have (4.1) when π_v is a supercuspidal representation, we make the following definition.

Definition 4.2. *An irreducible admissible representation Π_v of $GL_6(F_v)$ is said to be a local twisted exterior square lift of π_v if equations (4.1) with $\wedge_t^2(\pi_v)$ replaced by Π_v is valid* *for all irreducible admissible generic representations* σ_v *of* $GL_m(F_v)$*. For* $\pi = \otimes_v \pi_v$ *as above, an automorphic representation* $\Pi = \otimes_v \Pi_v$ *of* $GL_6(A_F)$ *is said to be a strong (resp. weak) twisted exterior square lift of* π *if* Π_v *is a local twisted exterior square lift of* π_v *for all (resp. almost all)* v*.*

From now on we simply say "local lift" instead of "local twisted exterior square lift". Observe that Proposition 4.1 shows that $\wedge_t^2(\pi_v)$ is a local lift of π_v for all $v \notin T$. Our goal is to prove the existence of a strong exterior square lift of π. One way to prove this is to use the following theorem which is a consequence of [CPSS, Corollary 6.2] and [Sh3].

Theorem 4.3 (Stability of γ-factors). *In the notation of Section 3 let us take* $n = 4$*; then* $\mathbf{G} = Spin_8^-$ *and* $\mathbf{M}_D = SU(2,2)$*. For* $v < \infty$*, let* $\pi_{1,v}, \pi_{2,v}$ *be two generic irreducible representations of* $GU(2,2)(F_v)$ *with the same central character. Then for every highly ramified character* μ *of* $F_v^\times$*,*

$$\gamma(s, \Sigma_{1v}^\mu, r_{1,v}, \psi_v) = \gamma(s, \Sigma_{2v}^\mu, r_{1,v}, \psi_v), \quad L(s, \Sigma_{1v}^\mu, r_{1,v}) = L(s, \Sigma_{2v}^\mu, r_{1,v}),$$

where, for $i = 1, 2$, Σ_{iv}^μ *is the representation of* $\mathbf{M}(F_v)$ *induced from the representation* $\mu \times \pi_{iv}$ *of* $GL_1(F_v) \times GU(2,2)(F_v)$ *as explained in Section 3.*

However as mentioned in the Introduction, since [CPSS] was not available while we were working on this paper, we use the descent method due to Ramakrishnan [Ra1]. Some of the intermediate results that we obtain in Section 5 in order to use the descent criterion seem to be of independent interest. In order to carry out the descent method, we first deal with the "good" case: The representation π is said to be "good" (with respect to E/F, of course) if the corresponding set T is empty. In Section 6, we deal with the general case.

Suppose π is such that T is empty. Then we can define $\wedge_t^2(\pi_v)$ for all v. Let $\wedge_t^2(\pi) = \otimes_v \wedge_t^2(\pi_v)$. It is an irreducible admissible representation of $GL_6(\mathbb{A}_F)$. Now by Proposition 4.1, we have $L(s, \sigma \times \wedge_t^2(\pi)) = L(s, \Sigma, r_1)$ for all cuspidal representations σ of $GL_m(\mathbb{A}_F)$, $1 \le m \le 4$. In particular, with χ as in Proposition 3.2, we have

$$L(s, (\sigma \otimes \chi) \times \wedge_t^2(\pi)) = L(s, \Sigma_\chi, r_1)$$

for all cuspidal representations σ of $GL_m(\mathbb{A}_F)$, $1 \le m \le 4$. Let $S = \{v\}$, where v is any finite place of F. By applying the converse theorem [Co-PS] to $\wedge_t^2(\pi)$ with respect to S, we obtain an automorphic representation Π of $GL_6(\mathbb{A}_F)$ such that $\Pi_u \simeq \wedge_t^2(\pi_u)$ for all $u \notin S$. Now, we can write Π as a subquotient of

$$Ind\,(|det|^{r_1}\sigma_1 \otimes \cdots \otimes |det|^{r_k}\sigma_k),$$

where $r_i \in \mathbb{R}$ and σ_i's are unitary cuspidal representations of $GL_{n_i}(\mathbb{A}_F)$. In order to conclude that $r_1 = \cdots = r_k = 0$, we use the weak Ramanujan property: If $\pi = \otimes_v \pi_v$ is a (unitary) cuspidal representation of $GU(2,2)(\mathbb{A}_F)$, and S is a finite set of places including the archimedean places such that both π_v and $GU(2,2)$ are unramified for $v \notin S$. Let $A_v \in GL_4(\mathbb{C}) \times GL_1(\mathbb{C})$, $v \notin S$, represent the semi-simple conjugacy class associated to π_v. If $v \notin S$ splits in E, we can write $A_v = \mathrm{diag}(\alpha_{1,v}, \cdots, \alpha_{4,v}) \times \lambda_v$ for suitable complex numbers $\alpha_{i,v}, 1 \leq i \leq 4$, and λ_v, $|\lambda_v| = 1$. If v is inert in E, we can write $A_v = \mathrm{diag}(\beta_{1,v}, \beta_{2,v}, 1, 1) \times \lambda_v$ for suitable complex numbers $\beta_{i,v}, i = 1, 2$, and $|\lambda_v| = 1$. Then we say that π satisfies the weak Ramanujan property if given $\epsilon > 0$, there exists a density zero set D of places of F such that $\max_i\{|\alpha_{i,v}|, |\alpha_{i,v}^{-1}|\} \leq q_v^{\epsilon}, \max_i\{|\beta_{i,v}|, |\beta_{i,v}^{-1}|\} \leq q_v^{\epsilon}$ for all $v \notin D$.

Now by [Ki-Kr, Theorem 9.3], any generic cuspidal representation of $U(2,2)(\mathbb{A}_F)$ satisfies the weak Ramanujan property. Given a generic cuspidal representation π of $GU(2,2)(\mathbb{A}_F)$, $\pi|_{U(2,2)(\mathbb{A}_F)}$ is a direct sum of cuspidal representations of $U(2,2)(\mathbb{A}_F)$. At least one of them is generic, which we denote it by π_0. The embedding $U(2,2) \subset GU(2,2)$ induces a map on the L-group ${}^L GU(2,2) \longrightarrow {}^L U(2,2)$, and the Satake parameter of π_v corresponds to that of π_{0v} [B1, page 43]. Hence it is clear that π satisfies the weak Ramanujan property since π_0 does.

Now, by proceeding as in the proof of [Ki1, Proposition 4.1.2], we see that $r_1 = \cdots = r_k = 0$ as desired. Hence Π is of the form

$$\Pi = Ind\,(\sigma_1 \otimes \cdots \otimes \sigma_k).$$

Pick two finite places v_1, v_2, where π_{iv_1}, π_{iv_2}, $i = 1, 2$, are unramified. Let $S_i = \{v_i\}$, $i = 1, 2$. We apply the converse theorem twice to $\wedge_t^2(\pi)$ with S_1 and S_2, and find two automorphic representations Π_1, Π_2 of $GL_6(\mathbb{A}_F)$ such that $\Pi_{1v} \simeq \wedge_t^2(\pi_v)$ for $v \neq v_1$, and $\Pi_{2v} \simeq \wedge_t^2(\pi_v)$ for $v \neq v_2$. Hence $\Pi_{1v} \simeq \Pi_{2v}$ for all $v \neq v_1, v_2$. Since Π_1, Π_2 are of the form $\tau_1 \boxplus \cdots \boxplus \tau_k$, where τ_i's are (unitary) cuspidal representations of GL, by the strong multiplicity one theorem, we conclude that $\Pi_1 \simeq \Pi_2$. In particular, $\Pi_{1v_i} \simeq \Pi_{2v_i} \simeq \wedge_t^2(\pi_v)$ for all v. Thus we have proved

Theorem 4.4. *Suppose $\pi = \otimes_v \pi_v$ is a generic cuspidal representation of $GU(2,2)(\mathbb{A}_F)$ and assume that it is good, i.e., the corresponding set T is empty. Then $\wedge_t^2(\pi)$ is an automorphic representation of $GL_6(\mathbb{A}_F)$, and is of the form $Ind\,(\sigma_1 \otimes \cdots \otimes \sigma_k)$, where σ_i's are unitary cuspidal representations of $GL_{n_i}(\mathbb{A}_F)$ and $n_1 + \cdots + n_k = 6$.*

5. Base change for unitary groups and unitary similitude groups.

In this section we will be using a family of quadratic extensions and the corresponding unitary and unitary similitude groups associated to them. We adopt the following notations. For any quadratic extension K/k, we write $\mathbf{H}_{K/k}$ to denote the quasi-split unitary group $U(n,n)_{K/k}$ and $\mathbf{G}_{K/k}$ to denote the quasi-split unitary similitude group $GU(n,n)_{K/k}$. We write $\delta_{K/k}$ for the quadratic character attached to K/k via class field theory. It is worth reminding the reader that the additive character ψ is fixed throughout this paper, and by generic representations we mean "generic" with respect to ψ.

Lemma 5.1. *Let K/k be any quadratic extension of number fields and let $\delta_{K/k}$ be the quadratic character attached to it. Let $Gal(K/k) = \{1, \theta\}$. Suppose τ is a cuspidal representation of $GL_n(\mathbb{A}_K)$ such that $L(s, \tau, As_{K/k} \otimes \delta_{K/k})$ has a pole at $s = 1$. Then $L(s, \tau, As_{K/k} \otimes \chi)$ has no pole at $s = 1$ for $\chi = 1$, or any $\chi \neq \delta_{K/k}$ such that $\tau \not\simeq \tau \otimes (\chi \circ N_{K/k})$.*

Proof. Since $L(s, \tau \times \tau^\theta) = L(s, \tau, As_{E/F})L(s, \tau, As_{E/F} \otimes \delta_{E/F})$ and $L(s, \tau, As_{E/F})$ is nonvanishing on $\mathrm{Re}(s) = 1$, our hypothesis implies that $L(s, \tau \times \tau^\theta)$ has a pole at $s = 1$ (known to be simple). Thus $\tau^\theta = \tilde{\tau}$, and further the simplicity of the pole proves our assertion that $L(s, \tau, As_{E/F})$ has no pole at $s = 1$. So we can assume that $\chi \neq 1$. Let χ' be any idele class character of E with restriction χ to F. Then $L(s, \tau \otimes \chi', As_{E/F}) = L(s, \tau, As_{E/F} \otimes \chi)$.

Now consider

$$\begin{aligned} L(s, (\tau \otimes \chi') \times (\tau \otimes \chi')^\theta) &= L(s, \tau \otimes \chi', As_{E/F})L(s, \tau \otimes \chi', As_{E/F} \otimes \delta_{E/F}) \\ &= L(s, \tau, As_{E/F} \otimes \chi)L(s, \tau, As_{E/F} \otimes \chi\delta_{E/F}). \end{aligned}$$

The left hand side is $L(s, \tau^\theta \times (\tau \otimes \chi'\chi'^\theta))$, and it has a pole at $s = 1$ if and only if $\tau \otimes \chi'\chi'^\theta \simeq \tau$. Here $\chi'\chi'^\theta = \chi \circ N_{E/F}$, and if $\chi \circ N_{E/F} = 1$, χ is a character on $\mathbb{A}_F^*/F^* N_{E/F}(\mathbb{A}_E^*)$. By class field theory, $\chi = \delta_{E/F}$ since $\chi \neq 1$. □

Remark. By [Ra2, Lemma 3.6.2], there are at most finitely many idele class characters η of K such that $\tau \simeq \tau \otimes \eta$.

Now, let L/F be a quadratic extension of F distinct from E/F. Let $K = LE$ and assume that K/L is a quadratic extension so that we have the canonical isomorphism $Gal(K/L) \longrightarrow Gal(E/F)$ given by $\sigma \longmapsto \sigma|_E$. Consider the L-group homomorphism

$$\phi : {}^L\mathbf{H}_{E/F} \longrightarrow {}^L R_{L/F}\left(\mathbf{H}_{E/F}\right)$$

induced by the diagonal embedding $x \longmapsto (x, x)$, $x \in GL_{2n}(\mathbb{C})$.

Let π be a globally generic cuspidal representation of $\mathbf{H}_{E/F}(\mathbb{A}_F)$ and let $\tau = BC(\pi)$ be the stable base change lift of π to $GL_{2n}(\mathbb{A}_E)$ (cf. [Ki-Kr, Theorem 8.8]). According to [Ki-Kr, Theorem 8.10], we can write $\tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_k$, where τ_i is cuspidal and $L(s, \tau_i, As_{E/F} \otimes \delta_{E/F})$ has a simple pole at $s = 1$, $1 \leq i \leq k$.

Theorem 5.2. *Let E/F, L/F, K, π, τ, $\tau_i, \cdots$ be as above. Suppose the base change lift $(\tau_i)_K$ is cuspidal for all i. Let us also suppose that each τ_i satisfies the property that $\tau_i \not\simeq \tau_i \otimes (\delta_{L/F} \circ N_{E/F})$. Then there exists a globally generic cuspidal representation π_L of $\mathbf{H}_{K/L}(\mathbb{A}_L)$ which is a functorial lift of π corresponding to the L-group homomorphism ϕ.*

Proof. For $1 \leq i \leq k$, we claim that the L-function $L(s, (\tau_i)_K, As_{K/L} \otimes \delta_{K/L})$ has a pole at $s = 1$. Let us assume the claim for the moment, then by [So, Theorem 14], there exists a globally generic cuspidal representation π_L of $\mathbf{H}_{K/L}(\mathbb{A}_L)$ which lifts to τ_K. It can be verified that π_L is a functorial lift of π corresponding to the L-group homomorphism ϕ. Thus it remains to prove the claim. To this end, let us set $\tau = \tau_i$, and prove the following L-function identity:

$$L^S(s, (\tau)_K, As_{K/L}) = L^S(s, \tau, As_{E/F})L^S(s, \tau, As_{E/F} \otimes \delta_{L/F})),$$

where S is a finite set of places of F containing the archimedean places such that all the local data outside of S are unramified. Here, it should be pointed out that we are viewing the identity as an identity of L-functions defined over F. We prove the above equality locally. (For an automorphic L-function $L(s, \eta, r)$ defined over any number field k, we use the notation $L(s, \eta, r)_u$ to denote its local component at a place u of k.) Now let v be a place of F outside S. There are four cases to consider:

Case 1: v inert in E, and $w|v$ inert in K. In this case, v is inert in L. Let $l|v$ be the place of L lying over v, and $u|w$, the place of K lying over w. Let $\mathrm{diag}(\alpha_1, ..., \alpha_n, 1, ..., 1)$ be the semi-simple conjugacy class associated to π_v. Then τ_w is parametrized by

$$\mathrm{diag}(\alpha_1, ..., \alpha_n, \alpha_1^{-1}, ..., \alpha_n^{-1}),$$

and $(\tau_K)_u$ is parametrized by $\mathrm{diag}(\alpha_1^2, ..., \alpha_n^2, \alpha_1^{-2}, ..., \alpha_n^{-2})$. Hence by [Ki-Kr, page 11],

$$L(s, \tau_K, As_{K/L})_l^{-1} = \prod_{i=1}^n (1-\alpha_i^2 q_{L_l}^{-s})(1-\alpha_i^{-2} q_{L_l}^{-s}) \prod_{i<j} (1-\alpha_i^2\alpha_j^2 q_{K_u}^{-s})(1-\alpha_i^2\alpha_j^{-2} q_{K_u}^{-s})(1-\alpha_i^{-2}\alpha_j^{-2} q_{K_u}^{-s}).$$

Here $q_{L_l} = q_{F_v}^2, q_{K_u} = q_{E_w}^2$. Hence we have

$$L(s, \tau_K, As_{K/L})_l^{-1} = \prod_{i=1}^n (1\pm\alpha_i q_{F_v}^{-s})(1\pm\alpha_i^{-1} q_{F_v}^{-s}) \prod_{i<j} (1\pm\alpha_i\alpha_j q_{E_w}^{-s})(1\pm\alpha_i\alpha_j^{-1} q_{E_w}^{-s})(1\pm\alpha_i^{-1}\alpha_j^{-1} q_{E_w}^{-s})$$

and its right hand side is precisely

$$L(s, \tau, As_{E/F} \otimes \delta_{L/F})_v L(s, \tau, As_{E/F})_v.$$

Case 2: v inert in E, $w|v$ splits in K. Let u_1, u_2 be places of K lying over w. In this case, v splits in L. Let l_1, l_2 be places of L lying over v. If $diag(\alpha_1, ..., \alpha_n, 1, ..., 1)$ represents the semi-simple conjugacy class associated to π_v, then τ_w is parameterized by $\mathrm{diag}(\alpha_1, ..., \alpha_n, \alpha_1^{-1}, ..., \alpha_n^{-1})$. By the property of base change, $(\tau_K)_{u_1} = (\tau_K)_{u_2} = \tau_w$. Hence

$$L(s, \tau_K, As_{K/L})_{l_i}^{-1} = \prod_{i=1}^n (1-\alpha_i q_{L_l}^{-s})(1-\alpha_i^{-1} q_{L_l}^{-s}) \prod_{i<j} (1-\alpha_i\alpha_j q_{K_{u_i}}^{-s})(1-\alpha_i\alpha_j^{-1} q_{K_{u_i}}^{-s})(1-\alpha_i^{-1}\alpha_j^{-1} q_{K_{u_i}}^{-s}),$$

where $q_{L_{l_i}} = q_{F_v}, q_{K_{u_i}} = q_{E_w}$. Thus we see that $L(s, \tau_K, As_{K/L})_{l_i}$ is exactly $L(s, \tau, As_{E/F})_v$ for $i = 1, 2$.

Case 3: v splits as (w_1, w_2) in E, and w_i is inert in K for each $i = 1, 2$. Let u_1, u_2 be places of K lying over w_1, w_2, resp. Note that v is inert in L, and let l be the place of L lying over v. In this case, π_v is a spherical representation of $GL_{2n}(F_v)$, and let $\mathrm{diag}(\alpha_1, ..., \alpha_{2n})$ represent the semi-simple conjugacy class associated to π_v. Then $\tau_{w_1} = \pi_v, \tau_{w_2} = \tilde{\pi}_v$. So $(\tau_K)_{u_1}$ is parametrized by $\mathrm{diag}(\alpha_1^2, ..., \alpha_{2n}^2)$, and $(\tau_K)_{u_2}$ is parametrized by $\mathrm{diag}(\alpha_1^{-2}, ..., \alpha_{2n}^{-2})$. Hence

$$L(s, \tau_K, As_{K/L})_l = L(s, (\tau_K)_{u_1} \times (\tau_K)_{u_2}).$$

Here $L(s, \tau, \mathrm{As}_{E/F})_v = L(s, \pi_v \times \tilde{\pi}_v)$ and it can be verified that

$$L(s, \tau_K, \mathrm{As}_{K/L})_l = L(s, \tau, \mathrm{As}_{E/F})_v L(s, \tau, \mathrm{As}_{E/F} \otimes \delta_{L/F})_v.$$

Case 4: v splits as (w_1, w_2) in E, and w_i splits in K for each $i = 1, 2$. Say w_1 splits as (u_1, u_2), and w_2 splits as (u_1', u_2'). In this case, v splits in L, say as (l_1, l_2). Since $Gal(K/L) \simeq Gal(E/F)$ acts transitively on $\{w_1, w_2\}$, one from each set $\{u_1, u_2\}$, $\{u_1', u_2'\}$ lie over l_1. In this case, π_v is a spherical representation of $GL_{2n}(F_v)$, $\tau_{w_1} = \pi_v$, and $\tau_{w_2} = \tilde{\pi}_v$. By the property of base change, $(\tau_K)_{u_1} = (\tau_K)_{u_2} = \pi_v$, $(\tau_K)_{u_1'} = (\tau_K)_{u_2'} = \tilde{\pi}_v$. Hence

$$L(s, \tau_K, \mathrm{As}_{K/L})_{l_i} = L(s, \pi_v \times \tilde{\pi}_v),$$

and therefore

$$L(s, \tau_K, \mathrm{As}_{K/L})_{l_1} L(s, \tau_K, \mathrm{As}_{K/L})_{l_2} = L(s, \tau, \mathrm{As}_{E/F})_v^2.$$

Thus we are done with the proof of the global L-function identity stated above. Now consider

$$L(s, \tau_K \times \tau_K^{\theta'}) = L(s, \tau_K, \mathrm{As}_{K/L}) L(s, \tau_K, \mathrm{As}_{K/L} \otimes \delta_{K/L}),$$

where θ' is the non-trivial element in $Gal(K/L)$, then $\theta = \theta'|_E$. Moreover, $\tau_K^{\theta'} = (\tau^\theta)_K$, and

$$L(s, \tau_K \times (\tau^\theta)_K) = L(s, \tau \times \tau^\theta) L(s, \tau \times \tau^\theta \otimes \delta_{K/E}).$$

Then the left hand side of this equation has a simple pole at $s = 1$. Since $L(s, \tau_K, \mathrm{As}_{K/L})$ has no pole at $s = 1$, we conclude that $L(s, \tau_K, \mathrm{As}_{K/L} \otimes \delta_{K/L})$ should have a simple pole at $s = 1$.

□

Remark. In [So], it is not proved that the choice of π_L is unique, even though it is expected. For our purpose, the existence is enough.

In order to ease our notation we write $\mathbf{G}$ (resp. $\mathbf{H}$) for $GU(n, n)_{E/F}$ (resp. $U(n, n)_{E/F}$). In the remainder of this section, we examine the relationship between the representations of $\mathbf{G}(\mathbb{A}_F)$ and those of $\mathbf{H}(\mathbb{A}_F)$. First, we record the following theorem [H-L, Proposition 1.8.1]:

Theorem 5.3. *Suppose π is a cuspidal representation of $\mathbf{G}(\mathbb{A}_F)$. Then $\pi|_{\mathbf{H}(\mathbb{A}_F)}$ is a direct sum of cuspidal representations of $\mathbf{H}(\mathbb{A}_F)$. Any cuspidal representation π of $\mathbf{H}(\mathbb{A}_F)$ occurs in the restriction $\pi^{\mathrm{G}}|_{\mathbf{H}(\mathbb{A}_F)}$ of some cuspidal representation π^{G} of $\mathbf{G}(\mathbb{A}_F)$. Moreover, if π is a constituent of both $\pi_1^{\mathrm{G}}|_{\mathbf{H}(\mathbb{A}_F)}$ and $\pi_2^{\mathrm{G}}|_{\mathbf{H}(\mathbb{A}_F)}$ if and only if $\pi_2^{\mathrm{G}} = \pi_1^{\mathrm{G}} \otimes \chi$, for some character χ of $\mathbb{A}_F^* \simeq \mathbf{G}(\mathbb{A}_F)/\mathbf{H}(\mathbb{A}_F)$.*

Second, we discuss the existence of a stable base change lift from unitary similitude groups to general linear groups. Let us write $\tilde{\mathbf{G}}$ (resp. $\tilde{\mathbf{H}}$) to denote the group $R_{E/F}(GL_{2n} \times GL_1)$ (resp. $R_{E/F}(GL_{2n})$). The stable base change map BC for unitary groups is defined in [Ki-Kr, §3]; we extend this definition to similitude unitary groups. Namely, the stable base change map for $\mathbf{G}$, which is also denoted as BC, is given by

$$BC : {}^L\mathbf{G} \longrightarrow {}^L\tilde{\mathbf{G}}; (g, \lambda, 1) \mapsto ((g, \lambda), \theta(g, \lambda), \theta); (g, \lambda, 1) \mapsto ((g, \lambda), \theta(g, \lambda), \theta).$$

Lemma 5.4. *Suppose π is a globally generic cuspidal representation of $\mathbf{G}(\mathbb{A}_F)$ and ω_π its central character. Let π' be a cuspidal constituent of $\pi|_{\mathbf{H}(\mathbb{A}_F)}$ (cf. Theorem 5.3) which is globally generic with respect to ψ. Then $BC(\pi) = BC(\pi') \otimes \bar{\omega}_\pi$ of π is a stable base change lift of π. Here $BC(\pi')$ is the stable base change lift of π' as established in [Ki-Kr].*

Proof. Note that $BC(\pi)$, if it exists, should be of the form $\pi_1 \otimes \chi$, where π_1 is an automorphic representation of $GL_{2n}(\mathbb{A}_E)$, and χ is an idele class character of E. Consider the diagram

$$\begin{array}{ccc} {}^L\mathbf{G} & \longrightarrow & {}^L\mathbf{H} \\ {\scriptstyle BC}\downarrow & & {\scriptstyle BC}\downarrow \\ {}^L\tilde{\mathbf{G}} & \longrightarrow & {}^L\tilde{\mathbf{H}} \end{array}$$

with the horizontal arrows being the natural projections (over the Galois group) onto the corresponding GL_{2n} factor. The commutativity of the diagram implies that we may take $\pi_1 = BC(\pi')$. Now, the central character of $BC(\pi)$ is $\omega_\pi \circ N$, where $N : Z(\tilde{\mathbf{G}}(\mathbb{A}_F)) \longrightarrow Z(\mathbf{G}(\mathbb{A}_F))$ is the norm map defined in [R2, Section 3.10]. Note that $Z(\tilde{\mathbf{G}}(\mathbb{A}_F))$ can be identified with $\mathbb{A}_E^\times \times \mathbb{A}_E^\times$. Then $N(z, \lambda)$, viewed as an element in $Z(\mathbf{G}(\mathbb{A}_F))$, is given by $(z, \lambda) \cdot \theta(z, \lambda)$, where

$$\theta(z, \lambda) = (\bar{\lambda} {J'_{2n}}^{-1} (\bar{z}^{-1} I_{2n}) J'_{2l}, \bar{\lambda}).$$

Consequently $N(z, \lambda) = (\bar{\lambda}(z/\bar{z}), \lambda\bar{\lambda})$. Therefore $\omega_\pi(N(z, \lambda)) = \omega_\pi(\bar{\lambda}(z/\bar{z}))$ and it follows that $\chi(\lambda) = \omega_\pi(\bar{\lambda})$. □

We record here the analogue of [So, Theorem 12] for unitary similitude groups.

Proposition 5.5. *Let $\Pi \otimes \chi$ be a cuspidal representation of $GL_{2n}(\mathbb{A}_E) \times GL_1(\mathbb{A}_E)$ such that $L(s, \Pi, As_{E/F} \otimes \delta_{E/F})$ has a pole at $s = 1$. Then there exists a generic cuspidal representation π of $\mathbf{G}(\mathbb{A}_F)$ such that $BC(\pi) = \Pi \otimes \chi$.*

Proof. By [So, Theorem 12], Π descends to a generic cuspidal representation π_0 of $\mathbf{H}(\mathbb{A}_F)$. We let π be a generic cuspidal representation of $\mathbf{G}(\mathbb{A}_F)$ whose restriction to $\mathbf{H}(\mathbb{A}_F)$ contains π_0 and whose central character is $\bar{\chi}$. [By Theorem 5.3, any two lifts differ by a character η of $\mathbb{A}_F^* \simeq \mathbb{A}_E^*/U_1(\mathbb{A}_F)$. Clearly, η is an idele class character of F. Hence by choosing an appropriate η, we can make the central character of π to be $\bar{\chi}$.] Then $BC(\pi) = \Pi \otimes \chi$. □

Now, we need to extend Theorem 5.2 to generic cuspidal representations of $\mathbf{G}(\mathbb{A}_F)$. Namely, let L/F be a quadratic extension distinct from E as in Theorem 5.1. Let us set $K = LE$ and assume that K/L is a quadratic extension. Consider the L-group homomorphism

$$\phi : {}^L\mathbf{G} \longrightarrow {}^L R_{L/F}\,(\mathbf{G})$$

induced by the diagonal embedding $x \longmapsto (x, x)$, $x \in GL_{2n}(\mathbb{C}) \times GL_1(\mathbb{C})$. Suppose π is a globally generic cuspidal representation of $\mathbf{G}(\mathbb{A}_F)$ with central character ω_π. Let π' be a generic cuspidal representation of $\mathbf{H}(\mathbb{A}_F)$ which is an irreducible constituent of $\pi|_{\mathbf{H}(\mathbb{A}_F)}$. The stable base change lift $BC(\pi')$ of π' is of the form

$$BC(\pi') = \tau_1' \boxplus \tau_2' \boxplus \cdots \boxplus \tau_k'.$$

Let us suppose that each $(\tau_i')_K$ is cuspidal. Then, by applying Theorem 5.2 to π', we obtain a generic cuspidal representation π_L' of $\mathbf{H}_{K/L}(\mathbb{A}_L)$. Now let π_L be a generic cuspidal representation of $\mathbf{G}_{K/L}(\mathbb{A}_L)$ such that π_L' is an irreducible constituent of $\pi_L|_{\mathbf{H}_{K/L}(\mathbb{A}_L)}$ and the restriction of π_L to the central split torus of $\mathbf{G}_{K/L}$ is an extension of ω_π. It can be verified that π_L is a functorial lift of π corresponding to the L-group homomorphism ϕ. We note that π_L is globally generic since all the unipotent elements of $\mathbf{G}_{K/L}$ are contained in $\mathbf{H}_{K/L}$. Thus we have established

Proposition 5.6. *Let E/F, L/F, K, π, π', τ_i', $\cdots$ be as above. Suppose that $(\tau_i')_K$ is cuspidal for all i. Then there exists a functorial lift π_L, which is a generic cuspidal representation of $\mathbf{G}_{K/L}(\mathbb{A}_L)$, corresponding to the L-group homomorphism ϕ.*

6. Twisted exterior square lift; the general case.

We continue with notations as in Section 4. Let π be a globally generic cuspidal representation of $GU(2,2)(\mathbb{A}_F)$ such that $T = \{v < \infty |\ v \text{ does not split in } E \text{ and } \pi_v \text{ is supercuspidal}\}$ is not empty. We proceed as in [Ki1, Kr]. Enumerate the finite places outside T as $\{v_1, v_2, \dots\}$. For each $j \geq 1$, by the Grunwald-Wang theorem [Pa, Theorem 2.23], we can find a quadratic extension F_j/F satisfying the following properties:

(1) v_j splits completely in F_j;
(2) for each $v \in T$, there is a unique place $v(j)$ of F_j such that $(F_j)_{v(j)} = E_w$, where $w|v$ is the unique place over v;
(3) the archimedean places of F split completely in F_j.

Let $E_j = EF_j$. By throwing away finitely many indices we may assume that E_j's are all distinct and that E_j/F_j is a quadratic extension for each j. Also, by throwing away finitely many indices if necessary, we may assume that the condition of Proposition 5.6 is satisfied. Namely, in the notation of Proposition 5.6, if $BC(\pi') = \tau_1' \boxplus \cdots \boxplus \tau_k'$, then except for finitely many indices j, $(\tau_i')_{E_j}$ is cuspidal and $\tau_i \not\simeq \tau_i \otimes (\delta_{F_j/F} \circ N_{E/F})$ for all $i = 1, ..., k$. (See the remark following Lemma 5.1.)

For each j, let π_j be the globally generic cuspidal representation of $GU(2,2)_{E_j/F_j}(\mathbb{A}_{F_j})$ associated to π as in Proposition 5.6. Then by construction, π_j is good with respect to E_j/F_j. Hence, by Theorem 4.4, there exists an automorphic representation Π_j of $GL_6(\mathbb{A}_{F_j})$ which is the strong twisted exterior square lift of π_j. We note that Π_j is isobaric for every j. Hence, by applying Ramakrishnan's descent criterion [Ra1], we obtain a unique automorphic descent Π (which is also isobaric) on GL_6/F such that $\Pi_{F_j} \simeq \Pi_j$ for almost all j. It is easy to see that Π is independent of the choice of the π_j's, and that it is a weak twisted exterior square lift of π. The point is that for almost all j, $\Pi_j \simeq (\Pi)_{F_j}$ and Π_j is the twisted exterior square lift of π_j; hence if v is a place of F that splits completely in F_j, then it follows t hat Π_v is the twisted exterior square lift of π_v. Further, the strong multiplicity one theorem for isobaric representations of general linear groups guarantees that Π is independent of the choice of π_j's. We note that our construction of Π is such that Π_v is the local lift of π_v for all $v|\infty$.

Next, we need to prove the existence of a strong twisted exterior square lift of π. In order to do so, we first need to construct a local lift of π_v when $v \in T$. Since the local Langlands correspondence is not available for $v \in T$, we need to construct an irreducible representation Π_v which satisfies the equalities in (4.1). By proceeding exactly as in the proof of [Ki1, Proposition 5.2.3] we obtain

Proposition 6.1. *Let K/k be a quadratic extension of p-adic fields. Let ρ be a generic supercuspidal representation of $GU(2,2)_{K/k}(k)$. Then a local lift τ of ρ exists. It is of the form $\tau = Ind(\rho_1 \otimes \cdots \otimes \rho_k)$, where ρ_i's are supercuspidal representations of $GL_{n_i}(k)$.*

Proof. We only give the outline of proof. By Proposition 5.1 of [Sh1], we can find a generic cuspidal representation π of $GU(2,2)(\mathbb{A}_F)$ which satisfies the following properties: (1) E/F is a quadratic extension of number fields with $F_v = k, E_w = K$, $w|v$; (2) $\pi = \otimes_u \pi_u$, π_u is unramified for all $u < \infty$ and $u \neq v$; (3) $\pi_v = \rho$.

Then, by the above result, there exists a weak exterior square lift Π of π such that $\Pi_u \simeq \wedge_t^2 \pi_u$ for $u \notin S$, where S is a finite set of finite places, containing v. Moreover, it follows from [Ta2] that all the local components of Π are irreducible, unitary, and generic. In particular, so is Π_v. Then by comparing the γ and L-factors as in [Ki1, Proposition 5.2.3], we can show that Π_v is a local twisted exterior square lift of π_v, and that Π_v is tempered follows as argued in [Ki1, Proposition 5.1.3]. Thus Π_v is of the form

$$\Pi_v = \mathrm{Ind}(\delta_{1,v} \otimes \cdots \otimes \delta_{d,v}),$$

with each $\delta_{i,v}$ is a discrete series representation. Finally, in order to show that each $\delta_{i,v}$ is in fact supercuspidal, we may argue as in [CKPSS, Theorem 7.3]. □

Theorem 6.2. *Let π be a globally generic cuspidal representation of $GU(2,2)(\mathbb{A}_F)$. Then a strong twisted exterior square lift Π of π exists and $\Pi_v \simeq \wedge_t^2(\pi_v)$ for $v \notin T$. It is of the form $\sigma_1 \boxplus \cdots \boxplus \sigma_k$, where σ_i's are unitary cuspidal representations of $GL_{n_i}(\mathbb{A}_F)$. Moreover, if Π_E denotes the base change of Π to $GL_4(\mathbb{A}_E)$, then $\Pi_E \simeq \wedge^2(BC(\pi))$.*

Proof. For each $v \in T$, let Π_v be the local lift of π_v as constructed in Proposition 6.1. We then have an irreducible admissible representation $\Pi' = \otimes \Pi'_v$ of $GL_6(\mathbb{A}_F)$, where $\Pi'_v = \wedge_t^2(\pi_v)$ for $v \notin T$ and $\Pi'_v = \Pi_v$ for $v \in T$. Then by applying the converse theorem twice to Π' as in [Ki1, Theorem 5.3.1], we obtain the result. We note that Π is uniquely determined by the strong multiplicity one theorem.

For the last equivalence, it is enough to prove the corresponding local identity. Suppose v is inert, and the semi-simple conjugacy class of π_v is represented by $(\mathrm{diag}(a,b,1,1),\lambda,\theta) \in (GL_4(\mathbb{C}) \times GL_1(\mathbb{C})) \rtimes \mathrm{Gal}(E/F)$. Then the semi-simple conjugacy class of Π_v is represented by $\lambda \mathrm{diag}(ab,a,b,1,\alpha,\beta) \in GL_6(\mathbb{C})$, where α,β are roots of $X^2 - ab$. Then the semi-simple conjugacy class of $(\Pi_E)_v$ is represented by $\lambda^2 \mathrm{diag}((ab)^2,a^2,b^2,1,\alpha^2,\beta^2) \in GL_6(\mathbb{C})$. On the other hand, the semi-simple conjugacy class of $BC(\pi_v)$ is represented by $(diag(a,b,b^{-1},a^{-1}),\lambda^2 ab) \in GL_4(\mathbb{C}) \times GL_1(\mathbb{C})$. Hence the semi-simple conjugacy class of $\wedge^2(BC(\pi_v))$ is represented by $(\lambda^2 ab)\mathrm{diag}(ab,ab^{-1},1,1,ba^{-1},b^{-1}a^{-1}) \in GL_6(\mathbb{C})$. Hence we have the equality by noting that $\alpha^2 = \beta^2 = ab$.

Suppose v splits as (w_1,w_2) in E, and let $\pi_v = \tau_v \otimes \chi_v$. Then $\Pi_v = \wedge^2(\tau_v) \otimes \chi_v$. Hence $(\Pi_E)_{w_1} = (\Pi_E)_{w_2} = \wedge^2(\tau_v) \otimes \chi_v$. On the other hand, $BC(\pi)_{w_1} = \tau_v \otimes \chi_v$, and $BC(\pi)_{w_2} = \tilde{\tau}_v \otimes \chi_v \omega_{\tau_v}$. Here $\wedge^2(BC(\pi)_{w_2}) = \wedge^2(\tilde{\tau}_v) \otimes \chi_v \omega_{\tau_v} \simeq \wedge^2(\tau_v) \otimes \chi_v$. □

7. The case of $PGU(2,2)$.

We continue with notations as in Section 2. For a generic cuspidal representation of $PGU(2,2)(\mathbb{A}_F)$, namely, a generic cuspidal representation of $GU(2,2)(\mathbb{A}_F)$ with trivial central character, we can obtain the twisted exterior square lift using the group SO_6^-. In this section we give a brief outline of the proof, skipping many details.

We have the maps $SU(2,2) \longrightarrow PGU(2,2)$, and $SO_6^- \longrightarrow PGSO_6^- \simeq PGU(2,2)$. The connected component of the dual group of $PGU(2,2)$ is $SL_4(\mathbb{C})$. Hence the map on the connected components of the dual groups is $SL_4(\mathbb{C}) \longrightarrow SO_6(\mathbb{C}) \hookrightarrow GL_6(\mathbb{C})$. If $\mathbf{G} = SO_{2n}^-$, then ${}^L\mathbf{G} = SO_{2n}(\mathbb{C}) \rtimes \mathrm{Gal}(E/F)$. Here the Galois action is given by $g \longmapsto BgB$, where B is any element in $O_{2n}(\mathbb{C}) - SO_{2n}(\mathbb{C})$ such that ${}^tB = B$. For example, take $B = \mathrm{diag}(1,...,1,\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},1,...,1)$. Therefore, ${}^L\mathbf{G} = O_{2n}(\mathbb{C})$. Hence, we have the L-group homomorphism

$$ {}^LPGU(2,2) = SL_4(\mathbb{C}) \rtimes \mathrm{Gal}(E/F) \longrightarrow {}^LSO_6^- = O_6(\mathbb{C}) \hookrightarrow GL_6(\mathbb{C}). $$

Given a cuspidal representation π of $PGU(2,2)(\mathbb{A}_F)$, we can think of π as a cuspidal representation of $SO_6^-(\mathbb{A}_F)$, and we expect a lift to GL_6/F according to the above L-group homomorphism.

Now consider the group $\mathbf{G} = SO_{2n}^-$. Let $\mathbf{P} = \mathbf{MN}$, $\mathbf{M} = GL_m \times SO_6^-$, where $m = n-3$. Let σ be a cuspidal representation of $GL_m(\mathbb{A}_F)$, and π be a generic cuspidal representation of $SO_6^-(\mathbb{A}_F)$. Then by applying the Langlands-Shahidi method to $\sigma \otimes \pi$, we obtain the L-functions in the constant term of the Eisenstein series: $L(s, \sigma \times \pi) L(2s, \sigma, \wedge^2)$.

We write the local factors explicitly when $\sigma_v \otimes \pi_v$ is spherical (cf. [G-PS-R, page 82]); Suppose the semi-simple conjugacy class of σ_v is given by $diag(\alpha_1, ..., \alpha_m) \in GL_m(\mathbb{C})$.

If v splits in E. Then $SO_6^-(F_v) = SO_6(F_v)$, and the semi-simple conjugacy class of π_v is given by $\mathrm{diag}(\beta_1, \beta_2, \beta_3, \beta_3^{-1}, \beta_2^{-1}, \beta_1^{-1}) \in GL_6(\mathbb{C})$. Consequently

$$L(s, \sigma_v \times \pi_v)^{-1} = \prod_{i=1}^m \prod_{j=1}^3 (1 - \alpha_i \beta_j q_v^{-s})(1 - \alpha_i \beta_j^{-1} q_v^{-s}).$$

If v is inert in E. Since a spherical representation of SO_2^- is trivial, the semi-simple conjugacy class of the spherical representation of SO_2^- is simply given by $(1, \theta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ under the identification $SO_2(\mathbb{C}) \rtimes \mathrm{Gal}(E/F) = O_2(\mathbb{C})$. Hence the semi-simple conjugacy class of π_v is given by $diag(\beta_1, \beta_2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \beta_2^{-1}, \beta_1^{-1}) \in O_6(\mathbb{C})$. Consequently

$$L(s, \sigma_v \times \pi_v)^{-1} = \prod_{i=1}^m \prod_{j=1}^2 (1 - \alpha_i \beta_j q_v^{-s})(1 - \alpha_i \beta_j^{-1} q_v^{-s}) \prod_{i=1}^m (1 - \alpha_i^2 q_v^{-2s}).$$

Now applying Langlands-Shahidi method, we can prove that $L(s, (\sigma \otimes \chi) \times \pi)$ is entire, satisfies a functional equation, and is bounded in vertical strips if χ is an idele class character which is highly ramified at one finite place. Moreover, in this case, we have the stability of γ-factors [CPSS, Corollary 6.2] and L-functions [Sh3]:

Theorem 7.1 (stability of local factors). *Let F be a p-adic field. Let π_1, π_2 be two generic irreducible representations of $SO_6^-(F)$. Then for every highly ramified character μ,*

$$\gamma(s, \mu \times \pi_1, \psi) = \gamma(s, \mu \times \pi_2, \psi), \quad L(s, \mu \times \pi_1) = L(s, \mu \times \pi_2) = 1.$$

Let S be a finite set of finite places such that if $v \notin S$, $v < \infty$, π_v is spherical. Then for $v \notin S$, we obtain the local lift Π_v by the local Langlands correspondence. Namely, suppose v splits in E. Then π_v is parametrized by $\phi_v : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow O_6(\mathbb{C})$, and the local lift Π_v is the one attached, by the local Langlands correspondence, to the composition $\iota \circ \phi_v : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow O_6(\mathbb{C}) \hookrightarrow GL_6(\mathbb{C})$. More specifically, suppose π_v is a quotient of a principal series given by $\mu_1 \otimes \mu_2 \otimes \mu_3$, where μ_i's are quasi-characters of F_v^*. Then $\Pi_v = \pi(\mu_1, \mu_2, \mu_3, \mu_3^{-1}, \mu_2^{-1}, \mu_1^{-1})$. Suppose v remains inert in E. Then a spherical representation of $SO_6^-(F_v)$ is given by a character $\mu_1 \otimes \mu_2 \otimes 1$. Hence the local lift Π_v is given by $\Pi_v = \pi(\mu_1, \mu_2, 1, -1, \mu_2^{-1}, \mu_1^{-1})$.

With Theorem 7.1 in hand, by applying the converse theorem [Co-PS], and using the techniques in [CKPSS] (see Sections 5, 6, and 7 in [CKPSS]), we obtain

Theorem 7.2. *Let π be a generic cuspidal representation of $SO_6^-(\mathbb{A}_F)$. Then it has a strong lift to an automorphic representation Π of $GL_6(\mathbb{A}_F)$.*

Corollary 7.3. *Let π be a generic cuspidal representation of $PGU(2,2)(\mathbb{A}_F)$. Then it has a strong lift to an automorphic representation Π of $GL_6(\mathbb{A}_F)$.*

Remark. This is similar to the $PGSp_4$ case. Given a cuspidal representation π of $PGSp_4(\mathbb{A}_F)$, in order to obtain a lift to $GL_4(\mathbb{A}_F)$ using the converse theorem, we need to identify $PGSp_4$ with SO_5, and consider the L-function $L(s, \sigma \times \pi)$ for a cuspidal representation σ of $GL_m(\mathbb{A}_F)$, $m = 1, 2$ for the case $GL_m \times SO_5 \subset SO_{2n+1}$. (See [Ki2].)

8. Twisted mth symmetric powers of $GU(1,1)$.

We fix E/F a quadratic extension of number fields as usual. We write $GU(1,1)$ to denote the group $GU(1,1)_{E/F}$. Let us define a map

$$\phi : GL_2(\mathbb{C}) \times GL_1(\mathbb{C}) \rtimes \mathrm{Gal}(E/F) \longrightarrow GL_{m+1}(\mathbb{C}) \times \mathrm{Gal}(E/F)$$

by $\phi(g, \lambda, \theta) = (Sym^m(g)\lambda^m, \theta)$. Then ϕ is an L-group homomorphism. This follows from the fact that

$$Sym^m(\theta(g))(\det(g))^m = Sym^m(g),\ \theta(g) = J_2'^{-1}\,{}^t g^{-1} J_2'.$$

(In order to see this, recall that for $g \in GL_2(\mathbb{C})$, $Sym^m(g) \in GSp_{m+1}(\mathbb{C})$ if m is odd, and $Sym^m(g) \in GO_{m+1}(\mathbb{C})$ if m is even [G-W, page 244]. For $m = 1$, we use the identification $GSp_2(\mathbb{C}) \simeq GL_2(\mathbb{C})$.) We call ϕ the twisted symmetric mth power map and denote it by Sym_t^m. We have the following commuting diagram

$$\begin{array}{ccc} {}^L\mathrm{GU(1,1)} & \xrightarrow{Sym_t^m} & {}^L\mathrm{GL}_{m+1} \\ {\scriptstyle BC}\downarrow & & {\scriptstyle B}\downarrow \\ {}^L\mathrm{R}_{E/F}(\mathrm{GL}_2 \times \mathrm{GL}_1) & \xrightarrow{Sym^m} & {}^L\mathrm{R}_{E/F}(\mathrm{GL}_{m+1}), \end{array}$$

where $Sym^m(g_1, \lambda_1, g_2, \lambda_2, \theta) = (Sym^m(g_1)\lambda_1^m, Sym^m(g_2)\lambda_2^m, \theta)$. Note that if $m = 1$, we have the L-group homomorphism $Sym_t^1 : {}^LG \longrightarrow {}^LGL_2$. It can be verified that $Sym_t^m = Sym^m \circ Sym_t^1$.

We need some facts about the structure of $GU(1,1)$ (cf. [Ko-Ko]). Recall that

$$GU(1,1)(\mathbb{A}_F) = \{g \in GL_2(\mathbb{A}_E) : \lambda_g\theta(\bar{g}) = g, \lambda_g \in \mathbb{A}_F^\times\}.$$

Hence any $g \in GU(1,1)(\mathbb{A}_F)$ satisfies the condition $\bar{g}g^{-1} = det(\bar{g})\lambda_g^{-1}$ (this uses the fact that $\theta(g) = g\,det(g)^{-1}$) and hence by Hilbert's 90 there exists $z_g \in \mathbb{A}_E^\times$ such that $det(\bar{g})\lambda_g^{-1} = \bar{z}_g z_g^{-1}$. This means that the element $z_g^{-1}g$ belongs to $GL_2(\mathbb{A}_F)$ or in other words there exists a $g' \in GL_2(\mathbb{A}_F)$ such that $\theta(g') = z_g^{-1}g$. Then the mapping $(z, g') \mapsto z\theta(g')$ yields the isomorphism $\mathbb{A}_E^\times \times GL_2(\mathbb{A}_F)/\mathbb{A}_F^\times \simeq GU(1,1)(\mathbb{A}_F)$, where $\mathbb{A}_F^\times$ embeds diagonally. In fact this argument, as discussed in [Ko-Ko, Section 4.3], gives the isomorphism (as algebraic groups) $GU(1,1) \simeq (GL_2 \times R_{E/F}GL_1)/GL_1$, where GL_1 is embedded diagonally. Hence any cuspidal representation π of $GU(1,1)(\mathbb{A}_F)$ may be written as $\pi = \sigma \otimes \chi$, where σ is a cuspidal representation of $GL_2(\mathbb{A}_F)$ and χ is an idele class character of E such that $\omega_\sigma \cdot (\chi|_{\mathbb{A}_F^*}) = 1$.

Let $\pi = \otimes \pi_v$ be a cuspidal representation of $GU(1,1)(\mathbb{A}_F)$ given by (σ, χ) under the above identification. Then for each place v of F, the local Langlands correspondence is available for $GU(1,1)(F_v)$. Let $\phi_v : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow {}^L(GU(1,1)/F_v)$ be the parametrization of π_v for each v. Let $Sym_t^m(\pi_v)$ be the irreducible admissible representation of $GL_{m+1}(F_v)$ attached to $Sym_t^m(\phi_v)$ by the local Langlands correspondence for GL_{m+1}. Let us set $Sym_t^m(\pi) = \otimes_v Sym_t^m(\pi_v)$; it is an irreducible admissible representation of $GL_{m+1}(\mathbb{A}_F)$. Langlands functoriality predicts that $Sym_t^m(\pi)$ is automorphic. For $1 \le m \le 4$, the functoriality of $Sym_t^m(\pi)$ follows from the deep results of [Ge-J, Ki1, Ki-Sh] on the functoriality of the usual symmetric powers. The point is that we have $Sym_t^m(\phi_v) = Sym^m(Sym_t^1(\phi_v))$ for each v. Hence $Sym_t^m(\pi) = Sym^m(Sym_t^1(\pi))$. Therefore if we show that $Sym_t^1(\pi)$ is automorphic, then the automorphy of $Sym_t^m(\pi)$, $1 \le m \le 4$, will follow from [loc. cit.]

Lemma 8.1. *$Sym_t^1(\pi)$ is an automorphic representation of $GL_2(\mathbb{A}_F)$. More explicitly, write $\pi = \sigma \otimes \chi$ as above. Then $Sym_t^1(\pi) = \sigma \otimes (\chi|_{\mathbb{A}_F^*})$.*

Proof. Let us describe the relationship between the parametrizations of (σ, χ) and π. Namely, let $\phi_1 : {}^LGU(1,1) \longrightarrow {}^LGL_2$ denote the L-group homomorphism given by $(g, \lambda, \gamma) \mapsto (\lambda^{-1}\theta(g), \gamma)$ and $\phi_2 : {}^LGU(1,1) \longrightarrow {}^L(R_{E/F}GL_1)$ be the L-group homomorphism given by $(g, \lambda, \gamma) \mapsto (\lambda, \lambda det(g), \gamma)$. Then from our description of the group isomorphism $GU(1,1)(\mathbb{A}_F) \simeq \mathbb{A}_E^\times \times GL_2(\mathbb{A}_F)/\mathbb{A}_F^\times$, it follows that if $\pi_v \leftrightarrow \phi_v$, the Langlands parametrizations for σ and χ are given by $\phi_1 \circ \phi_v$ and $\phi_2 \circ \phi_v$, respectively; let us write the above correspondence explicitly at an unramified place v of F where all the local data are also unramified. Suppose v doesn't split in E; if the semisimple conjugacy class of π_v is represented by $(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \lambda, \theta) \in {}^L(GU(1,1)/F_v)$, then $\begin{pmatrix} \lambda^{-1}\alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{C})$ represents the semisimple conjugacy class associated to σ and $\lambda^2\alpha \in GL_1(\mathbb{C})$ represents the semisimple conjugacy class associated to χ_v. Suppose v splits in E; if $(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \lambda) \in GL_2(\mathbb{C}) \times GL_1(\mathbb{C})$ represents the semi-simple conjugacy class of π_v, then the semi-simple conjugacy class associated with σ_v is represented by $\begin{pmatrix} \lambda^{-1}\alpha^{-1} & 0 \\ 0 & \lambda^{-1}\beta^{-1} \end{pmatrix} \in GL_2(\mathbb{C})$, that of χ at w_1 is represented by λ, and the semi-simple conjugacy class of χ at w_2 is represented by $\lambda\alpha\beta$.

Now it follows from the definition of $Sym_t^1(\pi)$ that $Sym_t^1(\pi) = \sigma \otimes (\chi|_{\mathbb{A}_F})$. □

Keeping π, σ and χ as in Lemma 8.1, since $Sym^m(\sigma)$ is automorphic for $m = 1, 2, 3, 4$ (see [Ge-J] for $m = 2$, [Ki-Sh] for $m = 3$, and [Ki1] for $m = 4$), it follows that $Sym_t^m(\pi)$ is automorphic for $m = 1, 2, 3, 4$. Moreover, if π' is an irreducible component of $\pi|_{U(1,1)}$, then by [Ko-Ko, Corollary 4.14] the stable base change $BC(\pi') = \sigma_E \otimes \chi$. Since the central character of π is χ, by Lemma 5.4, $BC(\pi) = \sigma_E \otimes \chi\bar{\chi}$. Hence combining this observation with Lemma 8.1 we have proved the following:

Proposition 8.2. *Let $\pi = \sigma \otimes \chi$ be a cuspidal representation of $GU(1,1)(\mathbb{A}_F)$ as above. Then $Sym_t^m(\pi)$ is an automorphic representation of $GL_m(\mathbb{A}_F)$ for $m = 1, 2, 3, 4$. More explicitly, $Sym_t^1(\pi) = \sigma \otimes (\chi|_{\mathbb{A}_F^*})$, and $Sym_t^m(\pi) = Sym^m(Sym_t^1(\pi)) = Sym^m(\sigma) \otimes (\chi|_{\mathbb{A}_F^*})^m$. Moreover, $Sym^m(BC(\pi)) = B(Sym_t^m(\pi)) = Sym^m(\sigma_E) \otimes (\chi\bar{\chi})^m$.*

Suppose that we have the functoriality of symmetric mth for all m. Then $Sym_t^m(\pi)$ and $Sym^m(BC(\pi))$ are automorphic representations of $GL_{m+1}(\mathbb{A}_F), GL_{m+1}(\mathbb{A}_E)$, resp. and $B(Sym_t^m(\pi)) = Sym^m(BC(\pi))$.

9. Twisted exterior square of $GU(2,2)$ and twisted symmetric fourth of $GU(1,1)$.

We start with the following L-function identity. It follows from the property of the base change: Let Π be a cuspidal representation of $GL_n(\mathbb{A}_F)$ and Π_E be its base change to $GL_n(\mathbb{A}_E)$, where E/F is as usual a quadratic extension. Then for χ, an idele class character of E,

$$L(s, \Pi_E \otimes \chi, \mathrm{As}) = L(s, \Pi, Sym^2 \otimes (\chi|_{\mathbb{A}_F^*}))L(s, \Pi, \wedge^2 \otimes \delta_{E/F}(\chi|_{\mathbb{A}_F^*})).$$

Let π be a cuspidal representation of $GU(1,1)(\mathbb{A}_F)$. Then by the result in Section 8, we can write $\pi = \sigma \otimes \chi$, where σ is a cuspidal representation of $GL_2(\mathbb{A}_F)$ and χ is an idele class character of E such that $\omega_\sigma \cdot (\chi|_{\mathbb{A}_F^*}) = 1$.

Consider $A^3(\sigma_E) \otimes \chi$. Here for a representation Π of GL_n, we set $A^m(\Pi) = Sym^m(\Pi) \otimes \omega_\Pi^{-1}$. We assume that $A^3(\sigma_E) \otimes \chi$ is a cuspidal representation of $GL_4(\mathbb{A}_E)$. By the property of the base change, we can see that $A^3(\sigma_E) = A^3(\sigma)_E$. Then from the above identity,

$$L(s, A^3(\sigma_E) \otimes \chi, \mathrm{As} \otimes \delta_{E/F}) = L(s, A^3(\sigma), Sym^2 \otimes \delta_{E/F}(\chi|_{\mathbb{A}_F^*}))L(s, A^3(\sigma), \wedge^2 \otimes (\chi|_{\mathbb{A}_F^*})).$$

By [Ki1, Theorem 7.3.2],

$$L(s, A^3(\sigma), \wedge^2 \otimes (\chi|_{\mathbb{A}_F^*})) = L(s, A^4(\sigma) \otimes (\chi|_{\mathbb{A}_F^*}))L(s, \omega_\sigma(\chi|_{\mathbb{A}_F^*})).$$

Since $\omega_\sigma(\chi|_{\mathbb{A}_F^*}) = 1$ and $L(s, A^4(\sigma) \otimes (\chi|_{\mathbb{A}_F^*}))$ has no zero at $s = 1$, $L(s, A^3(\sigma), \wedge^2 \otimes (\chi|_{\mathbb{A}_F^*}))$ has a pole at $s = 1$. Since $L(s, A^3(\sigma), Sym^2 \otimes \delta_{E/F}(\chi|_{\mathbb{A}_F^*}))$ has no zero at $s = 1$, $L(s, A^3(\sigma_E) \otimes \chi, \mathrm{As} \otimes \delta_{E/F})$ has a pole at $s = 1$. Hence by [So, Theorem 14], we have the following.

Theorem 9.1. *Let $\pi = \sigma \otimes \chi$ be a cuspidal representation of $GU(1,1)(\mathbb{A}_F)$, where σ is a cuspidal representation of $GL_2(\mathbb{A}_F)$ and χ is an idele class character of E such that $\omega_\sigma \cdot (\chi|_{\mathbb{A}_F^*}) = 1$. Assume that $A^3(\sigma_E) \otimes \chi$ is a cuspidal representation of $GL_4(\mathbb{A}_E)$. Then $A^3(\sigma_E) \otimes \chi$ descends to a generic cuspidal representation τ of $U(2,2)(\mathbb{A}_F)$.*

Here if v splits as $(w_1 w_2)$ in E, then τ_v is of the form $A^3(\sigma_v) \otimes \eta_v$, where $\eta_v = \chi_{w_1}$ or χ_{w_2}. For our purposes, it does not matter which one. Now, let τ' be an extension of τ to a cuspidal representation of $GU(2,2)(\mathbb{A}_F)$ such that the central character is χ^2. Then by Lemma 5.4, $BC(\tau') = BC(\tau) \otimes \overline{\chi^2}$ as a representation of $GL_4(\mathbb{A}_E) \times GL_1(\mathbb{A}_E)$. Then

$$\wedge^2(BC(\tau')) = \wedge^2(BC(\tau)) \otimes \overline{\chi^2} = \wedge^2(A^3(\sigma_E)) \otimes (\chi\bar{\chi})^2 = A^4(\sigma_E) \otimes (\chi\bar{\chi})^2 \boxplus \omega_{\sigma_E}(\chi\bar{\chi})^2.$$

By Theorem 6.2, $\wedge^2(BC(\tau')) = B(\wedge_t^2(\tau'))$. Since $\omega_\sigma(\chi|_{\mathbb{A}_F^*}) = 1$, this leads us to the claim:

$$\wedge_t^2(\tau') = A^4(\sigma) \otimes (\chi|_{\mathbb{A}_F^*})^2 \boxplus \delta_{E/F}(\chi|_{\mathbb{A}_F^*}). \tag{9.1}$$

Here, we note that the right hand side is equal to $Sym_t^4(\pi) \otimes (\chi|_{\mathbb{A}_F^*})^{-5} \boxplus \delta_{E/F}(\chi|_{\mathbb{A}_F^*})$.

Since both sides of (9.1) are automorphic representations of $GL_6(\mathbb{A}_F)$, it is enough to prove (9.1) for almost all places by the strong multiplicity one result: If v splits, $\tau_v' = \tau_v \otimes \mu$ for some character μ of F_v^*. Since the central character of τ' is χ^2, we get $\omega_{\tau'}|_{\mathbb{A}_F^*} = (\chi|_{\mathbb{A}_F^*})^2$. Hence $\omega_{\tau_v}\mu^2 = (\chi|_{\mathbb{A}_F^*})_v^2$. Here $\omega_{\tau_v} = \omega_{\sigma_v}^2 \eta_v^4$. Hence $\eta_v^2 \mu = (\chi|_{\mathbb{A}_F^*})_v^2$. So the left hand side of (9.1) is

$$\wedge^2(\tau_v) \otimes \mu = \wedge^2(A^3(\sigma_v) \otimes \eta_v) \otimes \mu = A^4(\sigma_v) \otimes \eta_v^2\mu \boxplus \omega_{\sigma_v}\eta_v^2\mu = A^4(\sigma_v) \otimes (\chi|_{\mathbb{A}_F^*})_v^2 \boxplus (\chi|_{\mathbb{A}_F^*})_v.$$

Suppose v is inert. Let the semi-simple conjugacy class of σ_v be given by $\mathrm{diag}(\alpha, \beta) \in GL_2(\mathbb{C})$, and $\chi(\varpi)\alpha\beta = 1$. Then the semi-simple conjugacy class of $(A^3(\sigma_E) \otimes \chi)_v$ is given by

$$\mathrm{diag}(\alpha^3\beta^{-3}, \alpha\beta^{-1}, \beta\alpha^{-1}, \alpha^{-3}\beta^3) \in GL_4(\mathbb{C}).$$

Hence the semi-simple conjugacy class of τ_v' is given by $(\mathrm{diag}(\alpha^3\beta^{-3}, \alpha\beta^{-1}, 1, 1), \lambda, \theta) \in (GL_4(\mathbb{C}) \times GL_1(\mathbb{C})) \rtimes \mathrm{Gal}(E/F)$, where $\alpha^4\beta^{-4}\lambda^2 = \chi(\varpi)^2 = \alpha^{-2}\beta^{-2}$. Then $\lambda^2 = \alpha^{-6}\beta^2$. We choose τ' so that $\lambda = \alpha^{-3}\beta$. Then the semi-simple conjugacy class of $\wedge^2(\tau_v')$ is given by

$$\begin{aligned}\lambda\,\mathrm{diag}(&\alpha^4\beta^{-4}, \alpha^3\beta^{-3}, \alpha^2\beta^{-2}, \alpha\beta^{-1}, 1, -\alpha^2\beta^{-2}) \\ &= \mathrm{Gal}(\alpha\beta^{-3}, \beta^{-2}, \alpha^{-1}\beta^{-1}, \alpha^{-2}, \alpha^{-3}\beta, -\alpha^{-1}\beta^{-1}) \in GL_6(\mathbb{C}).\end{aligned}$$

It represents the semi-simple conjugacy class of $A^4(\sigma_v) \otimes \omega_{\sigma_v}^{-2} \boxplus (\delta_{E/F})_v \omega_{\sigma_v}^{-1}$. Since $\omega_\sigma(\chi|_{\mathbb{A}_F^*}) = 1$, our result follows. Hence we have proved

Theorem 9.2. *Let π be a cuspidal representation of $GU(1,1)(\mathbb{A}_F)$, and τ' be as above. Then*

$$\wedge_t^2(\tau') = Sym_t^4(\pi) \otimes (\chi|_{\mathbb{A}_F^*})^{-5} \boxplus \delta_{E/F}(\chi|_{\mathbb{A}_F^*}).$$

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