A CONVERSE THEOREM FOR GL(n)

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Abstract. We complete the work of Cogdell and Piatetski-Shapiro [3] to prove, for \( n \geq 3 \), a converse theorem for automorphic representations of \( \text{GL}_n \) over a number field, with analytic data from twists by unramified representations of \( \text{GL}_{n-1} \).

1. Introduction

In this paper, we complete the work of Cogdell and Piatetski-Shapiro [3] to prove the following.

**Theorem 1.1.** Let \( F \) be a number field with adèlic ring \( \mathbb{A}_F \). Fix an integer \( n \geq 3 \), and let \( \pi = \bigotimes_v \pi_v \) be an irreducible, admissible representation of \( \text{GL}_n(\mathbb{A}_F) \) with automorphic central character. For every unitary, isobaric, automorphic representation \( \tau = \bigotimes_v \tau_v \) of \( \text{GL}_{n-1}(\mathbb{A}_F) \) which is unramified at all finite places, assume that the complete Rankin–Selberg \( L \)-functions

\[
\Lambda(s, \pi \times \tau) = \prod_v L(s, \pi_v \times \tau_v)
\]

converge absolutely in some right half plane, continue to entire functions of finite order, and satisfy the functional equation

\[
\Lambda(s, \pi \times \tau) = \epsilon(s, \pi \times \tau)\Lambda(1 - s, \pi \times \tau),
\]

where \( \epsilon(s, \pi \times \tau) \) is the product of the corresponding local \( \epsilon \)-factors defined in [7, Thm. 2.7] and [12]. Then \( \pi \) is quasiautomorphic, in the sense that there is a unique isobaric automorphic representation \( \Pi = \bigotimes_v \Pi_v \) of \( \text{GL}_n(\mathbb{A}_F) \) such that \( \pi_v \cong \Pi_v \) for all non-archimedean places \( v \) where \( \pi_v \) is unramified.

Remarks.

1. We recall the notion of an isobaric automorphic representation [14]: Given a partition \( n_1, \ldots, n_k \) of \( n \) and cuspidal representations \( \sigma_i \) of \( \text{GL}_{n_i}(\mathbb{A}_F) \), let \( P \) be the corresponding standard parabolic subgroup of \( \text{GL}_n \) and let \( \omega_i \) denote the central character of \( \sigma_i \). Then there is a real number \( t_i \) such that \( |\omega_i(z)| = \|z\|^{t_i} \) for \( z \) in the center of \( \text{GL}_{n_i}(\mathbb{A}_F) \). By re-ordering, if necessary, we may assume \( t_1 \geq \ldots \geq t_k \), and form globally the induced representation \( \Upsilon = \text{Ind}_{\text{P}(\mathbb{A}_F)}^{\text{GL}_n}(\sigma_1 \otimes \cdots \otimes \sigma_k) \). On the other hand, if \( \sigma_i = \bigotimes_v \sigma_{i,v} \) (here and throughout the paper, the symbol \( \bigotimes \) means a restricted tensor product with respect to a distinguished set of spherical vectors for almost all places), we may also form locally the induced representations \( \Upsilon_v = \text{Ind}_{\text{P}(\mathbb{F}_v)}^{\text{GL}_{n_i}(\mathbb{F}_v)}(\sigma_{1,v} \otimes \cdots \otimes \sigma_{k,v}) \) for each \( v \). (For archimedean \( v \), one has to pass to the smooth completion of \( \sigma_{i,v} \) in...
order to form the induced representation $\Upsilon_v$; see §3.1 and the references therein for details.) Then, by definition, $\sigma_1, v \boxplus \cdots \boxplus \sigma_{k,v}$ is the $(\mathfrak{g}_v, K_v)$-module associated to the Langlands quotient of $\Upsilon_v$. We may also form their (restricted) tensor product to obtain an automorphic representation of $\text{GL}_n(\mathbb{A}_F)$. This representation, denoted $\sigma_1 \boxplus \cdots \boxplus \sigma_n$, is called isobaric, and it satisfies the following properties:

(a) **Strong multiplicity one** [8]. If $n'_1, \ldots, n'_l$ is another partition of $n$ and $\sigma'_i$, $1 \leq i \leq l$, are cuspidal representations of $\text{GL}_{n'_i}(\mathbb{A}_F)$ such that

$$\sigma_1 \boxplus \cdots \boxplus \sigma_k \cong \sigma'_1 \boxplus \cdots \boxplus \sigma'_l$$

then $l = k$ and there is a permutation $\phi$ of $\{1, \ldots, k\}$ such that $n'_i = n_{\phi(i)}$ and $\sigma'_i \cong \sigma_{\phi(i)}$.

(b) **Multiplicativity of local factors.** The Rankin–Selberg method [7, 9] of associating local factors is bi-additive with respect to isobaric sums. In particular, for any automorphic representation $\tau = \bigotimes_v \tau_v$ of $\text{GL}_n(\mathbb{A}_F)$, one has

$$L(s, (\sigma_1, v \boxplus \cdots \boxplus \sigma_k, v) \times \tau_v) = \prod_{i=1}^{k} L(s, \sigma_{i,v} \times \tau_v),$$

$$\epsilon(s, (\sigma_1, v \boxplus \cdots \boxplus \sigma_k, v) \times \tau_v, \psi_v) = \prod_{i=1}^{k} \epsilon(s, \sigma_{i,v} \times \tau_v, \psi_v)$$

for each $v$.

We call an isobaric automorphic representation $\sigma = \sigma_1 \boxplus \cdots \boxplus \sigma_k$ unitary if each cuspidal representation $\sigma_i$ has unitary central character. In this case, for each $v$, it follows from the description of the unitary dual of $\text{GL}_n(F_v)$ [17, 16] that the corresponding parabolically induced representation $\text{Ind}_{P(F_v)}^{\text{GL}_n(F_v)}(\sigma_1, v \boxplus \cdots \boxplus \sigma_k, v)$ is irreducible, unitary and generic; therefore, $\sigma$ is the full induced representation $\text{Ind}_{P(\mathbb{A}_F)}^{\text{GL}_n(\mathbb{A}_F)}(\sigma_1 \boxplus \cdots \boxplus \sigma_k)$.

(2) The results in [3] are stated in terms of analytic properties of $\Lambda(s, \pi \times \tau)$ for cuspidal representations $\tau$ of $\text{GL}_r(\mathbb{A}_F)$ for all $r < n$. Our statement in terms of unitary isobaric representations of $\text{GL}_{n-1}(\mathbb{A}_F)$ generalizes this slightly (since one can pass from cuspidal twists to isobaric twists by multiplying), though as we show in Proposition 3.1 below, one can derive the properties of the twists by cuspidal representations of $\text{GL}_r(\mathbb{A}_F)$ from our hypotheses.

Alternatively, one could state the theorem in terms of twists by all unramified, generic, automorphic subrepresentations $\tau$ of $\text{GL}_{n-1}(\mathbb{A}_F)$, and in fact the proof proceeds along these lines, i.e. we first derive the properties of such twists from those for unitary isobaric representations in Proposition 3.1. (Here and throughout the paper, we say that a global representation or idèle class character is **unramified** if it is unramified at every finite place.)

(3) Note that the local constituents $\pi_v$ are not assumed to be generic. If they happen to be generic for all $v$ then, following the proof of [3, §7, Cor. 2], one can modify the argument to produce a unique generic (but not necessarily isobaric) automorphic representation $\Pi$ such that $\Pi_v \cong \pi_v$ for all unramified $v$. In this case, we also obtain $\Pi_v \cong \pi_v$ for all archimedean $v$. 

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Our proof closely follows the method of Cogdell and Piatetski-Shapiro, who established a version of Theorem 1.1 (cf. [3, Thm. 3]) under the assumption that \( F \) has class number 1. In fact, in full generality, their method exhibits a classical automorphic form (i.e. at the archimedean places) with the expected properties, but they encountered some combinatorial difficulties in relating it back to the representation \( \pi \) (via Hecke eigenvalues), and were only able to overcome them under the class number assumption. Our proof avoids attacking the combinatorics directly; rather, we rely on the classical fact, due to Harish-Chandra, that any \( K \)-finite, \( \mathcal{Z} \)-finite, automorphic form is a finite linear combination of Hecke eigenforms, from which we realize the \( L \)-function of \( \pi \) as a linear combination of automorphic \( L \)-functions. The final ingredient is multiplicativity—since each of the \( L \)-functions in question is given by an Euler product, only the trivial linear relation is possible. To make this precise, we adapt work of Kaczorowski, Molteni and Perelli [13] on linear independence in the Selberg class, generalizing it to number fields.

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It is remarkable that there is no analog of the Riemann Hypothesis for \( L \)-functions with \( \mathcal{Z} \)-finite coefficients, despite the rich theory of \( \mathcal{Z} \)-finite automorphic forms. Nonetheless, our proof closely follows the method of Cogdell and Piatetski-Shapiro, who established a version of Theorem 1.1 (cf. [3, Thm. 3]) under the assumption that \( F \) has class number 1. In fact, in full generality, their method exhibits a classical automorphic form (i.e. at the archimedean places) with the expected properties, but they encountered some combinatorial difficulties in relating it back to the representation \( \pi \) (via Hecke eigenvalues), and were only able to overcome them under the class number assumption. Our proof avoids attacking the combinatorics directly; rather, we rely on the classical fact, due to Harish-Chandra, that any \( K \)-finite, \( \mathcal{Z} \)-finite, automorphic form is a finite linear combination of Hecke eigenforms, from which we realize the \( L \)-function of \( \pi \) as a linear combination of automorphic \( L \)-functions. The final ingredient is multiplicativity—since each of the \( L \)-functions in question is given by an Euler product, only the trivial linear relation is possible. To make this precise, we adapt work of Kaczorowski, Molteni and Perelli [13] on linear independence in the Selberg class, generalizing it to number fields.

**2. Preliminaries**

Suppose \( F \) is a number field with ring of integers \( \mathfrak{o}_F \). For each place \( v \) of \( F \), let \( F_v \) denote the completion of \( F \) at \( v \). For finite \( v \), let \( \mathfrak{o}_v \), \( q_v \) the cardinality of \( \mathfrak{o}_v / \mathfrak{p}_v \), and \( \varpi_v \) a generator of \( \mathfrak{p}_v \) with absolute value \( \| \varpi_v \|_v = q_v^{-1} \). Put \( F_\infty = \prod_{v|\infty} F_v \), and let \( \mathbb{A}_F = F_\infty \times \mathbb{A}_{F, f} \) denote the ring of adèles of \( F \).

Recall that a Größencharakter of conductor \( q \) is a multiplicative function \( \chi \) of non-zero integral ideals satisfying \( \chi(a\mathfrak{o}_F) = \chi_f(a)\chi_\infty(a) \) for associated characters \( \chi_f : (\mathfrak{o}_F / q)^\times \to \mathbb{C}^\times \) and \( \chi_\infty : F_\infty^\times \to \mathbb{C}^\times \), with \( \chi_f \) primitive and \( \chi_\infty \) continuous, and all \( a \in \mathfrak{o}_F \) relatively prime to \( q \). By convention we set \( \chi(a) = 0 \) for any ideal \( a \) with \( (a, q) \neq 1 \). The Größencharakters are in one-to-one correspondence with idèle class characters \( \omega : F_\times \setminus \mathbb{A}_F^\times \to \mathbb{C}^\times \), and the correspondence is such that \( \chi_\infty = \omega_{\infty}^{-1} \) and \( \chi(p_v \cap \mathfrak{o}_F) = \omega(\varpi_v) \) at each finite place \( v \) co-prime to \( q \). For any idèle class character \( \omega \), we write \( \chi_\omega \) to denote the associated Größencharakter.

For any \( r > 1 \) and any commutative ring \( R \), let \( B_r(R) = T_r(R)U_r(R) \subset GL_r(R) \) be the Borel subgroup of upper triangular matrices, \( P_r^\prime(R) \) the parabolic subgroup of type \( (r-1, 1) \), and \( N_r(R) \) its unipotent radical. Let \( P_r(R) \subset P_r^\prime(R) \) denote the mirabolic subgroup consisting of matrices whose last row is of the form \( (0, \ldots, 0, 1) \), i.e.

\[
P_r(R) = \left\{ \begin{pmatrix} h & y \\ 0 & 1 \end{pmatrix} : h \in GL_{r-1}(R), y \in R^{r-1} \right\} \cong GL_{r-1}(R) \times N_r(R).
\]

Let \( w_r \) denote the long Weyl element in \( GL_r(R) \), and put \( \alpha_r = \left( \begin{smallmatrix} 1 \\ & & 1 \\ & & & & & & & & & \end{smallmatrix} \right) \).
From now on, we fix an integer \( n \geq 3 \) and consider \( \text{GL}_n \) along with certain distinguished subgroups. For each \( v < \infty \), we will consider certain compact open subgroups of \( \text{GL}_n(F_v) \); namely, let \( K_v = \text{GL}_n(o_v) \), and for any integer \( m \geq 0 \), set

\[
K_{1,v}(p_v^m) = \left\{ g \in \text{GL}_n(o_v) : g \equiv \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \pmod{p_v^m} \right\},
\]

\[
K_{0,v}(p_v^m) = \left\{ g \in \text{GL}_n(o_v) : g \equiv \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \pmod{p_v^m} \right\},
\]

so that \( K_{1,v}(p_v^m) \) is a normal subgroup of \( K_{0,v}(p_v^m) \), with quotient \( K_{0,v}(p_v^m)/K_{1,v}(p_v^m) \cong (o_v/p_v^m)^* \). Next, define \( K_f = \prod_{v<\infty} K_v \), and for an integral ideal \( \mathfrak{a} \) of \( F \), set

\[
K_i(\mathfrak{a}) = \prod_{v<\infty} K_{i,v}(p_v^{m_v}) \quad \text{for } i = 0, 1,
\]

where \( m_v \) are the unique non-negative integers such that \( \mathfrak{a} = \prod_v (p_v \cap o_v)^{m_v} \). Then \( K_1(\mathfrak{a}) \subseteq K_0(\mathfrak{a}) \subseteq K_f \) are compact open subgroups of \( \text{GL}_n(A_{\mathfrak{a}, F}) \). We consider also the corresponding principal congruence subgroups of \( \text{GL}_n(F_{\infty}) \), embedded diagonally, namely,

\[
\Gamma_i(\mathfrak{a}) = \{ \gamma \in \text{GL}_n(F) : \gamma_f \in K_i(\mathfrak{a}) \} \subseteq \text{GL}_n(F_{\infty}) \quad \text{for } i = 0, 1,
\]

where \( \gamma_f \) denotes the image of \( \gamma \) in \( \text{GL}_n(A_{\mathfrak{a}, F}) \).

From strong approximation for \( \text{GL}_n \), one knows that \( \text{GL}_n(F) \backslash \text{GL}_n(A_{\mathfrak{a}, F}) / \text{GL}_n(F_{\infty}) K_1(\mathfrak{a}) \) is finite, with cardinality \( h \), the class number of \( F \). Let us write

\[
\text{GL}_n(A_{\mathfrak{a}, F}) = \prod_{j=1}^h \text{GL}_n(F) g_j \text{GL}_n(F_{\infty}) K_1(\mathfrak{a}),
\]

where each \( g_j \in \text{GL}_n(A_{\mathfrak{a}, F}) \). In particular,

\[
\text{GL}_n(F) \backslash \text{GL}_n(A_{\mathfrak{a}, F}) / K_1(\mathfrak{a}) \cong \prod_{j=1}^h \text{GL}_n(F_{\infty}) \Gamma_{1,j}(\mathfrak{a}) \backslash \text{GL}_n(F_{\infty}),
\]

where \( \Gamma_{1,j}(\mathfrak{a}) = \{ \gamma \in \text{GL}_n(F) : \gamma_f \in g_j K_1(\mathfrak{a})g_j^{-1} \} \subseteq \text{GL}_n(F_{\infty}) \), embedded diagonally. Replacing \( K_1(\mathfrak{a}) \) by \( K_0(\mathfrak{a}) \) in this definition, we get the corresponding groups \( \Gamma_{0,j}(\mathfrak{a}) \).

For groups \( H \subseteq G \), let \( \mathcal{F}(H \backslash G) \) denote the vector space of all complex-valued functions \( f : G \to \mathbb{C} \) that are left invariant under \( H \). Further, for any subgroup \( L \subseteq G \), let \( \mathcal{F}(H \backslash G)^L \) denote the subspace of right \( L \)-invariant functions in \( \mathcal{F}(H \backslash G) \). Then we have an isomorphism of vector spaces

\[
(2.1) \quad \mathcal{F}(\text{GL}_n(F) \backslash \text{GL}_n(A_{\mathfrak{a}, F}))^{K_1(\mathfrak{a})} \cong \prod_{j=1}^h \mathcal{F}(\Gamma_{1,j}(\mathfrak{a}) \backslash \text{GL}_n(F_{\infty})),
\]

given by \( f \mapsto (f_j) \), where \( f_j(x) = f(xg_j) \) for \( x \in \text{GL}_n(F_{\infty}) \).

3. The method of Cogdell and Piatetski-Shapiro

3.1. Initial setup. For convenience, we write \( G_{\infty} \) to denote the group \( \text{GL}_n(F_{\infty}) \). Let \( g_{\infty} \) be the real Lie algebra of \( G_{\infty} \), and let \( U \) denote the universal enveloping algebra of its complexification, \( g_{\infty}^C \). Let \( K_\infty = \prod_{v|\infty} K_v \), which is a maximal compact subgroup of \( G_{\infty} \). If
$S$ is a finite set of places of $F$, we write $\mathbb{A}_F^S$ to denote the restricted product $\prod_{v \in S} F_v$ and $G^S = \text{GL}_n(\mathbb{A}_F^S) = \prod_{v \in S} \text{GL}_n(F_v)$.

Let us recall the notion of a (smooth) $\text{GL}_n(\mathbb{A}_F)$-module, which is a complex vector space $X$ equipped with an action of $U$, $K_\infty$, and $\text{GL}_n(\mathbb{A}_{F,f})$ satisfying the following usual conditions:

1. the actions of $U$ and $K_\infty$ commute with that of $\text{GL}_n(\mathbb{A}_{F,f})$;
2. each $u \in X$ is fixed by some compact open subgroup of $\text{GL}_n(\mathbb{A}_{F,f})$;
3. $X$ has the structure of a $(\mathfrak{g}_\infty, K_\infty)$-module under the actions of $U$ and $K_\infty$.

Suppose $\delta$ is an irreducible representation of $K_\infty$ and $H$ is any compact open subgroup of $\text{GL}_n(\mathbb{A}_{F,f})$. Let $X(\delta, H)$ denote the subspace consisting of elements in $X$ which are fixed by $H$ and of isotypic type $\delta$. For $\Delta$ a finite collection of irreducible representations of $K_\infty$, let $X(\Delta, H) = \sum_{\delta \in \Delta} X(\delta, H)$. Then $X$ is said to be admissible if $X(\Delta, H)$ is finite dimensional for every $\Delta$ and $H$. We will also use the notation $X^H$ to denote the subspace of $H$-fixed vectors in $X$.

For each place $v$, let $\mathcal{H}_v$ denote the Hecke algebra of $\text{GL}_n(F_v)$ (defined with respect to $K_v$ for $v \mid \infty$, and let $\ast$ denote the multiplication operation in $\mathcal{H}_v$. Set $\mathcal{H}_\infty = \bigotimes_{v \mid \infty} \mathcal{H}_v$ and $\mathcal{H}_f = \bigotimes_{v < \infty} \mathcal{H}_v$, so that the global Hecke algebra $\mathcal{H}$ satisfies $\mathcal{H} = \mathcal{H}_\infty \otimes \mathcal{H}_f$. Given a smooth $\text{GL}_n(\mathbb{A}_F)$-module $X$ as above, it inherits an action of the Hecke algebra $\mathcal{H}_v$ for each $v$ and hence becomes a module for $\mathcal{H}$. It is well known that a smooth irreducible admissible $\text{GL}_n(\mathbb{A}_F)$-module is factorizable in the sense of [5, Thm. 3]. As is customary, by a smooth irreducible admissible representation of $\text{GL}_n(\mathbb{A}_F)$ we mean a smooth irreducible admissible $\text{GL}_n(\mathbb{A}_F)$-module.

Now, let $\pi = \bigotimes_v \pi_v$ be an irreducible, admissible representation of $\text{GL}_n(\mathbb{A}_F)$ with automorphic central character $\omega_\pi$, as in the statement of Theorem 1.1. We fix an additive character $\psi = \bigotimes_v \psi_v$ of $F \backslash \mathbb{A}_F$ whose conductor is the inverse different $\varpi^{-1}$ of $F$. For each $v$, let $\Xi_v$ be the induced representation of “Langlands type” having $\pi_v$ as the unique irreducible quotient [7, 9]. For $v \mid \infty$, by definition, $\Xi_v$ is actually a smooth admissible representation of $\text{GL}_n(F_v)$ of moderate growth and $\pi_v$ is the underlying $(\mathfrak{g}_v, K_v)$-module, also known as the Harish-Chandra module, of the unique irreducible quotient of $\Xi_v$. Each $\Xi_v$ is an induced representation of Whittaker type in the sense of [3, p. 159], also called a generic induced representation [9, p. 4]. In particular, $\Xi_v$ is admissible of finite type and admits a non-zero $\psi_v$-Whittaker form which is unique up to scalar factor. In general, such a representation is said to be of Whittaker type. It should be noted that any constituent of $\Xi_v$ has the same central character as that of $\pi_v$.

For any representation $\tau_v$ of Whittaker type, we write $\mathcal{W}(\tau_v, \psi_v)$ to denote its Whittaker model with respect to $\psi_v$ [7, 9]. An important feature of the $\Xi_v$ defined in the previous paragraph is that, although it may not be irreducible, the usual map $f \mapsto W_f$ from its space to $\mathcal{W}(\Xi_v, \psi_v)$ is bijective [11, 9]. The local factors associated to $\pi_v$ (not necessarily generic) are then defined via integral representations using the Whittaker model of $\Xi_v$. More precisely, by definition (cf. [7, 9]), for every representation $\tau_v$ of $\text{GL}_r(F_v)$ that is induced of Whittaker type one has

$$L(s, \pi_v \times \tau_v) = L(s, \Xi_v \times \tau_v),$$

with a similar equality for the local $\epsilon$-factors.

In the following two paragraphs, we rely heavily on [9] and refer the reader to that paper for any unexplained notation or terminology. For a fixed $v \mid \infty$, suppose $V$ is the representation space of $\Xi_v$, let $V_0 \subset V$ be the unique minimal invariant subspace which is generic, and let $\pi'_v$
be the corresponding underlying Harish-Chandra module. By [9, Lemma 2.4], any non-zero \( \psi_{\nu} \)-form \( \lambda \) on \( V \) will restrict to a non-zero \( \psi_{\nu} \)-form on \( V_0 \). Therefore, for any generic induced representation \( \tau_v \) of \( \text{GL}_n(F_v) \), the family of integrals defining \( L(s, \nu_\nu \times \tau_v) \) is a subspace of those defining \( L(s, \Xi_v \times \tau_v) \). By passing to the projective tensor product and applying [9, Thm. 2.6], we see that the factor \( L(s, \nu_\nu \times \tau_v) \) is equal to an integral of the form

\[
\int W \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, g \right] \| \det g \|_{\nu}^{-\frac{s-1}{2}} dg,
\]

where \( W \) corresponds to a smooth vector in the projective completion of the representation \( \nu_\nu \otimes \tau_v \).

Then, by continuity and using the extension of [9, Prop. 11.1] to the complete tensor product (see §12.3 of loc. cit.), we see that \( L(s, \nu_\nu \times \tau_v) \) must be a holomorphic multiple of \( L(s, \pi_\nu \times \tau_v) \). Let us realize \( \pi_\nu \) (in the above sense) as the Langlands quotient of an induced representation \( (\Xi_\nu, V') \) of Langlands type. Then, it follows from [9, Lemma 2.5] that \( \Xi_\nu' \) is irreducible and consequently \( \pi_\nu \) is the Harish-Chandra module of \( \Xi_\nu' \). Now, as in [9, §11], \((\Xi_\nu, V)\) (and hence \((\Xi_\nu', V')\)) is a subrepresentation of a principal series representation \( I_{\mu,t} \), where \( \mu \) is an \( n \)-tuple of characters and \( t \) is an \( n \)-tuple of complex numbers. Similarly, \( \tau_v \) is a subrepresentation of an \( I_{\mu',t'} \). Then, from the proof of Proposition 11.1 of loc. cit., it follows that both \( L(s, \nu_\nu \times \tau_v) \) and \( L(s, \pi_\nu \times \tau_v) \) are polynomial multiples of

\[
\prod_{i,j} L(s + t_i + t'_j, \mu_i \mu'_j).
\]

Thus \( L(s, \nu_\nu \times \tau_v) \) is a rational multiple of \( L(s, \pi_\nu \times \tau_v) \). Since it is a holomorphic multiple as well, we conclude that

\[
(3.1) \quad L(s, \nu_\nu \times \tau_v) = f(s)L(s, \pi_\nu \times \tau_v),
\]

where \( f(s) \) is a polynomial. Finally, it also follows from loc. cit. that the \( \gamma \)-factors associated with the pairs \((\nu_\nu, \tau_v)\) and \((\pi_\nu, \tau_v)\), respectively, are the same.

Now, let \( \nu' = \bigotimes_{v \mid \infty} \nu_\nu' \otimes \bigotimes_{v < \infty} \pi_\nu \), which is an irreducible admissible representation of \( \text{GL}_n(A_F) \) with the same central character as \( \pi \). Suppose \( \tau \) is a unitary isobaric automorphic representation of \( \text{GL}_{n-1}(A_F) \). As noted in the remarks following Theorem 1.1, \( \tau_v \) is a generic induced representation for each \( v \mid \infty \), so it follows from the above that \( \Lambda(s, \nu' \times \tau) = P(s)\Lambda(s, \pi \times \tau) \) for some polynomial \( P(s) \). Further,

\[
\gamma(s, \pi' \times \tau) := \frac{\epsilon(s, \pi' \times \tau)\Lambda(1 - s, \pi' \times \tau)}{\Lambda(s, \pi' \times \tau)} = \prod_{v \mid \infty} \gamma(s, \nu_\nu' \times \tau_v, \psi_v) \prod_{v < \infty} \epsilon(s, \nu_\nu' \times \tau_v, \psi_v) \cdot \frac{L(1 - s, \nu_\nu' \times \tau)}{L(s, \pi' \times \tau)} = \prod_{v \mid \infty} \gamma(s, \nu_\nu \times \tau_v, \psi_v) \prod_{v < \infty} \epsilon(s, \nu_\nu \times \tau_v, \psi_v) \cdot \frac{L(1 - s, \nu_\nu \times \tau)}{L(s, \pi \times \tau)} = \gamma(s, \pi \times \tau).
\]

Since \( \gamma(s, \pi \times \tau) = 1 \) identically if and only if \( \Lambda(s, \pi \times \tau) \) satisfies the functional equation (1.1), the functional equation for \( \Lambda(s, \pi' \times \tau) \) is equivalent to that of \( \Lambda(s, \pi \times \tau) \). In summary, \( \pi' \) satisfies the hypotheses of Theorem 1.1 if \( \pi \) does. The representation \( \pi' \) has the added
feature that its archimedean local components have a Whittaker model, and if \( \pi_v \) is generic for all \( v \mid \infty \) to begin with then \( \pi = \pi' \).

In general, we may replace \( \pi \) by \( \pi' \), and assume without loss of generality that \( \pi_v \) is generic for all archimedean \( v \), at the expense of losing compatibility between \( \pi_v \) and \( \Pi_v \) for those \( v \). In [3], the authors work with the full induced representation \( \Xi_v \) instead of \( \pi_v \) at archimedean places, but we have made the above modification in order to preserve irreducibility, which is essential in Lemma 3.4 (see §3.4 below).

Next, for \( v < \infty \), choose \( \xi_v^0 \) in the space of \( \Xi_v \) as in [3, p. 203]. In particular, for \( v < \infty \) where \( \pi_v \) (and hence \( \Xi_v \)) is unramified, \( \xi_v^0 \) is the unique \( K_v \)-fixed vector that projects onto the distinguished spherical vector of \( \pi_v \). For \( v < \infty \) where \( \pi_v \) is ramified, the choice of \( \xi_v^0 \) is such that it is fixed by \( K_{1,v}(p_v^{m_v}) \) for some \( m_v > 0 \). Set \( n = \prod_{v < \infty} (p_v \cap \mathfrak{o}_F)^{m_v} \). We note that when \( \pi \) is generic, we may choose each \( \xi_v^0 \) to be the essential vector [6, 10] and \( n \) to be the conductor of \( \pi \). At any rate, for \( v < \infty \) where \( \pi_v \) is unramified, and \( \tau_v \) any unramified representation of \( \text{GL}_{n-1}(F_v) \) of Langlands type with normalized spherical function \( W_{\tau_v}^0 \in W(\tau_v, \psi_v^{-1}) \), one has (cf. [8, §1, (3)])

\[
(3.2) \quad \int_{U_{n-1}(F_v) \setminus \text{GL}_{n-1}(F_v)} W_{\tau_v}^0 \left( \begin{pmatrix} g & 0 \\ \mathbf{1} \end{pmatrix} \right) W_{\tau_v}(g) \| \det g \|^{-\frac{1}{2}} \, dg = L(s, \pi_v \times \tau_v).
\]

As pointed out in [10, Remark 2] as well as in [15, §1.5], the above equality is derived for generic unramified representations in [8] but the proof extends verbatim to unramified representations of Langlands type.

### 3.2. The functions \( U_\xi \) and \( V_\xi \)

In this subsection, we summarize the construction in [3] of the functions \( U_\xi \) and \( V_\xi \) associated to \( \pi \), and describe their properties, culminating in the identity given in Proposition 3.1; we defer to [3] for detailed proofs.

First note that if we take \( \tau \) to be the isobaric sum of \( n - 1 \) copies of the trivial character then, by hypothesis, the product \( \prod_v L(s, \pi_v \times \tau_v) \) converges absolutely for \( s \) in a right half plane. For each \( v \), let \( \pi_v \boxtimes \tau_v \) denote the functorial tensor product defined via the local Langlands correspondence. For an unramified finite place \( v \), let \( \alpha_{v,1}, \ldots, \alpha_{v,n} \) denote the Satake parameters of \( \pi_v \). Then \( \pi_v \boxtimes \tau_v \) has the same parameters, repeated with multiplicity \( n - 1 \). Applying [3, Lemma 2.2] to the representation \( \pi \boxtimes \tau = \bigotimes_v (\pi_v \boxtimes \tau_v) \), we obtain an estimate of the form \( \alpha_{v,i} = O(q_v^\sigma) \) for some \( \sigma \in \mathbb{R} \). Hence, the Euler product defining the standard \( L \)-function, \( \prod_v L(s, \tau_v) \), also converges absolutely for \( s \) in a right half plane, which is the form that this hypothesis takes in [3].

Next, as discussed in [3, §8], \( \omega_\pi \) determines a character \( \chi_\pi = \bigotimes_v \chi_{\pi_v} \) of \( K_0(\mathfrak{n}) \) which is trivial on \( K_1(\mathfrak{n}) \). Moreover, it follows from loc. cit. that \( K_0(\mathfrak{n}) \) acts on the space of \( K_1(\mathfrak{n}) \)-fixed vectors via \( \chi_\pi \). In particular, for \( v < \infty \) and \( g \in K_{0,v}(p_v^{m_v}) \), we have

\[
\Xi_v(g) \xi_v^0 = \chi_{\pi_v}(g) \xi_v^0.
\]

Let \( V_{\pi_\infty} \) denote the space of \( \pi_\infty \), and fix \( \xi_{\infty} = \bigotimes_{v \mid \infty} \xi_v \in V_{\pi_\infty} \). Let \( \xi = \xi_\infty \otimes \xi_f \), where \( \xi_f = \bigotimes_{v < \infty} \xi_v^0 \), and consider

\[
U_\xi(g) = \sum_{\gamma \in U_n(F) \setminus P_n(F)} W_\xi(\gamma g) = \sum_{\gamma \in U_{n-1}(F) \setminus \text{GL}_{n-1}(F)} W_\xi(\gamma \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} g).
\]

This sum converges absolutely and uniformly on compact subsets to a continuous function on \( \text{GL}_n(\mathbb{A}_F) \) which is cuspidal along the unipotent radical of any standard maximal parabolic
subgroup of $GL_n(\mathbb{A}_F)$. Since $\omega_\pi$ is assumed to be automorphic, as a function of $GL_n(\mathbb{A}_F)$, $U_\xi$ is left invariant under both $P_n(F)$ and the center $Z_n(F)$.

We also consider a second function $V_\xi$ attached to $\xi$, which will be related to $U_\xi$ through the functional equation. Namely, let $\widetilde{U}_\xi(g) = W_\xi(w_n^{-1}g^{-1})$, put

$$\widetilde{U}_\xi(g) = \sum_{\gamma \in U_n(F) \backslash P_n(F)} \widetilde{W}_\xi(\gamma g),$$

and define $V_\xi(g) = \widetilde{U}_\xi(\alpha_n^{-1} g^{-1})$, where $\alpha_n = (w_n^{-1})$, as defined in \S 2. In other words,

$$V_\xi(g) = \sum_{\gamma \in U_n(F) \backslash P_n(F)} \widetilde{W}_\xi(\gamma \alpha_n^{-1} g^{-1}) = \sum_{\gamma \in U_{n-1}(F) \backslash GL_{n-1}(F)} \widetilde{W}_\xi\left(\begin{pmatrix} \gamma^{-1} & 0 \\ 0 & 1 \end{pmatrix} \alpha_n^{-1} g^{-1}\right).$$

Hence, if $Q_n = {}^t P_n^{-1}$, then $V_\xi(g)$ is invariant on the left by both $Q_n(F)$ and $Z_n(F)$.

We record another formula for $V_\xi(g)$ which shows that it agrees with the definition given in [3]. By definition, since $^t \alpha_n^{-1} = \alpha_n$, we have

$$V_\xi(g) = \sum_{\gamma \in U_{n-1}(F) \backslash GL_{n-1}(F)} W_\xi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \alpha_n^{-1} g^{-1}\right) = \sum_{\gamma \in U_{n-1}(F) \backslash GL_{n-1}(F)} W_\xi\left(\begin{pmatrix} \gamma^{-1} & 0 \\ 0 & 1 \end{pmatrix} w_{n-1}^{-1} g\right).$$

Here, the last equality follows since the transformation $\gamma \mapsto w_{n-1}^{-1} \gamma^{-1} w_{n-1}$ permutes the set of right cosets $U_{n-1}(F) \gamma$ in $GL_{n-1}(F)$. Thus,

$$V_\xi(g) = \sum_{\gamma \in U_{n-1}(F) \backslash GL_{n-1}(F)} W_\xi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right),$$

where $\alpha_n' = (I_{n-1}^{-1})$.

Now, let $\tau$ be an automorphic subrepresentation of $GL_{n-1}(\mathbb{A}_F)$, and let $\phi$ be an automorphic form in the space of $\tau$. Suppose $\tau \cong \bigotimes_v \tau_v$ and $\phi$ corresponds to a pure tensor $\bigotimes_v \phi_v$ under this isomorphism. Let

$$I(s; U_\xi, \phi) = \int_{GL_{n-1}(F) \backslash GL_{n-1}(\mathbb{A}_F)} U_\xi\left(\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right) \phi(h) \| \det h \|^{s - \frac{1}{2}} dh.$$

The above integral is absolutely convergent for $\Re(s) \gg 1$ and if $\tau$ is cuspidal, it converges for all $s$. Further, the integral unfolds to give

$$I(s; U_\xi, \phi) = \prod_v \int_{U_{n-1}(F_v) \backslash GL_{n-1}(F_v)} W_{\xi_v}\left(\begin{pmatrix} h_v & 0 \\ 0 & 1 \end{pmatrix}\right) W_{\phi_v}(h_v) \| \det h_v \|^{s - \frac{1}{2}} dh_v,$$

(3.3)

$$= \prod_v \Psi_v(s; W_{\xi_v}, W_{\phi_v}),$$
where $W_\phi(h) = \int_{U_{n-1}(F)\backslash U_{n-1}(A_F)} \phi(wh) \, \psi(n) \, dn$, i.e. $W_\phi \in \mathcal{W}(\tau, \psi^{-1})$. In particular, the integral vanishes unless $\tau$ is generic. Further, from the theory of local $L$-functions [7, 9],

$$E_v(s) = \frac{\Psi_v(s; W_{\xi_v}, W_{\phi_v})}{L(s, \tau_v \times \tau_v)}$$

is entire for all $v$. If $v$ is non-archimedean then $E_v(s) \in \mathbb{C}[q_v^s, q_v^{-s}]$, and for almost all such $v$ we have $E_v(s) = 1$. Thus, setting $E(s) = \prod_v E_v(s)$, we have

$$I(s; U_{\xi}, \phi) = E(s) \prod_v L(s, \pi_v \times \tau_v).$$

Similarly, we define the integral

$$I(s; V_{\xi}, \phi) = \int_{GL_{n-1}(F) \backslash GL_{n-1}(A_F)} V_{\xi}(h) \det h^{\frac{s-1}{2}} \, dh,$$

which converges for $-\Re(s) \gg 1$. If we unfold this integral, we get

$$I(s; V_{\xi}, \phi) = \prod_v \int_{U_{n-1}(F_v) \backslash GL_{n-1}(A_F)} W_{\xi_v}(h_v) \det h_v^{\frac{1}{2} - s} \, dh_v$$

$$= \prod_v \Psi_v(1 - s; \widehat{W}_{\xi_v}, \widehat{W}_{\phi_v}),$$

where $\widehat{W}_{\phi}(h) = W_\phi(w_{n-1}^{-1}h^{-1})$. In passing, we mention that this is the $\psi$-Whittaker coefficient of the dual function $\phi(h) = \phi(w_{n-1}^{-1}h^{-1})$, i.e. $\widehat{W}_{\phi}(h) = \int_{U_{n-1}(F) \backslash U_{n-1}(A_F)} \phi(wh) \psi^{-1}(u) \, du$.

Now, for every $v$, just as we defined $E_v(s)$, let

$$\widehat{E}_v(s) = \frac{\Psi_v(s; \widehat{W}_{\xi_v}, \widehat{W}_{\phi_v})}{L(s, \widehat{\pi}_v \times \widehat{\tau}_v)}$$

denote the corresponding entire function attached to the pair of dual representations $(\widehat{\pi}_v, \widehat{\tau}_v)$. Then

$$I(s; V_{\xi}, \phi) = \widehat{E}(1 - s) \prod_v L(1 - s, \widehat{\pi}_v \times \widehat{\tau}_v),$$

where $\widehat{E}(s) = \prod_v \widehat{E}_v(s)$.

Hence the two integrals $I(s; U_{\xi}, \phi)$ and $I(s; V_{\xi}, \phi)$ continue to meromorphic or analytic functions of $s$ if the respective $L$-functions $\Lambda(s, \pi \times \tau)$ and $\Lambda(1 - s, \widehat{\pi} \times \widehat{\tau})$ do. In addition, if these $L$-functions satisfy the standard functional equation, together with the local functional equation (cf. [3, p. 169]), it follows that the two analytically-continued integrals are in fact equal.

In what follows, we abuse notation and identify $GL_{n-1}$ with its image in $P_n$ via the embedding $h \mapsto (h_{11})$. We now prove a slight generalization of [3, Prop. 10.2]:

**Proposition 3.1.** We have $U_{\xi}(g) = V_{\xi}(g)$ for all $g \in GL_n(F_{\infty}) \cdot GL_{n-1}(A_F) Z_n(A_F)K_0(n)$. 

Proof. Since \( K_0(n) \) (resp. \( Z_n(\mathcal{A}_F) \)) acts on \( \xi_f^0 \) (resp. \( \xi_\infty \otimes \xi_f^0 \)) through the central character, it is sufficient to prove the identity for \( g \in \text{GL}_n(F_\infty) \text{GL}_{n-1}(\mathcal{A}_F) \). First, we prove it for \( g \in \text{GL}_{n-1}(\mathcal{A}_F) \). As in the proof of [3, Prop. 10.2], this follows from the Langlands spectral theory, provided that one knows the expected analytic properties of \( \Lambda(s, \pi \times \tau) \) for all unramified, generic, automorphic subrepresentations \( \tau \) of \( \text{GL}_{n-1}(\mathcal{A}_F) \). Here we only assume these for unitary isobaric representations. Our proof proceeds by passing from unitary to non-unitary isobaric representations, and then to generic subrepresentations.

To that end, we first take \( \tau \) to be the isobaric sum of \( n-1 \) trivial characters, so that \( \Lambda(s, \tau_1) = \xi_F(s)^{1-n} \), where \( \xi_F(s) \) is the complete Dedekind zeta-function, and \( \Lambda(s, \pi \times \tau_1) = \Lambda(s, \pi)^{1-n} \). (This follows from the multiplicativity property in Remark 1, which will be put to repeated use throughout this proof without further mention.) Thus, by our hypotheses, \( \Lambda(s, \pi)^{1-n} \) continues to an entire function of finite order and does not vanish identically. Hence,

\[
\frac{\Lambda'(s, \pi)}{\Lambda(s, \pi)} = \frac{1}{n-1} \frac{d}{ds} \log(\Lambda(s, \pi)^{1-n})
\]

has meromorphic continuation to \( \mathbb{C} \).

Next, we take \( \tau_2 \) to be the isobaric sum of one trivial character and \( n-2 \) copies of the character \( \| \cdot \|^t \) for a fixed \( t \in \mathbb{R} \), so that \( \Lambda(s, \tau_2) = \xi_F(s)\xi_F(s+it)^{n-2} \) and \( \Lambda(s, \pi \times \tau_2) = \Lambda(s, \pi)\Lambda(s+it, \pi)^{n-2} \). Then as above we find that

\[
\frac{\Lambda'(s, \pi)}{\Lambda(s, \pi)} + (n-2) \frac{\Lambda'(s+it, \pi)}{\Lambda(s+it, \pi)}
\]

has meromorphic continuation to \( \mathbb{C} \) and has non-negative integral residues at every point. Now, since \( \frac{\Lambda'}{\Lambda}(s, \pi) \) has at most countably many poles, there exists \( t \in \mathbb{R} \) such that \( \frac{\Lambda'}{\Lambda}(s+it, \pi) \) and \( \frac{\Lambda'}{\Lambda}(s, \pi) \) have no poles in common. Hence, \( \frac{\Lambda'}{\Lambda}(s, \pi) \) has non-negative integral residues, and therefore \( \Lambda(s, \pi) \) continues to an entire function. Moreover, from the functional equation

\[
\Lambda(s, \pi)^{1-n} = \Lambda(s, \pi \times \tau_1) = \epsilon(s, \pi \times \tau_1)\Lambda(1-s, \tilde{\pi} \times \tilde{\tau}_1) = \epsilon(s, \pi)^{1-n} \Lambda(1-s, \tilde{\pi})^{1-n},
\]

we derive

\[
\Lambda(s, \pi) = \mu e(s, \pi \times \tau_3) \Lambda(1-s, \tilde{\pi} \times \tilde{\tau}),
\]

where \( \mu \) is an \((n-1)^{st}\) root of unity (which may depend on \( \pi \)), and similarly the finite order of \( \Lambda(s, \pi) \) follows from that of \( \Lambda(s, \pi)^{1-n} \).

Now, let \( \sigma \) be an unramified unitary cuspidal representation of \( \text{GL}_r(\mathcal{A}_F) \) for some \( r < n \), and let \( \tau_3 \) be the isobaric sum of \( \sigma \) and \( n-1-r \) copies of \( \| \cdot \|^t \) for \( t \in \mathbb{R} \). Then arguing as above we see that \( \Lambda(s, \pi \times \sigma) \) continues to an entire function of finite order, and from the functional equations for \( \Lambda(s, \pi \times \tau_3) \) and \( \Lambda(s, \pi) \), we derive

\[
\Lambda(s, \pi \times \sigma) = \mu^r e(s, \pi \times \sigma) \Lambda(1-s, \tilde{\pi} \times \tilde{\sigma}).
\]

Since every cuspidal representation is unitary up to twisting by a power of the determinant, by shifting \( s \) in this equation by a real displacement, we conclude the same properties of \( \Lambda(s, \pi \times \sigma) \) for every cuspidal representation \( \sigma \), not necessarily unitary.

For \( i = 1, \ldots, k \), let \( \sigma_i \) be an unramified (not necessarily unitary) cuspidal representation of \( \text{GL}_{r_i}(\mathcal{A}_F) \), assume that \( r_1 + \ldots + r_k = n-1 \), and put \( \tau = \sigma_1 \boxplus \cdots \boxplus \sigma_k \). Then by the above we see that \( \Lambda(s, \pi \times \tau) \) continues to an entire function of finite order and satisfies the functional equation

\[
\Lambda(s, \pi \times \tau) = \mu^{r_1+\ldots+r_k} e(s, \pi \times \sigma_1) \cdots e(s, \pi \times \sigma_k) \Lambda(1-s, \tilde{\pi} \times \tilde{\tau}) = \epsilon(s, \pi \times \tau) \Lambda(1-s, \tilde{\pi} \times \tilde{\tau}).
\]
Now let \( \tau' \) be an unramified, generic, automorphic subrepresentation of \( \text{GL}_{n-1}(\mathbb{A}_F) \). By Langlands’ classification, it can be realized as a subquotient of an induced (parabolic) representation of the form \( \text{Ind}_P^{\text{GL}_{n-1}(\mathbb{A}_F)}(\sigma_1 \otimes \cdots \otimes \sigma_k) \) for some cuspidal automorphic representations \( \sigma_i \). Since \( \tau' \) is a subrepresentation to begin with, as explained in the proof of [3, Prop. 6.1], it is in fact a subrepresentation of the induced representation. Now, at a finite place \( v \) where \( \tau'_v \) is unramified, by [3, p. 201, proof of Prop. 10.5] \( \tau'_v \) is the full induced representation; in particular, \( \tau'_v = \tau_v \), where \( \tau \) is the isobaric sum \( \sigma_1 \boxplus \cdots \boxplus \sigma_k \). This need not be the case at archimedean places, but by the argument leading up to (3.1) (with the roles of \( \pi \) and \( \tau \) reversed), we see that \( L(s, \pi_v \times \tau'_v) \) is a polynomial multiple of \( L(s, \pi_v \times \tau_v) \) and that the corresponding local \( \gamma \)-factors are the same. Thus, the analytic properties of \( \Lambda(s, \pi \times \tau') \) follow from those of \( \Lambda(s, \pi \times \tau) \).

It remains to prove the assertion for any \( g = (g_\infty, g_f) \) with \( g_\infty \in \text{GL}_n(F_\infty), g_f = (g'_f) \), \( g'_f \in \text{GL}_{n-1}(\mathbb{A}_{F,f}) \). To this end, let \( F_{\xi_\infty} = U_\xi - V_\xi \), where \( \xi = \xi_\infty \otimes \xi_0^f \). Then we have \( F_{\xi_\infty}((1, g_f)) = 0 \) from the above conclusion. In other words, the linear functional on \( V_{\pi_\infty} \) given by

\[
\xi_\infty \mapsto F_{\xi_\infty}((1, g_f))
\]

is trivial. Since \( \xi_\infty \mapsto W_{\xi_\infty} \) is continuous on the Casselman–Wallach completion of \( V_{\pi_\infty} \) and the \( K_\infty \)-finite vectors are dense in this completion, it follows that \( F_{\xi_\infty}((1, g_f)) = 0 \) for all smooth vectors \( \xi_\infty \). Finally, fixing a pure tensor \( \xi_\infty \) in \( V_{\pi_\infty} \) as in the statement of the Proposition, we have \( F_{\xi_\infty}(g) = F_{g_\infty \xi_\infty}((1, g_f)) = 0 \).

### 3.3. Congruence subgroups and classical automorphic forms.

Let \( \{t_1, \ldots, t_h\} \subset \mathbb{A}_{F,f}^\times \) be a set of representatives for the ideal class group of \( F \), with \( t_1 = 1 \), and let \( a_j \) denote the ideal generated by \( t_j \), which we assume to be integral. Put \( g_j = \text{diag}(t_1, 1, \ldots, 1) \in \text{GL}_n(\mathbb{A}_{F,f}) \). For \( \xi_\infty \in V_{\pi_\infty} \), we associate the \( h \)-tuple of functions \( (\Phi_{\xi_\infty,1}, \ldots, \Phi_{\xi_\infty,h}) \) given by

\[
\Phi_{\xi_\infty,j}(g) = U_{\xi_\infty \otimes \xi_0^f}((g, g_j)) = V_{\xi_\infty \otimes \xi_0^f}((g, g_j)) \quad \text{for } g \in \text{GL}_n(F_\infty).
\]

In the notation of §2, for \( j = 1, \ldots, h \), let us set \( G_j = \Gamma_{1,j}(\mathfrak{o}_F) = \Gamma_{0,j}(\mathfrak{o}_F) \). In concrete terms,

\[
G_j = \left\{ \gamma \in \left( \begin{array}{ccc}
\ast & a_j & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast
\end{array} \right) : \det \gamma \in \mathfrak{o}_F^\times \right\}.
\]

For each \( j \), \( \Gamma_{1,j}(n) \subseteq \Gamma_{0,j}(n) \) are then subgroups of \( G_j \). For instance, if \( \gamma = (\gamma_{kl}) \in \Gamma_{1,j}(n) \), then the congruence condition on its last row is given by

\[
\gamma_{nj} \equiv 1 \pmod{n a_j^{-1}}, \gamma_{n1} \in n, \ldots, \gamma_{nn} - 1 \in n.
\]

Now, for \( i = 0, 1, j = 1, \ldots, h \), let

\[
\Gamma_{ij}^n(n) = Z_n(F)P_n(F) \cap \text{GL}_n(F_\infty)g_j K_i(n)g_j^{-1},
\]

\[
\Gamma_{ij}^0(n) = Z_n(F)Q_n(F) \cap \text{GL}_n(F_\infty)g_j K_i(n)g_j^{-1},
\]

which are subgroups of \( \Gamma_{ij}(n) \subseteq G_j \subset \text{GL}_n(F_\infty) \). Since the functions \( U_{\xi_\infty \otimes \xi_0^f} \) and \( V_{\xi_\infty \otimes \xi_0^f} \) are left invariant under \( Z_n(F)P_n(F) \) and \( Z_n(F)Q_n(F) \), respectively, and \( \xi_0^f \) is fixed by \( K_1(n) \), we see that \( \Phi_{\xi_\infty,j} \) is invariant on the left by both \( \Gamma_{ij}^n(n) \) and \( \Gamma_{ij}^0(n) \) for \( j = 1, \ldots, h \). We now need the following result, generalizing [3, Prop. 9.1].
Proposition 3.2. For \( j = 1, \ldots, h \), the groups \( \Gamma_{1,j}^P(n) \) and \( \Gamma_{1,j}^Q(n) \) together generate the congruence subgroup \( \Gamma_{1,j}(n) \).

Proof. Let \( \gamma \in \Gamma_{1,j}(n) \) be a typical element, and let \( (a_1, \ldots, a_n) \) be its bottom row. Recall that \( \det \gamma \) is a unit in \( \mathfrak{o}_F \). Expanding the determinant along the bottom row, we find that

\[
1 = \sum_{i=1}^{n} c_i a_i
\]

for some \( c_1 \in \mathfrak{a}_j, c_2, \ldots, c_n \in \mathfrak{o}_F \). In particular, \((c_1 a_1, a_2, \ldots, a_n) \in \mathfrak{o}_F^r\) is unimodular. Since \( n \geq 3 \), it follows from the Bass stable range theorem [1, Thm. 11.1] that there are \( b_2, \ldots, b_n \in \mathfrak{o}_F \) such that if \( a'_i = a_i + b_i c_1 a_1 \) then \((a'_2, \ldots, a'_n)\) is unimodular. Put

\[
\sigma = \begin{pmatrix}
1 & b_2 c_1 & \cdots & b_n c_1 \\
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 1
\end{pmatrix}
\]

so that \( \sigma \in \Gamma_{1,j}^P(n) \) and \((a_1, \ldots, a_n) \sigma = (a_1, a'_2, \ldots, a'_n) \). Hence, replacing \( \gamma \) by \( \gamma \sigma \) if necessary, we may assume without loss of generality that \((a_2, \ldots, a_n)\) is unimodular.

Next let \( \mathfrak{m} = a_2 \mathfrak{o}_F + \cdots + a_{n-1} \mathfrak{o}_F \), so that \( \mathfrak{m} + a_n \mathfrak{o}_F = \mathfrak{o}_F \). In particular, if \( \mathfrak{m} \) is the zero ideal then \( a_n \) is a unit; but then right-multiplying by the matrix

\[
\tau = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 1
\end{pmatrix}
\]

reduces the bottom row of \( \gamma \) to \((0, \ldots, 0, 1)\), so that \( \gamma \tau \in \Gamma_{1,j}^P(n) \), and we are finished. Hence, we may assume that \( \mathfrak{m} \) is non-zero.

Choose \( y_1 \in \mathfrak{a}_j \setminus \bigcup_{p \mid m} \mathfrak{p}_j \) and \( z \in \mathfrak{a}_j^{-1} \setminus \bigcup_{p \geq m} \mathfrak{p}_j^{-1} \). Then \( z y_1 \in \mathfrak{o}_F \) is invertible modulo \( \mathfrak{m} \). Let \( z' \in \mathfrak{o}_F \) be a multiplicative inverse of \( z y_1 \) (mod \( \mathfrak{m} \)), and set \( u_1 = z z' \in \mathfrak{a}_j^{-1} \), so that \( u_1 y_1 \equiv 1 \) (mod \( \mathfrak{m} \)). Further, since \( \mathfrak{a}_j^{-1} \mathfrak{a}_j = \mathfrak{o}_F \), there are elements \( u_2, \ldots, u_K \in \mathfrak{a}_j^{-1} \) and \( v_2, \ldots, v_K \in \mathfrak{a}_j \) such that \( 1 = \sum_{k=2}^{K} u_k v_k \). Setting \( y_k = (1 - u_1 y_1) v_k \in \mathfrak{a}_j \mathfrak{m} \) for \( k \geq 2 \), we have

\[
1 = \sum_{k=1}^{K} u_k y_k \quad \text{and} \quad a_n + (1 - a_n) \sum_{k=1}^{K'} u_k y_k \equiv 1 \pmod{\mathfrak{m}}
\]

for \( K' = 1, \ldots, K \). Next we set \( x_k = (a_n - 1) u_k \in \mathfrak{a}_j^{-1} \mathfrak{n} \) so that

\[
a_n - 1 = \sum_{k=1}^{K} x_k y_k,
\]

and \( a_n - \sum_{k=1}^{K'} x_k y_k \) is invertible modulo \( \mathfrak{m} \) for \( K' = 0, \ldots, K \).

By unimodularity there exist \( d_2, \ldots, d_n \in \mathfrak{o}_F \) such that \( d_2 a_2 + \cdots + d_n a_n = 1 \). If we put

\[
\tau_1 = \begin{pmatrix}
d_2 (x_1 - a_1) & 1 \\
\vdots & \ddots \\
d_n (x_1 - a_1) & \cdots & 1
\end{pmatrix}, \quad \sigma_1 = \begin{pmatrix}
1 & -y_1 \\
\vdots & \ddots \\
0 & \cdots & 1
\end{pmatrix},
\]

then \( \tau_1 \in \Gamma_{1,j}^P(n) \), \( \sigma_1 \in \Gamma_{1,j}^P(n) \), and \((a_1, \ldots, a_n) \tau_1 \sigma_1 = (x_1, a_2, \ldots, a_n - x_1 y_1, a_n - x_1 y_1) \). Moreover, since \( a_n - x_1 y_1 \) is invertible modulo \( \mathfrak{m} \), we have \( \mathfrak{m} + (a_n - x_1 y_1) \mathfrak{o}_F = \mathfrak{o}_F \), so that \((a_2, \ldots, a_n - 1) \mathfrak{n} = x_1 y_1 \) is again unimodular. Thus, we may repeat the construction with
$a_n$ replaced by $a_n - \sum_{k=1}^{K'} x_k y_k$ for $K' = 1, \ldots, K - 1$, obtaining matrices $\tau_2, \sigma_2, \ldots, \tau_K, \sigma_K$ such that

$$(a_1, \ldots, a_n) \tau_1 \sigma_1 \cdots \tau_K \sigma_K = (x_K, a_2, \ldots, a_{n-1}, 1).$$

Finally, applying the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ -x_K & -a_2 & \cdots & -a_{n-1} \\ 1 & 1 & \cdots & 1 \end{pmatrix} \in \Gamma_{1,j}^Q(n)$$

reduces the bottom row to $(0, \ldots, 0, 1)$, and we are finished.

Thus, for each $j$, the function $\Phi_{\xi_\infty,j}$ is left invariant under $\Gamma_{1,j}(n)$. Indeed, these are classical automorphic forms. To be precise, observe that for each $j$, $\chi_\pi$ also determines a character of $\Gamma_{0,j}(n)$ that is trivial on $\Gamma_{1,j}(n)$, which we continue to denote by $\chi_\pi$. Let $\mathcal{A}(\Gamma_{0,j}(n) \setminus \text{GL}_n(F_\infty); \omega_{\pi,\infty}, \chi_\pi^{-1})$ denote the space of classical automorphic forms $f$ on $\text{GL}_n(F_\infty)$ satisfying

$$f(\gamma g) = \chi_\pi^{-1}(\gamma) f(g) \quad \text{for all } \gamma \in \Gamma_{0,j}(n) \subset \text{GL}_n(F_\infty),$$

$$f(zg) = \omega_{\pi,\infty}(z) f(g) \quad \text{for all } z \in Z_n(F_\infty).$$

Then it follows that $\Phi_{\xi_\infty, j}$ belongs to $\mathcal{A}(\Gamma_{0,j}(n) \setminus \text{GL}_n(F_\infty); \omega_{\pi,\infty}, \chi_\pi^{-1})$. (The relevant growth properties follow from [4].) The character $\chi_\pi^{-1}$ is usually referred to as the Nebentypus character.

Now, let $\mathcal{A}(\omega_\pi)$ denote the space of automorphic forms on $\text{GL}_n(\mathbb{A}_F)$ which transform under the central character $\omega_\pi$. Then the isomorphism (2.1) induces a topological isomorphism

$$\mathcal{A}(\omega_\pi)^{\mathcal{K}(n)} \cong \prod_{j=1}^{h} \mathcal{A}(\Gamma_{0,j}(n) \setminus \text{GL}_n(F_\infty); \omega_{\pi,\infty}, \chi_\pi^{-1}).$$

In particular, the family of functions $\{\Phi_{\xi_\infty,j}\}_{j=1}^{h}$ determine a global automorphic form $\Phi_{\xi_\infty}$ through this isomorphism. Explicitly, given $g \in \text{GL}_n(\mathbb{A}_F)$, choose $j$ (which is uniquely determined) and $\gamma \in \text{GL}_n(F)$ so that $\gamma g \in g_j \text{GL}_n(F_\infty) \mathcal{K}_1(n)$; then $\Phi_{\xi_\infty}(g) = \Phi_{\xi_\infty,j}(\gamma g_\infty)$. One checks that this is well defined, in the sense that it is independent of the choice of $\gamma$. In the reverse direction, as mentioned above, we have

$$\Phi_{\xi_\infty,j}(g) = \Phi_{\xi_\infty}(g, g_j) \quad \text{for all } g \in \text{GL}_n(F_\infty), \quad j = 1, \ldots, h.$$ 

For later reference, we note that $\Phi_{\xi_\infty}$ satisfies the relation

$$\Phi_{\xi_\infty, j}(g) = U_{\xi_\infty \otimes \xi_j}(g) \quad \text{for } g \in P_n(\mathbb{A}_F),$$

which readily follows from its construction.

### 3.4. Hecke eigenforms and automorphic representations

We continue with the notation of §3.1. It is clear that (3.4) is an isomorphism of $(\mathfrak{g}_\infty, K_\infty)$-modules. Then the mapping $\xi_\infty \mapsto \Phi_{\xi_\infty}$ gives us a canonical embedding $\iota : V_{\pi,\infty} \rightarrow \mathcal{A}(\omega_\pi)$ of $(\mathfrak{g}_\infty, K_\infty)$-modules. Moreover, since $\pi_\infty$ is irreducible, the center $Z$ of $\mathcal{U}$ acts on $V_{\pi,\infty}$ through a character, say $\lambda$. Let $(\Pi, \mathcal{W})$ be the smallest $\text{GL}_n(\mathbb{A}_F)$-submodule of $\mathcal{A}(\omega_\pi)$ containing $\iota(V_{\pi,\infty})$, which is admissible according to [2, Prop. 4.5]. Indeed, for a fixed $\xi_\infty \neq 0$, $\mathcal{W} = \mathcal{H} \ast \Phi_{\xi_\infty}$. In [3, Prop. 10.4], Cogdell and Piatetski-Shapiro show that $\Pi$ consists of Hecke eigenforms for an
appropriate Hecke algebra, under the assumption that $h = 1$. Our proof departs from their approach in what follows.

For each $v$, as mentioned in §3.1, $\pi_v$ induces an action of $\mathcal{H}_v$, which we continue to denote by $\pi_v$. For $\Delta$ a finite collection of irreducible representations of $K_\infty$, let $e_\Delta \in \mathcal{H}_\infty$ be the corresponding idempotent. We write $V_{\pi_\infty}(\Delta)$ to denote the image of the operator $\pi_\infty(e_\Delta)$, i.e. $V_{\pi_\infty}(\Delta)$ is the sum of the $\delta$-isotopic components $V_{\pi_\infty}(\delta)$ for $\delta \in \Delta$. This is a finite-dimensional vector space since $\pi_\infty$ is admissible. Further, if $V_{\pi_\infty}(\Delta) \neq 0$, then it is an irreducible module for the (unital) subalgebra $\mathcal{H}_\infty(\Delta) = e_\Delta * \mathcal{H}_\infty * e_\Delta$. Before we proceed further, we need the following basic result from the theory of finite-dimensional representations of unital algebras.

Lemma 3.3. Suppose $A$ and $B$ are unital algebras over $\mathbb{C}$. Set $C = A \otimes B$, and suppose $(\rho, E)$ is a finite-dimensional representation of $C$. Let $M$ be a simple $A$-module, and set $M' = \text{Hom}_A(M, E)$. Consider the left $B$-module structure of $M'$ coming from that of the left $B$-module $E$, and regard $M \otimes M'$ as a $C$-module. Then the natural map $\alpha : M \otimes M' \to E$ induced by $v \otimes f \mapsto f(v)$ is a monomorphism of $C$-modules.

Proof. First, it is straightforward to see that $\alpha$ is a morphism of $C$-modules. Let $I = \text{Ann}_A(M)$; then $I$ is cofinite and, since $M$ is a simple $A$-module, it follows that $A/I \cong M_n(\mathbb{C})$ for some $n$. Thus, by reducing to $A/I \otimes B$, we may assume that $A$ is simple Artinian. Now, let $\{v_1, \ldots, v_n\}$ be a basis for $M$, and let $e_{kl} \in A = M_n(\mathbb{C})$ be the matrix units with respect to that basis. If $x = \sum_{i=1}^n v_i \otimes f_i \in M \otimes M'$ is such that $\alpha(x) = 0$, then $0 = e_{kl} \cdot \alpha(x) = \alpha(e_{kl} \cdot \sum_{i=1}^n v_i \otimes f_i) = \alpha(v_k \otimes f_i) = f_i(v_k)$, for all $k, l$. Thus, $f_i = 0$ for all $l$, i.e. $x = 0$, and hence $\alpha$ is injective. \hfill \Box

Next we show that one can split $\pi_\infty$ off from $\Pi$ in the following sense.

Lemma 3.4. There exists a smooth admissible $\GL_n(\mathbb{A}_F, f)$-module $(\Pi_f, U)$ such that

$$\pi_\infty \otimes \Pi_f \cong \Pi$$

as $\GL_n(\mathbb{A}_F)$-modules. Further, there exists $\Phi_f \in U$ such that for any $\xi_\infty \in V_{\pi_\infty}$, $\xi_\infty \otimes \Phi_f \mapsto \Phi_{\xi_\infty}$ under this isomorphism.

Proof. Let $(\Pi_f, U)$ denote the $\GL_n(\mathbb{A}_F, f)$-module $\text{Hom}_{\mathcal{H}_\infty}(V_{\pi_\infty}, W)$ and let $\Phi_f \in U$ be the element $\xi_\infty \mapsto \Phi_{\xi_\infty}$. Then $\Phi_f \in U^{K_1(n)}$ as every $\Phi_{\xi_\infty}$ is right $K_1(n)$-invariant. Since $W$ is smooth and admissible, and $V_{\pi_\infty}$ is a cyclic $\mathcal{H}_\infty$-module, it follows that $U$ is a smooth, admissible $\GL_n(\mathbb{A}_F, f)$-module. There is a natural homomorphism of $\GL_n(\mathbb{A}_F)$-modules from

$$(3.6) \quad V_{\pi_\infty} \otimes \text{Hom}_{\mathcal{H}_\infty}(V_{\pi_\infty}, W) \to W,$$

and it is clear that $\xi_\infty \otimes \Phi_f \mapsto \Phi_{\xi_\infty}$, for $\xi_\infty \in V_{\pi_\infty}$, under this morphism. It is surjective since $W$ is cyclic and generated by any nonzero $\Phi_{\xi_\infty}$. Finally, in order to show that $3.6$ is injective, it is sufficient to do so after fixing a level $H = \prod_{v \infty} H_v$, a compact open subgroup of $\GL_n(\mathbb{A}_F)$, and an infinity type $\Delta$. To this end, take $A = \mathcal{H}_\infty(\Delta)$, $B = \mathcal{H}(\GL_n(\mathbb{A}_F), H) = \bigotimes_{v \infty} \mathcal{H}(\GL_n(F_v), H_v)$, $M = V_{\pi_\infty}(\Delta)$, and $E = W(\Delta, H)$, and apply Lemma 3.3. \hfill \Box

Now, let $T$ be the smallest finite set of places of $F$ containing the archimedean places such that $\pi_v$ is unramified at all $v \notin T$. By construction $\Pi$ is also unramified at every $v \notin T$. Let $\mathcal{H}^T = \bigotimes_{v \notin T} \mathcal{H}(\GL_n(F_v), K_v)$ denote the spherical Hecke algebra, which is known to be commutative. Then, as explained in [3, §2, Appendix], $\mathcal{H}^T$ is naturally a subalgebra of
\( \mathcal{H}(\text{GL}_n(\mathbb{A}_{F,f}), K_1(n)) \), and hence acts on the space of \( K_1(n) \)-fixed vectors \( U^{K_1(n)} \). Therefore, \( U^{K_1(n)} \) has a basis consisting of Hecke eigenvectors for the action of the algebra \( \mathcal{H}^T \). In particular, we may write

\[
(3.7) \quad \Phi_f = \sum_{i=1}^{m} \eta_i,
\]

where \( \eta_i \) is a Hecke eigenvector with eigencharacter \( \Lambda_i \), say.

For \( 1 \leq i \leq m \), put \( V_i = \mathbb{C} \eta_i \), and let \( U_i \) be the smallest \( \text{GL}_n(\mathbb{A}_{F,f}) \)-submodule of \( U \) that contains \( \eta_i \), viz. \( U_i = \mathcal{H}_f \ast V_i \). Then \( U_i \) is admissible and \( U_i^{K_1} = V_i \), where \( K_1 = \prod_{v \notin T} K_v \) is the maximal compact open subgroup of \( G^T \). Now, let \( (\pi_i', U_i') \) denote the unique spherical representation of \( G^T \) associated to \( \Lambda_i \). Then, by an argument identical to that in Lemma 3.4, it follows that there is an admissible representation \( U''' \) of \( \prod_{v \in \mathcal{T}} \text{GL}_n(F_v) \) such that \( U_i \cong U_i' \otimes U''' \) as \( \mathcal{H}_f \)-modules, or equivalently, as representations of \( \text{GL}_n(\mathbb{A}_{F,f}) \). Therefore, we may assume that each \( \eta_i \) is of the form \( \eta_{i,T} \otimes \bigotimes_{v \notin T} \eta_{i,v}^0 \), where \( \eta_{i,v}^0 \) is the spherical vector at \( v \) (normalized to give the correct local \( L \)-factor, as in (3.2)), and \( \eta_{i,T} \) is a vector belonging to the space of an admissible representation of \( \prod_{v < \infty} \text{GL}_n(F_v) \).

**Lemma 3.5.** There exists a unique isobaric automorphic representation \( \pi_i \) such that \( \pi_i^T = \bigotimes_{v \notin T} \pi_{i,v} \) is the unique irreducible admissible representation of \( \text{GL}_n(\mathbb{A}_{F}) \) associated to the character \( \Lambda_i \).

**Proof.** By [3, Thm. A], there exists an irreducible (but not necessarily isobaric) automorphic representation \( \Pi_i \) with the required property and also satisfying \( \Pi_{i,\infty} \cong \pi_\infty \). We may realize \( \Pi_i \) as a component of an induced representation \( \Xi = \text{Ind}(\sigma_1 \otimes \cdots \otimes \sigma_k) \) of Langlands type, where the \( \sigma_j \) are cuspidal representations of \( \text{GL}_{r_j}(\mathbb{A}_F) \) with \( r_1 + \cdots + r_k = n \). Since \( \Pi_{i,v} \) is unramified for \( v \notin T \), it follows that the representation \( \Xi_v = \text{Ind}(\sigma_{1,v} \otimes \cdots \otimes \sigma_{k,v}) \) is also unramified and that \( \Pi_{i,v} \) is the unique spherical constituent of \( \Xi_v \). Let \( \pi_i \) be the isobaric representation \( \sigma_1 \boxplus \cdots \boxplus \sigma_k \). Then, since the Langlands quotient of \( \Xi_v \) is the same as the unique spherical constituent for an unramified place \( v \), it follows that \( \pi_{i,v} \cong \Pi_{i,v} \) for \( v \notin T \). The uniqueness of \( \pi_i \) follows from the strong form of multiplicity one (cf. Remark 1).

**3.5. A linear relation of \( L \)-functions.** Next, given any automorphic form \( \phi \) on \( \text{GL}_n(\mathbb{A}_F) \), we recall that its Whittaker–Fourier coefficient \( W_\phi \) (for a fixed \( \psi \)) is defined as

\[
W_\phi(g) = \int_{U_n(F)\backslash U_n(\mathbb{A}_F)} \phi(ug)\psi(u) \, du.
\]

**Lemma 3.6.** Assume the measure is normalized so that \( F \backslash \mathbb{A}_F \) has unit volume. For \( \xi_\infty \in V_\pi_\infty \), let \( \Phi_{\xi_\infty} \) be as defined after (3.4). Then its Whittaker–Fourier coefficient satisfies

\[
W_{\Phi_{\xi_\infty}}(g) = W_\xi(g) \quad \text{for } g \in P_n(\mathbb{A}_F),
\]

where \( \xi = \xi_\infty \otimes \xi_f^0 \).

**Proof.** For any subgroup \( N \subseteq U_n \), we write \( [N] \) to denote the adelic quotient \( N(F) \backslash N(\mathbb{A}_F) \). Also, for \( m < n \), we view \( \text{GL}_m \) as a subgroup of \( \text{GL}_n \) via the diagonal embedding \( h \mapsto \)
It is straightforward to check that the first integral in the above expression vanishes unless
\[ \gamma \in U_{n-1}(F) \backslash \text{GL}_{n-1}(F) \]

Let us write \( u = u_1 u_2, \ u_1 \in N_n(A_F), \ u_2 \in U_{n-1}(A_F) \subset \text{GL}_{n-1}(A_F) \subset \text{GL}_n(A_F) \). We further write elements of \( N_n \) as \( u(X) = (I_{n-1} \ X) \), where \( X \) is a column vector. Since \( (\gamma_{1}) \) normalizes \( N_n \), we get

\[
W_{\Phi_{\infty}}(g) = \sum_{\gamma \in U_{n-1}(F) \backslash \text{GL}_{n-1}(F)} \int_{(F \backslash A_F)^{n-1}} \psi(u(\gamma X-X)) \, dX \int_{[U_{n-1}]} W_{\xi}(\left( \begin{array}{c} \gamma \\ 1 \end{array} \right) u_2 g) \overline{\psi}(u_2) \, du_2.
\]

Let \( \Phi \in W \) be the automorphic form corresponding to \( \xi_\infty \otimes \eta_1 \). Then it follows from the above lemma and (3.7) that

\[
W_{\xi}(\left( \begin{array}{c} h \\ 1 \end{array} \right)) = \sum_{i=1}^{m} W_{\Phi_i}(\left( \begin{array}{c} h \\ 1 \end{array} \right)) \quad \text{for } h \in \text{GL}_{n-1}(A_F).
\]

Choosing \( \xi_\infty \) so that \( W_{\xi_\infty}(I_n) \neq 0 \), we evaluate this at \( (I_{n-1}, h) \) for \( h \in \text{GL}_{n-1}(A_{F,I}) \) and cancel the factor of \( W_{\xi_\infty}(I_n) \) on both sides to get

\[
\prod_{v < \infty} W_{\xi_v}(h_v) = \sum_{i=1}^{m} W_{\eta_i}(h) \quad \text{for } h \in \text{GL}_{n-1}(A_{F,I}).
\]

For a finite \( v \in T \), according to [11, Prop. 3.2], we can choose \( h_v^0 \in \text{GL}_{n-1}(F_v) \) so that \( W_{\xi_v}(h_v^0) \neq 0 \). Then since \( \eta_i \) is of the form \( \eta_{k,T} \otimes \bigotimes_{v \not\in T} \eta_{i,v}^0 \), we may evaluate the above at \( (\prod_{v \in T} h_v^0, h) \) for \( h \in \text{GL}_{n-1}(A_{F,I}) \) and divide by \( \prod_{v \not\in T} W_{\xi_v}(h_v^0) \), to obtain

\[
\prod_{v \in T} W_{\xi_v}(h_v) = \sum_{i=1}^{m} c_i \prod_{v \not\in T} W_{\eta_i,v}(h_v) \quad \text{for } h \in \text{GL}_{n-1}(A_{F,I}),
\]

for some constants \( c_i \in \mathbb{C} \).

Note that \( \text{GL}_{1}(A_F) = A_F^\times \) is a subgroup of \( \text{GL}_{n-1}(A_F) \) as described at the beginning of the proof of Lemma 3.6. So, for any unramified idèle class character \( \omega \) of \( F \), we may multiply (3.8) by \( \omega(h) \| \det h \|^{-\frac{s-n+1}{2}} \) for \( h \in (A_F^\times)^n \), integrate over \( h \) and use (3.2) to get

\[
L_T^T(s, \pi \otimes \omega) = \sum_{i=1}^{m} c_i L_T^T(s, \pi_i \otimes \omega).
\]
Here $L^T$ denotes the partial Euler product over all places $v \notin T$, viz. (3.10)
\[ L^T(s, \pi \otimes \omega) = \prod_{v \notin T} \prod_{j=1}^{n} \frac{1}{1 - \alpha_{0,v,j} \omega_v(D_v) q_v^{-s}}, \quad L^T(s, \pi_i \otimes \omega) = \prod_{v \notin T} \prod_{j=1}^{n} \frac{1}{1 - \alpha_{i,v,j} \omega_v(D_v) q_v^{-s}}, \]
where $\alpha_{0,v,j}$ (resp. $\alpha_{i,v,j}$) are the Satake parameters of $\pi_v$ (resp. $\pi_{i,v}$).

4. SOME MULTIPLICATIVE NUMBER THEORY

In this section, we extend some of the basic notions of multiplicative number theory to number fields. First, let $\mathcal{I}_F$ denote the set of non-zero integral ideals of $\mathcal{O}_F$ and $\mathcal{P}_F \subset \mathcal{I}_F$ the set of prime ideals. Let $D_F$ denote the set of all functions $f : \mathcal{I}_F \to \mathbb{C}$. Given $f, g \in D_F$, we define their Dirichlet convolution $f \ast g \in D_F$ via
\[ f \ast g(a) = \sum_{b \in \mathcal{I}_F \atop b \geq a} f(b)g(ab^{-1}), \quad \forall a \in \mathcal{I}_F. \]
This gives $D_F$ the structure of a commutative ring with multiplicative identity
\[ 1_{D_F}(a) = \begin{cases} 1 & \text{if } a = \mathcal{O}_F, \\ 0 & \text{otherwise}. \end{cases} \]
In fact, one can show that $D_F$ is an integral domain, though we will not need that in what follows.

For any $f \in D_F$, we say that
(i) $f$ is multiplicative if $f(\mathcal{O}_F) = 1$ and $f(ab) = f(a)f(b)$ for every $a, b \in \mathcal{I}_F$ satisfying $a + b = \mathcal{O}_F$;
(ii) $f$ has polynomial growth if there exists $\sigma \in \mathbb{R}$ such that $f(a) = O(N(a)^\sigma)$ for all $a \in \mathcal{I}_F$;
(iii) $f$ is $p$-finite if there is a finite set $S \subset \mathcal{P}_F$ such that $f(a) = 0$ for every $a \in \mathcal{I}_F$ which is contained in a prime ideal $p \in \mathcal{P}_F \setminus S$.

Further, we call multiplicative functions $f, g \in D_F$ equivalent if there is a finite set $S \subset \mathcal{P}_F$ such that $f(p^k) = g(p^k)$ for all $p \in \mathcal{P}_F \setminus S$ and all $k \geq 1$, and inequivalent otherwise.

Finally, let $M_F \subset D_F$ and $R_F \subset D_F$ denote the subsets of multiplicative and $p$-finite elements, respectively. It is easy to verify that $R_F$ is a subring of $D_F$ and $M_F$ is a subgroup of the unit group $D_F^\times$.

**Lemma 4.1** (adapted from [13], Thm. 2). Let $m$ be a positive integer and let $f_1, \ldots, f_m \in M_F$ be pairwise inequivalent, multiplicative functions. Then $f_1, \ldots, f_m$ are linearly independent over $R_F$, i.e. if $c_1, \ldots, c_m \in R_F$ satisfy $\sum_{j=1}^{m} c_j f_j = 0$, then $c_1 = \ldots = c_m = 0$ identically.

**Proof.** Suppose otherwise, and let $f_1, \ldots, f_m \in M_F$ and $c_1, \ldots, c_m \in R_F$ be a counterexample with $m$ minimal; in particular, none of $c_1, \ldots, c_m$ vanishes identically. Since all elements of $M_F$ are units in $D_F$, we must have $m > 1$. Let $S \subset \mathcal{P}_F$ be a finite set of primes such that $c_j(a) = 0$ for $j = 1, \ldots, m$ whenever $a$ has a prime factor outside of $S$. Since $f_1$ and $f_2$ are inequivalent, there exists $p \in \mathcal{P}_F \setminus S$ and $k \in \mathbb{Z}_{>0}$ such that $f_1(p^k) \neq f_2(p^k)$. 


We consider the equation

\[ \sum_{j=1}^{m} c_j \ast f_j(a) = 0 \]

with \( a = p^k b \) for all \( b \) co-prime to \( p \), obtaining

\[ \sum_{j=1}^{m} f_j(p^k) \sum_{n \geq b} c_j(n) f_j(bn^{-1}) = 0. \]

Next we replace \( a \) by \( b \) in (4.1), multiply by \( f_1(p^k) \), and subtract (4.2) to get

\[ \sum_{j=2}^{m} (f_1(p^k) - f_j(p^k)) \sum_{n \geq b} c_j(n) f_j(bn^{-1}) = 0. \]

Finally, for \( j = 2, \ldots, m \) we define

\[ \tilde{c}_j(a) = (f_1(p^k) - f_j(p^k))c_j(a) \quad \text{and} \quad \tilde{f}_j(a) = \begin{cases} 0 & \text{if } a \subseteq p, \\ f_j(a) & \text{otherwise,} \end{cases} \]

so that \( \tilde{f}_j \in M_F, \tilde{c}_j \in R_F, \tilde{c}_2 \) is not identically 0, and

\[ \sum_{j=2}^{m} \tilde{c}_j \ast \tilde{f}_j = 0. \]

This contradicts the minimality of \( m \) and completes the proof. \( \square \)

When \( F = \mathbb{Q} \), it is well known that one can identify any \( f \in D_F \) of polynomial growth with its Dirichlet series \( \sum_{n=1}^{\infty} f(n\mathbb{Z})n^{-s} \), which defines a holomorphic function in a right half-plane. When \( F \neq \mathbb{Q} \), the map \( f \mapsto \sum_{a \in I_F} f(a)N(a)^{-s} \) is still a ring homomorphism, but it is no longer injective since there may be multiple ideals with the same norm. However, we recover a one-to-one correspondence if we include the twists by unramified Größencharakteres, as the following lemma shows.

**Lemma 4.2.** Let \( f \in D_F \) be a function of polynomial growth. Suppose that there exists \( \sigma \in \mathbb{R} \) such that

\[ \sum_{a \in I_F} f(a)\chi_\omega(a)N(a)^{-s} = 0 \]

for every unramified, unitary, idèle class character \( \omega \) and all \( s \in \mathbb{C} \) with \( \Re(s) > \sigma \). Then \( f = 0 \) identically.

**Proof.** Collecting the terms with a common value of \( N(a) \), we have

\[ \sum_{n=1}^{\infty} n^{-s} \sum_{\substack{a \in I_F \\ N(a)=n}} f(a)\chi_\omega(a) = 0 \]

for all unramified unitary characters \( \omega \) and all \( s \) with \( \Re(s) \) sufficiently large. Considering the asymptotic behavior as \( s \to \infty \), we find that \( \sum_{a \in I_F \atop N(a)=n} f(a)\chi_\omega(a) \) vanishes for all unramified \( \omega \).
Fix a choice of \( n \), and let \( a_1, \ldots, a_m \in \mathcal{I}_F \) be the ideals of norm \( n \). It suffices to show that \( \mathbb{C}^m \) is spanned by the vectors \( (\chi_\omega(a_1), \ldots, \chi_\omega(a_m)) \), with \( \omega \) running through all unramified characters. If that is not the case then there exist \( c_1, \ldots, c_m \in \mathbb{C} \), not all zero, such that \( c_1 \chi_\omega(a_1) + \ldots + c_m \chi_\omega(a_m) = 0 \) for all such \( \omega \). Reordering if necessary, we may assume that \( c_j \neq 0 \) for \( 1 \leq j \leq k \) and \( c_j = 0 \) for \( k < j \leq m \). Further, by scaling we may assume that \( c_1 = 1 \), so that

\[
1 + c_2 \chi_\omega(a_2 a_1^{-1}) + \ldots + c_k \chi_\omega(a_k a_1^{-1}) = 0.
\]

Since this holds for all unramified characters \( \omega \), we are free to replace \( \omega \) by any unramified twist \( \omega \omega' \). In particular, letting \( \omega' \) run through all characters of the class group and taking the average, all terms for which \( a_j a_i^{-1} \) is not a principal fractional ideal vanish. Thus, we may assume without loss of generality that \( a_j a_i^{-1} \) is principal for each \( j = 2, \ldots, k \), so that \( a_j a_i^{-1} = (\gamma_j) \) for some \( \gamma_j \in F^\times \). Since \( \omega \) is unramified, we have \( \chi_\omega(a_j a_i^{-1}) = \chi_\omega((\gamma_j)) = \omega_\infty(\gamma_j)^{-1} \), so the above becomes

\[
1 + c_2 \omega_\infty(\gamma_2)^{-1} + \ldots + c_k \omega_\infty(\gamma_k)^{-1} = 0.
\]

Next we replace \( \omega \) by the twist \( \omega(\omega')^\ell \), take the average over \( \ell \in \{0, 1, \ldots, L - 1\} \), and let \( L \to \infty \). Recall that any idèle class character is unitary up to a power of \( \| \cdot \| \); since \( \| \gamma_j \|_\infty = N(a_j a_i^{-1}) = 1 \), it follows that \( \omega_\infty'(\gamma_j) \) is a complex number of modulus 1, so that

\[
\lim_{L \to \infty} \frac{1}{L} \sum_{\ell=0}^{L-1} \omega_\infty'(\gamma_j)^{-\ell} = \begin{cases} 1 & \text{if } \omega_\infty(\gamma_j) = 1, \\ 0 & \text{otherwise.} \end{cases}
\]

Now, for any \( j = 2, \ldots, k \), \( a_j \) and \( a_i \) are distinct ideals, so \( \gamma_j \) is not an element of \( \mathfrak{a}_F^\times \). Since we are free to choose any \( \omega_\infty' \) in the dual of \( \mathfrak{a}_F^\times \setminus \{ y \in F_\infty^\times : \| y \|_\infty = 1 \} \), we may always arrange it so that \( \omega_\infty'(\gamma_j) \neq 1 \) for a particular \( j \). Thus, by repeating the above averaging procedure, all of the terms for \( j = 2, \ldots, k \) vanish, so we are left with the absurd conclusion \( 1 = 0 \). This completes the proof. \( \square \)

We conclude this section with two consequences of the above for automorphic \( L \)-functions that may be of independent interest.

**Corollary 4.3** (Linear independence of automorphic \( L \)-functions). Let \( n_1, \ldots, n_m \) be positive integers, and for each \( i = 1, \ldots, m \), let \( \pi_i = \bigotimes_v \pi_{i,v} \) be an irreducible automorphic representation of \( \text{GL}_{n_i}(\mathbb{A}_F) \). For each pair \( i \neq j \), assume that there is a finite place \( v \) such that \( \pi_{i,v} \) and \( \pi_{j,v} \) are both unramified and \( \pi_{i,v} \not\cong \pi_{j,v} \). Let \( S \) be a finite set of places containing all archimedean places, and consider the partial \( L \)-functions

\[
L^S(s, \pi_i \otimes \omega) = \prod_{v \notin S} L(s, \pi_{i,v} \otimes \omega_v),
\]

where \( \omega \) is an unramified idèle class character. Then, if \( c_0, \ldots, c_m \in \mathbb{C} \) are such that

\[
(4.3) \quad c_0 + c_1 L^S(s, \pi_1 \otimes \omega) + \ldots + c_m L^S(s, \pi_m \otimes \omega) = 0
\]

for every unramified \( \omega \), then \( c_0 = \ldots = c_m = 0 \).

**Proof.** Let \( \lambda_{\pi_i}(a) \) denote the Dirichlet coefficients of \( L(s, \pi_i) \), so that

\[
L(s, \pi_i \otimes \omega) = \sum_{a} \lambda_{\pi_i}(a) \chi_\omega(a) N(a)^{-s}
\]
for \( \Re(s) > 0 \) sufficiently large. Next let \( \mathfrak{m} \) be the product of the prime ideals corresponding to the finite places in \( S \), and define

\[
\lambda^S_{\pi_i}(a) = \begin{cases} 
\lambda_{\pi_i}(a) & \text{if } a + \mathfrak{m} = \mathfrak{o}_F, \\
0 & \text{otherwise,}
\end{cases}
\]

so that

\[
L^S(s, \pi_i \otimes \omega) = \sum_a \lambda^S_{\pi_i}(a)\chi_\omega(a)N(a)^{-s}.
\]

By Lemma 4.2, the linear relation (4.3) implies that \( c_01_{DF}(a) + c_1\lambda^S_{\pi_1}(a) + \ldots + c_m\lambda^S_{\pi_m}(a) = 0 \) identically. Moreover, by restricting \( a \) in this equality to the ideals co-prime to a fixed modulus, we are free to replace \( S \) by any larger finite set of places. In particular, we may assume without loss of generality that \( S \) contains all finite places of ramification of \( \pi_1, \ldots, \pi_m \).

Next, following the proof of Lemma 3.5, for each \( i \) there is a unique isobaric automorphic representation \( \Pi_i = \bigotimes \Pi_{i,v} \) of \( \text{GL}_{n_i}(\mathbb{A}_F) \) such that \( \Pi_{i,v} \cong \pi_i \) for every finite place \( v \) at which \( \pi_{i,v} \) is unramified. Then for any \( i \neq j \), by hypothesis there is an unramified finite place \( v \) for which \( \Pi_{i,v} \cong \pi_{i,v} \neq \pi_{j,v} \cong \Pi_{j,v} \), so that \( \Pi_i \not\cong \Pi_j \).

Finally, since \( S \) contains all ramified finite places, we are free to replace \( \pi_i \) by \( \Pi_i \), so we may assume without loss of generality that \( \pi_i \) is isobaric. Then, by strong multiplicity one for isobaric representations, \( \lambda^S_{\pi_1}, \ldots, \lambda^S_{\pi_m} \) are pairwise inequivalent, multiplicative elements of \( DF \). Moreover, for every unramified place \( v \), \( L(s, \pi_{i,v}) \) is not identically \( 1 \), so each \( \lambda^S_{\pi_i} \) is also inequivalent to the identity \( 1_{DF} \). The conclusion now follows from Lemma 4.1. \( \square \)

**Corollary 4.4** (Algebraic independence of cuspidal automorphic \( L \)-functions). Assume the hypotheses of Corollary 4.3, and suppose that \( \pi_1, \ldots, \pi_m \) are cuspidal. Then, if \( P \in \mathbb{C}[x_1, \ldots, x_m] \) is such that

\[
P(L^S(s, \pi_1 \otimes \omega), \ldots, L^S(s, \pi_m \otimes \omega)) = 0
\]

for every unramified \( \omega \), then \( P = 0 \) identically.

**Proof.** We may write \( P = \sum_{e_1, \ldots, e_m} c_{e_1, \ldots, e_m} x_1^{e_1} \cdots x_m^{e_m} \) as a linear combination of monomials. For each non-zero \( m \)-tuple \( (e_1, \ldots, e_m) \), we may define

\[
\Pi_{e_1, \ldots, e_m} = \pi_1^{e_1} \boxplus \cdots \boxplus \pi_1^{e_1} \boxplus \cdots \boxplus \pi_m^{e_m} \boxplus \cdots \boxplus \pi_m^{e_m}.
\]

Then, by [8], the \( \Pi_{e_1, \ldots, e_m} \) are pairwise non-isomorphic isobaric representations satisfying

\[
L^S(s, \Pi_{e_1, \ldots, e_m} \otimes \omega) = L^S(s, \pi_1 \otimes \omega)^{e_1} \cdots L^S(s, \pi_m \otimes \omega)^{e_m},
\]

and the conclusion follows from Corollary 4.3 applied to these. \( \square \)

**Remark.** This result should be compared to that of Jacquet and Shalika [8], who proved the multiplicative independence of cuspidal \( L \)-functions, and thus showed the existence of the class of isobaric representations. In our notation, this means that for any solution to

\[
L^S(s, \pi_1 \otimes \omega)^{c_1} \cdots L^S(s, \pi_m \otimes \omega)^{c_m} = 1,
\]

where the \( \pi_i \) are pairwise non-isomorphic cuspidal representations and \( c_1, \ldots, c_m \in \mathbb{C} \), one has \( c_1 = \ldots = c_m = 0 \). (Here we interpret \( L^S(s, \pi_i \otimes \omega)^{c_i} \) to mean \( \exp(c_i \log L^S(s, \pi_i \otimes \omega)) \), where \( \log L^S(s, \pi_i \otimes \omega) \) is the unique logarithm with zero constant term in its expansion as a Dirichlet series.) In particular, taking the \( c_i \) to be integers, one sees that for any isobaric
representation $\pi$, $L^S(s, \pi \otimes \omega)$ has a unique factorization into products of cuspidal $L$-functions $L^S(s, \pi_i \otimes \omega)$. Note that Corollary 4.4 constitutes a strengthening of this particular case, from multiplicative independence to algebraic independence.

5. CONCLUSION OF THE PROOF

By Lemma 4.2 and (3.9)–(3.10), there are unique functions $f_0, f_1, \ldots, f_m \in M_F$ such that

$$\sum_{a \in I_F} f_0(a) \chi_\omega(a) N(a)^{-s} = L^T(s, \pi \otimes \omega)$$

and

$$\sum_{a \in I_F} f_j(a) \chi_\omega(a) N(a)^{-s} = L^T(s, \pi_j \otimes \omega)$$

for $j = 1, \ldots, m$ and all unramified idèle class characters $\omega$, and they are related by the identity

$$f_0 = \sum_{j=1}^m c_j f_j. \quad (5.1)$$

By collecting common terms of (3.9) if necessary, we may assume without loss of generality that the $\pi_j$ are pairwise non-isomorphic. Then strong multiplicity one for isobaric representations (see Remark 1) implies that, for any $i \neq j$, the local $L$-factors $L(s, \pi_{i,v})$ and $L(s, \pi_{j,v})$ differ at infinitely many places, and it follows that $f_1, \ldots, f_m$ are pairwise inequivalent. Thus, by Lemma 4.1, $f_0$ must be equivalent to $f_j$ for some $j \in \{1, \ldots, m\}$, and by reordering if necessary we may assume that $f_0$ is equivalent to $f_1$.

Let $S \subset \mathcal{P}_F$ be the finite set of primes $p$ for which $f_0(p^k) \neq f_1(p^k)$ for some $k \geq 1$. For $j = 0, 1$ we factor $f_j$ as $f_j^\flat \ast f_j^\sharp$, where $f_j^\flat, f_j^\sharp \in M_F$ are the unique multiplicative functions satisfying

$$f_j^\flat(p^k) = \begin{cases} f_j(p^k) & \text{if } p \in S, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_j^\sharp(p^k) = \begin{cases} 0 & \text{if } p \in S, \\ f_j(p^k) & \text{otherwise}. \end{cases}$$

Note that $f_0^\flat$ and $f_1^\flat$ are $p$-finite, and $f_0^\sharp = f_1^\sharp$, so we may rewrite (5.1) in the form

$$(c_1 f_1^\flat - f_0^\flat) \ast f_1^\sharp + \sum_{j=2}^m c_j f_j = 0.$$

Invoking Lemma 4.1 again, we see that $c_j = 0$ for $j = 2, \ldots, m$, and thus (5.1) becomes

$$f_0 = c_1 f_1.$$ 

Evaluating both sides at $\mathfrak{a}_F$, we find that $c_1 = 1$, and it follows that $L(s, \pi_v) = L(s, \pi_{1,v})$ for all places $v \notin T$. Since $\pi_v$ is unramified for all $v \notin T$, we conclude that $\pi_v \cong \pi_{1,v}$, as desired.
References


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