ON THE FOURIER COEFFICIENTS OF AUTOMORPHIC FORMS ON
GL₂

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Abstract. Given $E/F$ a quadratic extension of number fields and a cuspidal representation
$p$ of $GL₂(𝔸_E)$, we give a full description of the fibers of the Asai transfer of $p$. We then
determine the extent to which the Hecke eigenvalues of all the Hecke operators indexed by
integral ideals in $F$ determine the representation $p$.

1. Introduction

Let us fix $E/F$ a quadratic extension of number fields and let $Γ_F$ denote the absolute
Galois group of $F$. For any number field $k$, we write $𝔸_k$ to denote its ring of adeles, and $ℂ_k$
to denote its idèle class group. Consider the representation

$$r : GL₂(ℂ) \times GL₂(ℂ) \rtimes Γ_F \longrightarrow GL₄(ℂ) \times Γ_F$$

given by

$$r(x, y, γ) = \begin{cases} (x \otimes y, γ) & \text{if } γ \text{ restricted to } E \text{ is trivial} \\ (y \otimes x, γ) & \text{if } γ \text{ restricted to } E \text{ is not trivial}. \end{cases}$$

This is a special case of a general construction known as tensor induction. (See [5, §13].)
We call $r$ the Asai representation (with respect to $E/F$) in this paper. Since
$GL₂(ℂ) \times GL₂(ℂ) \rtimes Γ_F$ is the Langlands dual group of the restriction of scalars $R_{E/F}GL₂$, we can view
$r$ as a homomorphism of $L$-groups. The corresponding global functoriality is established in
the complementary, but independant works of the author [12] and Dinakar Ramakrishnan
[19]. Given a cuspidal representation $p$ of $GL₂(𝔸_E)$, let $As(p)$ denote this functorial transfer,
called the Asai transfer of $p$; it is an isobaric automorphic representation of $GL₄(𝔸_F)$. We
refer the reader to §2 for more details.

In [12], we described the fibers of the Asai transfer when $p$ is not dihedral, i.e., not
automorphically induced from a character. In this paper, we address the dihedral case, and
thereby completing the description of the fibers of the Asai transfer. Then, as a consequence,
we obtain a “multiplicity one” type result for cuspidal representations of $GL₂(𝔸_E)$. To be
precise, if $p$ and $p'$ are two cuspidal representations of $GL₂(𝔸_E)$ such that their Hecke
eigenvalues for all the Hecke operators coming from integral ideals of $F$ are identical, we
then show that $p^\gamma \otimes ν \simeq p', \, γ \in Gal(E/F)$, for some idèle class character $ν$ of $E$ with
$(ν|ℂ_F) = 1$. (See Theorem 4.0.2.) For any unitary cuspidal representation $p$ of $GL₂(𝔸_E)$
with conductor $c$, let $λ_p(m), \, m ⊂ ℓₕE$, denote the Hecke eigenvalues of $p$ for the Hecke
operators $T_c(m)$. (See §3.2 for the definition of $T_c(m)$ and $λ_p(m)$..) If $ω_p$ (resp. $ω_p'$) denotes
the central character of $p$ (resp. $p'$), we note that our hypothesis on $p$ and $p'$ simply mean
that $ω_p|ℂ_F = ω_{p'}|ℂ_F$ and that $λ_p(n) = λ_{p'}(n)$ for all $n ⊂ ℓₕF$. 

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The key point is that our hypothesis on \( \pi \) and \( \pi' \) guarantee that both \( \pi \) and \( \pi' \) have the same Asai transfer. The problem then becomes one of describing the fibers of the Asai transfer. Theorems 4.0.2 and Theorem 4.0.3 seem to be of considerable interest to researchers in analytic number theory. In fact the problem was brought to the author’s attention by Dinakar Ramakrishnan based on his conversations with Duke and Kowalski.

We will now elaborate on the methods used. Suppose \( \pi \) and \( \pi' \) are such that \( \pi^\gamma \otimes \nu \simeq \pi' \) for some idèle class character of \( \nu \) of \( E \) with \( \nu|_{C_F} = 1 \), then it is easy to see that \( \text{As}(\pi) \simeq \text{As}(\pi') \). In [12], we proved the converse, at least when both \( \pi \) and \( \pi' \) are not dihedral. As mentioned earlier, we address the dihedral case in this paper. Namely, if \( \pi \) is a dihedral cuspidal representation of \( GL_2(A_E) \), let \( \sigma \) be the corresponding two dimensional representation of the global Weil group \( W_E \). If \( \pi' \) is another cuspidal representation of \( GL_2(A_E) \) such that \( \text{As}(\pi) \simeq \text{As}(\pi') \), then \( \pi' \) is also dihedral as was shown in [77, [12]]; let \( \sigma' \) be the two dimensional representation of \( W_E \) corresponding to \( \pi' \). Now, the determination of the fibers of the Asai transfer can be carried out on the “Weil-group” side. Namely, our hypothesis yields that \( \text{As}(\sigma) \simeq \text{As}(\sigma') \). Here \( \text{As}(\sigma) \) (resp. \( \text{As}(\sigma') \)) is the four dimensional representation of \( W_F \) obtained via tensor induction from \( \sigma \) (resp. \( \sigma' \)) (c.f. §2). By restricting these representations to \( W_E \), we get \( \sigma \otimes \sigma^\theta \simeq \sigma' \otimes \sigma'^\theta \), where \( \theta \) is the non-trivial Galois automorphism of \( E/F \). A result by Rajan [17] asserts that \( H^2(W_E, C^\times) = 1 \) for the trivial \( W_E \) action, (note that \( W_E \) is not a profinite group), it is then a standard argument to see that the pair \( (\sigma, \sigma^\theta) \) is determined uniquely up to a twist by a character \( (\chi, \chi^{-1}) \). (See Lemma 2.0.1.) Thus one concludes that \( \pi^\gamma \otimes \chi \simeq \pi', \gamma \in \{1, \theta\} \), for some idèle class character \( \chi \) of \( E \).

The rest of the argument is to show that one can in fact choose \( \chi \) so that \( \chi|_{C_F} = 1 \). The idea here is to analyze the poles of the partial \( L \) function \( L^S(s, \Pi \times \bar{\Pi}) \) at \( s = 1 \), where \( \Pi = \text{As}(\pi) \simeq \text{As}(\pi') \), through the decomposition of \( \Pi \) into its isobaric constituents. On the one hand, we have \( \text{ord}_{s=1} L^S(s, \Pi \times \bar{\Pi}) = 2, 3, \) or \( 4 \). On the other hand, we have the identity \( L^S(\Pi \times \bar{\Pi}) = L^S(s, \pi \times \bar{\pi}', R) \), where \( R \) is the obvious analogue of \( r \) defined from the \( L \)-group of \( R_{E/F}GL_4 \) to that of \( GL_{16} \) via tensor induction. This identity is simply a reflection of the fact that tensor induction commutes with taking tensor products and duals. Now, one can factorize \( L^S(s, \pi \times \bar{\pi}', R) \) further using some general properties of tensor induction, namely its behaviour with respect to sums of representations, and analyze the order of the pole at \( s = 1 \). Finally, a comparison of the order using the resulting identity of \( L \)-functions yields the desired result. (See the proof of Theorem 2.0.4.) Needless to say, a key ingredient in our analysis is the cuspidality criterion of \( \text{As}(\pi) \) [19, 20, 16].

More recently, Dipendra Prasad and U.K. Anadavaradhan have given a group-theoretic proof of Theorem 2.0.4 in [1, Section 5] which in fact is a better proof. However, our proof here which is more in line with the proof of the analogous theorem in the non-dihedral case [12] has its own benefits. For instance, it gives a full account of the isobaric constituents of \( \Pi \). (This basically amounts to giving the multiplicity of the trivial representation in \( \Pi \times \bar{\Pi} \).)

We now briefly describe the organization of the paper. In §2, we recall the notion of the Asai transfer, both on the automorphic and on the “Weil-group” side. In the dihedral case, we discuss the properties of the Asai transfer and its fibers in detail. In §3, which can be read independent of §2, we present two kinds of Dirichlet series that one can attach to a cuspidal representation of \( GL_2(A_E) \). To be precise, in §3.1, we review the theory of new vector for \( GL_2 \) and introduce the normalized Fourier coefficients \( \lambda_n(m) \) associated to a cuspidal representation \( \pi \). In §3.2, we recall the notion of Hecke operators, and in §3.3, we
introduce the relevant Dirichlet series associated to a cuspidal representation of $GL_2(\mathbb{A}_E)$. We prove an important identity of $L$-functions (cf. Lemma 3.3.1) that allows us to reduce our problem to the determination of the fibers of the Asai transfer. We note that §3 can be read independent of §2. Finally, in §4, using the multiplicity one theorem for isobaric automorphic representations [11, 10], we prove the main result alluded to at the beginning of this introduction. It is worth pointing out that it is sufficient to require that $\lambda_\pi(p) = \lambda_{\pi'}(p)$ for almost all prime ideals $p$ of $F$ (cf. Theorem 4.0.2 and Remark 4.0.1).

A word about the notation: Since $E/F$ is a fixed quadratic extension, “Asai” or “tensor induction” is always taken with respect to $E/F$. In §2, for any character $\xi$, we write $\xi_0$ for its restriction to $C_F$ whenever it makes sense. Also, while writing the Hecke $L$-functions in this section, we sometimes include the field of definition as a subscript, for example $L_k(s, \chi)$, where $\chi$ is a idèle class character of $k$.

History and Acknowledgments. This paper has an interesting history. An earlier draft of this paper was submitted for publication in the Shahidi volume on 14 May 2008 and was accepted for publication on August 1, 2008. However, due to some purely technical reasons, it was not included in the volume which came out in print in June, 2011. In the meantime (April, 2011 to be precise), Dipendra Prasad pointed out to the author that Dinakar Ramakrishnan’s cuspidality criterion [19, 18] is incomplete in the dihedral case. Since then, both Dinakar Ramakrishnan and Dipendra Prasad have provided the author with different proofs of the revised cuspidality criterion (see Theorem 2.0.3). This is the subject matter of their forthcoming paper [16]. Here, we have fixed our (earlier) proof of Theorem 2.0.4 by incorporating the new cuspidality criterion.

I thank Dinakar Ramakrishnan for bringing the problem to the author’s attention. The author is indebted to both Dinakar Ramakrishnan and Dipendra Prasad for communicating a detailed proof of the cuspidality criterion in the dihedral case. I would also like to thank Phil Kutzko and Freydoon Shahidi for useful discussions while writing this paper. It gives me great pleasure to dedicate this article to Freydoon Shahidi. I have been fortunate to have studied mathematics under his tutelage and I am grateful for his support and guidance over the years. Thanks are also due to the referee for useful comments towards improved presentation of the paper. Finally, the author is grateful to James Cogdell for his courtesy.

2. The Asai transfer

We fix $E/F$ a quadratic extension of number fields. For any number field $k$, we write $A_k$ for its ring of adèles and $A_k^\times$ for its group of idèles. For a place $u$ of $k$, we write $k_u$ to denote the completion of $k$ at $u$. Now, consider the algebraic group $R_{E/F}GL_2$ given by the restriction of scalars; then its dual group $L(R_{E/F}GL_2)$ is given by $GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \rtimes \Gamma_F$, where $\Gamma_F$ denotes the absolute Galois group of $F$. There is a natural four dimensional representation

$$r : GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \rtimes \Gamma_F \longrightarrow GL(\mathbb{C}^2 \otimes \mathbb{C}^2)$$

given by

$$r(x, y, \gamma) = \begin{cases} (x \otimes y) & \text{if } \gamma \text{ restricted to } E \text{ is trivial} \\ (y \otimes x) & \text{if } \gamma \text{ restricted to } E \text{ is not trivial.} \end{cases}$$
This representation in turn yields a $L$-group homomorphism, also denoted as $r$, from $L(R_{E/F}GL_2)$ to $^LG_4$. For each place $v$ of $F$, we let $r_v$ denote the corresponding local $L$-group homomorphism obtained from $r$ via restriction. Let $W'_{F_v}$ denote the Weil group if $v$ is archimedean, and the Weil-Deligne group if $v$ is non-archimedean.

Let $\pi$ be an irreducible cuspidal representation of $GL_2(\mathbb{A}_F)$. Then we may consider $\pi$ as a representation of $R_{E/F}GL_2(\mathbb{A}_F)$ and factorize it as a restricted tensor product, namely, $\pi = \otimes_v \pi_v$, where each $\pi_v$ is an irreducible admissible representation of $GL_2(E \otimes F_v)$. For every place $v$ of $F$, let $\phi_v : W'_{F_v} \longrightarrow GL_2(C) \times GL_2(C) \times \Gamma_{F_v}$ be the local parameter attached to $\pi_v$ under the local Langlands correspondence. Now, for each place $v$ of $F$, let $As(\pi_v)$ be the irreducible admissible representation of $GL_4(F_v)$ corresponding to the parameter $r_v \circ \phi_v$ under the local Langlands correspondence. Now, set $As(\pi) := \otimes_v As(\pi_v)$ – it is an irreducible admissible representation of $GL_4(\mathbb{A}_F)$. By [12, Theorem 6.7], or see [19], one knows that $As(\pi)$ is an isobaric automorphic representation of $GL_4(\mathbb{A}_F)$. We refer to $As(\pi)$ as the Asai transfer of $\pi$.

For each place $v$ of $F$, let $L(s, \pi_v, r_v)$ be the local $L$-function obtained by applying the Langlands-Shahidi method to the case $^2A_3 = 2$ (see [12, §4]), then by Proposition 6.2, 6.5, and Corollary 6.8 of [12], we have

\[(2.0.1) \quad L(s, \pi_v, r_v) = L(s, As(\pi_v))\]

for all $v$. If $v$ splits as $(w_1, w_2)$ in $E$, then $L(s, As(\pi_v))$ is the usual Rankin-Selberg $L$-function $L(s, \pi_{w_1} \times \pi_{w_2})$. Further, if $v$ is archimedean, or if $v$ is unramified and $\pi_v$ is spherical, then it follows from the works of Shahidi [22], [23] that the local factor $L(s, \pi_v, r_v)$ equals $L(s, r_v \circ \phi_v)$, where $L(s, r_v \circ \phi_v)$ is the local factor attached to the Weil representation $r_v \circ \phi_v$ [26]. (We also refer the reader to [12] for the relevant unramified calculations.)

If $v$ is a finite place that does not split in $E$, say $w|v$ is the unique place of $E$ lying over $v$, then $\pi_v = \pi_w$ is an irreducible unitary generic representation of $GL_2(E_w)$. It is well known that $\pi_v$ must be one of the following type: supercuspidal, Steinberg, or an irreducible principal series representation. If $\pi_v$ is supercuspidal or Steinberg, then $L(s, \pi_v, r_v)$ is computed by Goldberg [7, Theorems 5.2 and 5.6]. If $\pi_v$ is an irreducible principal series representation, namely, $\pi_v = I(\chi_1, \chi_2); \chi_1, \chi_2$, are characters of $E_w^\times$, then using Shahidi’s result on multiplicativity of local factors [24] one checks that

$$L(s, \pi_v, r_v) = L_{F_v}(s, \chi_1)L_{F_v}(s, \chi_2)L_{E_w}(s, \chi_1 \chi_2),$$

where the local factors on the right hand side are those of Hecke (cf. [26]). It is worth mentioning that for our purposes in this paper it suffices to just know the unramified computation of the local factors.

Next, we consider the case when $\pi$ is dihedral – our discussion in [12, §7] in this case is incomplete and warrants more explanation. So, from now on, unless mentioned otherwise, let us assume that $\pi$ is dihedral, i.e., $\pi = \pi \otimes \chi$, for some idèle class character $\chi$ of $E$. (For any number field $k$, we write $W_k$ to denote its global Weil group.) Then there is a two dimensional representation $\sigma : W_E \longrightarrow GL(V)$ such that

$$L(s, \pi) = L(s, \sigma),$$

where $L(s, \sigma)$ is the global $L$-function defined as in [26, §3]. In fact, $\sigma$ is irreducible and is induced from a character of an open subgroup $W_M \subset W_E$ of index 2, namely $\sigma = ind_{W_M}^{W_E} \mu$, $\mu^\tau \neq \mu$, $\tau \in Gal(M/E)$, $\tau \neq 1$ and $\pi$ is obtained by automorphic induction of $\mu$ [9]. (Note
that \( \mu \) can be identified with a character of \( C_M \) via the reciprocity isomorphism \( C_M \simeq W_{ab}^M \).

Using this \( \sigma \), one can then define a four dimensional representation \( As(\sigma) \) of \( W_F \) in the following two ways:

First, we define \( As(\sigma) \) via tensor induction \([5, \S 13]\). (Recall for any number field \( k \), the global Weil group \( W_k \) comes equipped with a continuous homomorphism \( \phi_k : W_k \rightarrow \Gamma_k \) with dense image.) Let us write \( W_F = W_E \cup w_0 W_E \), then \( \phi_F(w_0) \) restricts to \( \theta \) on \( E \), where \( \theta \) is the non trivial element in \( Gal(\mathbb{F} | E) \). After fixing a basis for \( V \), we can identify \( GL(V) \) with \( GL_2(\mathbb{C}) \) and hence we have the homomorphism \( \sigma : W_E \rightarrow GL_2(\mathbb{C}) \). We then define the homomorphism

\[
\tilde{\sigma} : W_F \rightarrow GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \rtimes \Gamma_F
\]

over \( \Gamma_F \) by

\[
\tilde{\sigma}(x) = \begin{cases} 
(\sigma(x), \sigma(w_0 x w_0^{-1}), \phi_F(x)) & \text{if } x \in W_E \\
(\sigma(x w_0^{-1}), \sigma(w_0 x), \phi_F(x)) & \text{if } x \notin W_E.
\end{cases}
\]

We now let \( As(\sigma) = r \circ \tilde{\sigma} \).

Alternately, we may proceed as in \([8, \S 4]\). Namely, given an irreducible two dimensional representation \( \sigma : W_E \rightarrow GL(V) \), consider \( V \) as a symplectic space via the usual determinant form

\[
det : V \times V \rightarrow \mathbb{C},
\]

then identify \( V \) with \( V^\vee = Hom(V, \mathbb{C}) \) by \( v \mapsto l_v \); \( l_v(w) = det(v, w) \); let \( E = End(V) \), then we have the isomorphisms \( V \otimes V \simeq V \otimes V^\vee \simeq E \) given by

\[
v \otimes w \mapsto v \otimes l_w \mapsto z \mapsto v l_w(z).
\]

Let \( \phi \mapsto \phi^t \) be the involution of \( E \) corresponding to \( x \otimes y \mapsto -y \otimes x \); it is easy to verify that \( \phi^t \phi = \phi \phi = det(\phi) \); now let \( q \) be the quadratic form on \( E \) defined by

\[
q(\phi, \psi) = tr(\phi^t \psi).
\]

Let \( GO(E, q) = \{g \in GL(E) : q(g \phi, g \psi) = \lambda(g) q(\phi, \psi), \lambda(g) \in \mathbb{C}^\times \} \), then \( d(g) = det(g)/\lambda(g)^2 \) defines a homomorphism from \( GO(E, q) \rightarrow \{\pm 1\} \). We let \( SGO(E, q) = \{g \in GO(E, q) : d(g) = 1\} \).

Let us define the homomorphism \( \Phi : GL(V) \times GL(V) \rightarrow SGO(E, q) \) by \( \Phi(\alpha, \beta) : \phi \mapsto \alpha \cdot \phi \cdot \beta \); it is an exercise in elementary linear algebra to check that the following diagram is commutative:

\[
\begin{array}{ccc}
GL(V) \times GL(V) & \xrightarrow{\Phi} & SGO(E, q)
\\
det \times det & \searrow & \downarrow \lambda
\\
& \downarrow \chi
\\
& \mathbb{C}^\times
\end{array}
\]

and that the kernel of \( \Phi \) is \( \mathbb{C}^\times \), embedded in \( GL(V) \times GL(V) \) via \( x \mapsto (x, x^{-1}) \). Let \( \Theta \) denote the involution \( \phi \mapsto -\phi \). We can define \( As(\sigma) : W_F \rightarrow GL(E) \) by

\[
As(\sigma)(x) = \begin{cases} 
\Phi(\sigma(x), \sigma(w_0 x w_0^{-1})) & \text{if } x \in W_E \\
\Phi(\sigma(x w_0^{-1}), \sigma(w_0 x)) \Theta & \text{if } x \notin W_E.
\end{cases}
\]

It is clear that the image of \( As(\sigma) \subset GO(E, q) \) and that

\[
As(\sigma)|_{W_E} : W_E \rightarrow SGO(E, q).
\]
Further, if we take a basis \( \{ v_1, v_2 \} \) for \( V \) and then fix the basis
\[
\{ v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2 \}
\]
for \( E \), it can be verified that the matrix of \( \Phi(\alpha, \beta) \) with respect to this basis is \( M(\alpha) \otimes M(\beta) \), where \( M(\alpha) \) (resp. \( M(\beta) \)) is the matrix of \( \alpha \) (resp. \( \beta \)) with respect to \( \{ v_1, v_2 \} \). Thus we see that the above two definitions of \( As(\sigma) \) are in fact equivalent. In particular
\[
As(\sigma)|_{W_E} \simeq \sigma \otimes \sigma^\theta,
\]
where \( \sigma^\theta \) is the representation of \( W_E \) given by \( \sigma^\theta(x) = \sigma(w_\theta x w_\theta^{-1}) \). The argument in the following Lemma is from [19, §3].

**Lemma 2.0.1.** Suppose \( \sigma, \sigma' \) are two dimensional irreducible representations of \( W_E \) such that \( As(\sigma) \simeq As(\sigma') \), where the Asai representation is with respect to the quadratic extension \( E/F \). Then there exists a character \( \chi \) of \( W_E \) such that \( \sigma^\eta \simeq \sigma' \otimes \chi \), for some \( \eta \in \text{Gal}(E/F) = \{1, \theta\} \).

**Proof.** Suppose \( V \) is the representation space of \( \sigma \); then from our discussions above, we have the exact sequence
\[
1 \longrightarrow \mathbb{C}^\times \longrightarrow GL(V) \times GL(V) \longrightarrow SGO(E, q) \longrightarrow 1;
\]
viewing this as an exact sequence of trivial \( W_E \)-modules we get the induced long exact sequence
\[
1 \longrightarrow \text{Hom}(W_E, \mathbb{C}^\times) \longrightarrow \text{Hom}(W_E, GL(V) \times GL(V)) \longrightarrow \text{Hom}(W_E, SGO(E, q)) \longrightarrow H^2(W_E, \mathbb{C}^\times).
\]
The main theorem in [17], which is a generalization of a theorem of Tate [21, §6] to Weil groups, asserts that \( H^2(W_E, \mathbb{C}^\times) \) is trivial. Consequently, the representations \((\sigma, \sigma^\theta)\) are unique upto twisting \( \sigma \rightarrow \sigma \otimes \chi, \sigma^\theta \rightarrow \sigma^\theta \otimes \chi^{-1} \) with \( \chi \) a character of \( W_E \).

Now, if \( \pi' \) is another cuspidal representation of \( GL_2(\mathbb{A}_E) \) such that \( As(\pi) \simeq As(\pi') \). By [12, Lemma 7.3], we know that \( \pi' \) is also dihedral. Say \( \sigma' \) is the corresponding two dimensional representation of \( W_E \) attached to \( \pi' \). Then by hypothesis we have \( As(\sigma) \simeq As(\sigma') \) and by restriction to \( W_E \) we also have the equivalence \( \sigma \otimes \sigma^\theta \simeq \sigma' \otimes \sigma'^\theta \). By Lemma 2.0.1, there exists an idèle class character \( \chi \) of \( E \) such that \( \sigma^\gamma \otimes \chi \simeq \sigma', \gamma \in \text{Gal}(E/F) \). Then by moving to the automorphic side we get
\[
\pi^\gamma \otimes \chi \simeq \pi'.
\]
Having taken care of the details in the dihedral case, we now recall [12, Theorem 7.1]:

**Theorem 2.0.2.** Suppose \( \pi \) and \( \pi' \) are cuspidal representations of \( GL_2(\mathbb{A}_E) \) with \( As(\pi) \simeq As(\pi') \). Then there is an idèle class character \( \nu \) of \( E \) such that
\[
\pi^\gamma \otimes \nu \simeq \pi'
\]
for some \( \gamma \in \Gamma_{E/F} = \{1, \theta\} \). If \( \pi \) is nondihedral, then \( \nu|_{C_F} = 1 \).

The main goal for the remainder of this section is to complete the description of the fibers in the dihedral case, namely, that \( \nu \) can be chosen so that \( \nu|_{C_F} = 1 \), under the assumption that the central characters of \( \pi \) and \( \pi' \) agree upon restriction to \( C_F \). We adopt the following notation in what follows: For any quadratic extension \( K/k \) of number fields and a cuspidal representation \( \tau \) of \( GL_n(\mathbb{A}_K) \), we write \( I_K^k(\tau) \) to denote the automorphic representation of
we write \((\tau)_K\) to denote the automorphic representation of \(GL_n(A_K)\) obtained by base change transfer. (See [2].) We will use the basic properties of automorphic induction and base change (in the cyclic case), such as their fibers and cuspidality criterion, with no further mention. (See [18, Proposition 2.3.1] for a summary of these properties.) Also, we remind the reader that the notation \(\xi_0\) refers to the restriction of \(\xi\) to \(C_F\), whenever the notion makes sense.

We need a theorem due to Dinakar regarding the cuspidality of \(\text{As}(\pi)\). As noted earlier, there is a minor error in Dinakar’s proof [20] in the dihedral case which was pointed out to the author by Dipendra Prasad. Subsequently, both Dipendra and Dinakar have independently communicated to the author the correct version of the cuspidality criterion which is stated below. This is expected to appear in a forthcoming paper of theirs.

**Theorem 2.0.3.** [16] Suppose \(\pi\) is a dihedral cuspidal representation of \(GL_2(A_E)\). Then \(\text{As}(\pi)\) is not cuspidal if and only if one of the following happens:

1. \(\pi^\theta\) is an abelian twist of \(\pi\)
2. \(\pi\) is automorphically induced from an idèle class character \(\mu\) of a quadratic extension \(M/E\), with \(M\) bi-quadratic over \(F\).

**Remark 2.0.1.** In fact, suppose \(\pi^\theta\) is an abelian twist of \(\pi\), say \(\pi^\theta \simeq \pi \otimes \omega\). Then \(\omega_0^2 = 1\). If \(\omega_0 = 1\), then, as explained in [12, Section 7, p 2249], some twist of \(\pi\) is Gal\((E/F)\)-invariant and \(\text{As}(\pi)\) is not cuspidal. Moreover, in this case, the fibers of the Asai transfer is described in loc. cit. On the other hand, if \(\omega_0 \neq 1\), then \(\omega_0 \neq \delta_{E/F}\) and \(\pi\) is induced from the quadratic extension cut out by \(\omega \omega^\theta\) which is necessarily a bi-quadratic extension of \(F\). So, for the rest of this section, whenever \(\pi^\theta \simeq \pi \otimes \omega\) for some \(\omega\), we will assume \(\omega_0 \neq 1\).

**Theorem 2.0.4.** Let \(E/F\) be a quadratic extension of number fields and let \(\pi, \pi'\) be dihedral cuspidal representations of \(GL_2(A_E)\) whose central characters satisfy \(\omega|_{C_F} = \omega'|_{C_F}\). Suppose \(\text{As}(\pi) \simeq \text{As}(\pi')\). Then there exists an idèle class character \(\nu\) of \(E\) such that \(\pi^\gamma \otimes \nu \simeq \pi'\), \(\gamma \in \text{Gal}(E/F)\), and satisfying \(\nu|_{C_F} = 1\).

**Proof.** From Lemma 2.0.1, we know that there exists an idèle class character \(\chi\) of \(E\) such that

\[
\pi^\gamma \otimes \chi \simeq \pi', \gamma \in \text{Gal}(E/F) = \{1, \theta\}.
\]

Since \(\text{As}(\pi^\theta) \simeq \text{As}(\pi)\), we may suppose that \(\pi' \simeq \pi \otimes \chi\) by replacing \(\pi\) with \(\pi^\theta\), if necessary, and our hypotheses on the central characters imply that \(\chi_0^2 = 1\). Let \(\Pi = \text{As}(\pi) \simeq \text{As}(\pi')\), it is an isobaric automorphic representation of \(GL_4(A_F)\). A routine calculation with the Hecke-Frobenius parameters yields

\[
L^S(s, \Pi \times \Pi) = L^S(s, \pi \times \pi', R),
\]

where \(S\) is a finite set of places including the archimedean ones so that all the relevant local data are unramified outside of \(S\). (This a natural identity since “tensor” or “Asai” induction commutes with taking tensor products and duals [5].) Here \(R\) is defined (similar to the definition of \(r\)) from \(GL_4(\mathbb{C}) \times GL_4(\mathbb{C}) \rtimes \Gamma_F \longrightarrow GL_{16}(\mathbb{C}) \times \Gamma_F\) via tensor induction. We consider two main cases (A) and (B):

(A). \(\Pi\) is cuspidal: Then \(L(s, \Pi \times \Pi)\) has a simple pole at \(s = 1\) [15]. On the other hand, we may compute \(L^S(s, \pi \times \pi', R)\) using \(\pi' \simeq \pi \otimes \chi\). We know that \(\pi \boxtimes \pi' = \text{Ad}(\pi)\chi^{-1} \boxplus \chi^{-1}\).
where $Ad(\pi) = sym^2(\pi) \otimes \omega^{-1}_\pi$ is an automorphic representation of $GL_3(\mathbb{A}_E)$. In fact, since $\pi$ is dihedral, say $\pi = I_{M}^E(\mu)$, $Ad(\pi)$ is given by

$$Ad(\pi) = \tau \boxplus \delta, \quad \tau := I_{M}^E\left(\frac{\mu}{\mu^\alpha}\right),$$

where $\alpha$ is the non-trivial Galois automorphism of $M/E$, and $\delta$ is the quadratic character attached to the extension $M/E$ via class field theory. By Theorem 2.0.3, the cuspidality of $\Pi$ implies that either $M/F$ is not Galois in which case $\delta$ is not $\theta$-invariant or $M/F$ is a cyclic extension in which case $\delta_0 = \delta_{E/F}$. In any event, we have

$$L$$

then $\Pi$ implies that either $M/F$ is not Galois in which case $\delta$ is not $\theta$-invariant or $M/F$ is a cyclic extension in which case $\delta_0 = \delta_{E/F}$. In any event, we have

$$L$$

Then $L^S(s, \pi \times \tilde{\pi}', R)$ is given by

$$L^S(s, \pi \times \tilde{\pi}', R) = L^S(s, As(\tau) \otimes \chi_0^{-1})L^S(s, I_{E}^F(\tau \otimes \chi^{-1}\chi^{-\theta}\delta_0))L^S(s, I_{E}^F(\tau \otimes \chi^{-1}\chi^{-\theta}) \times$$

$$L^S(s, I_{E}^F(\chi^{-1}\chi^{-\theta})\L(s, \chi_0^{-1})L^S(s, \chi_0^{-1}\delta_0).$$

Here, the fourth $L$-function has a simple pole at $s = 1$ if and only if $\delta = \chi\chi^\theta$. However, $\delta \neq \chi\chi^\theta$ since $\chi_0^2 = 1$. Consequently this $L$-function is entire. We now consider the following two sub-cases.

(i) $\tau$ is cuspidal. Since $\tau$ is also induced from the quadratic extension $M/E$, we conclude from Theorem 2.0.3 that $As(\tau)$ is cuspidal. Hence the first $L$-function in (2.0.5) is entire and so are the second and the third $L$-function in that expression. Therefore, for (2.0.5) to have a simple pole at $s = 1$, either $\chi_0 = 1$ or $\chi_0^{-1}\delta_0 = 1$ and only one of them can hold. If $\chi_0 = 1$, we are done. Otherwise $\chi_0^{-1}\delta_0 = 1$ in which case (since $\pi \simeq \pi \otimes \delta$) we may replace $\chi$ by $\chi\delta$ in $\pi' \simeq \pi \otimes \chi$. This completes the proof of the theorem with $\nu = \chi$ or $\nu = \chi\delta$.

(ii) $\tau$ is not cuspidal. Then $\frac{\mu}{\mu^\alpha} = \eta \circ N_{M/E}$ for some idèle class character $\eta$ of $E$, and $\tau = \eta \boxplus \eta\delta$. Thus we get

$$\pi \boxplus \tilde{\pi}' = \eta\chi^{-1} \boxplus \eta\delta\chi^{-1} \boxplus \delta\chi^{-1} \boxplus \chi^{-1},$$

and the right hand side of (2.0.5) now takes the form

$$\begin{cases}
L^S_E(s, \eta\delta\chi^{-1}\chi^{-\theta}) \\
L^S_E(s, \eta\chi^{-1}\chi^{-\theta}) \\
L^S_E(s, \eta\delta\chi^{-1}\chi^{-\theta}) \\
L^S_E(s, \delta\chi^{-1}\chi^{-\theta}) \\
L^S_E(s, \eta\theta\chi^{-1}\chi^{-\theta}) \\
L^S_E(s, \eta\theta\chi^{-1}\chi^{-\theta}) \\
L^S_E(s, \delta\theta\chi^{-1}) \\
L^S_E(s, \eta\delta\theta\chi^{-1}) \\
L^S_E(s, \delta\theta\chi^{-1}) \\
L^S_E(s, \chi^{-1}).
\end{cases}$$

Once again, since either $\delta \neq \delta^\theta$ or $\delta_0 = \delta_{E/F} \neq 1$ and $\chi_0^2 = 1$, we see that $\delta\chi^{-1}\chi^{-\theta}$ and $\eta\theta\chi^{-1}\chi^{-\theta}\delta^\theta$ are both non-trivial. Therefore $L^S_E(s, \eta\theta\chi^{-1}\chi^{-\theta})$ and $L^S(s, \eta\theta\chi^{-1}\chi^{-\theta}\delta^\theta)$ are both entire. Further, since $\pi \simeq \pi \otimes \eta$ and $\pi$ is cuspidal, $\eta \neq 1, \delta$. In fact, $\pi$ is precisely induced from the three quadratic extensions of $E$ cut out by $\delta, \eta$ and $\eta\delta$. 

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Next, we claim that the first four $L$-functions in (2.0.7) are entire. Otherwise, using the fact that $\pi$ is induced from three quadratic extensions, it is easy to obtain a contradiction. For example, say the first $L$-function has a simple pole at $s = 1$, then $\eta \delta^\theta = \chi \chi^\theta$ and consequently $\eta \delta_0 = \chi^2 = 1$, in particular the quadratic extension of $E$ cut out by $\eta \delta$ is bi-quadratic over $F$. Since $\pi$ is also induced from the quadratic extension cut out by $\eta \delta$ this contradicts the cuspidality of $\Pi$. Therefore we see that the right hand side of (2.0.2) has a simple pole at $s = 1$ precisely when one of the last four $L$-functions in (2.0.7) has a simple pole. This in turn implies that exactly one of the characters $\{\chi_0, \chi_0 \delta_0, \chi_0 \eta_0, \chi_0 \eta_0 \delta_0\}$ is trivial. Since $\pi \simeq \pi \otimes \delta \simeq \pi \otimes \eta \simeq \pi \otimes \eta \delta$, we are done with the proof of the theorem in case (A) by taking $\nu \in \{\chi, \chi \delta, \chi \eta, \chi \eta \delta\}$.

(B). $\Pi$ is not cuspidal: In view of Theorem 2.0.3 and Remark 2.0.1 we may assume $\pi$ is induced from a quadratic extension $M/E$ with $M/F$ bi-quadratic. In particular, $\delta^\theta = \delta$ and $\delta_0 = 1$. Let $\text{Gal}(M/E) = \{1, \alpha\}$ and let $\tilde{\theta}$ denote the extension of $\theta$ to a Galois automorphism of $M/F$. Then $\text{Gal}(M/E) = \{1, \alpha, \tilde{\theta}, \alpha \tilde{\theta}\}$ with $\tilde{\theta}^2 = 1$ and $\tilde{\theta} \alpha = \alpha \tilde{\theta}$. Moreover,

\begin{equation}
\Pi_E \simeq \pi \otimes \pi^\theta = \tau_1 \boxplus \tau_2,
\end{equation}

where $\tau_1 = I_M^E(\mu \mu^\theta), \tau_2 = I_M^E(\mu \mu^\theta \alpha)$. It is then straightforward to verify that the order of the pole

\begin{equation}
\text{ord}_{s=1} L^S(s, \Pi \times \bar{\Pi}) \begin{cases} = 2, 3, & \text{if either } \tau_1 \text{ or } \tau_2 \text{ is cuspidal} \\
= 2, 3, 4, & \text{otherwise.}
\end{cases}
\end{equation}

With $\tau$ as in (2.0.3), we also have

\begin{equation}
\text{As}(\tau)_E \simeq \tau \otimes \tau^\theta = \tau_1' \boxplus \tau_2',
\end{equation}

where $\tau_1' = I_M^E(\mu \mu^\theta \alpha), \tau_2' = I_M^E(\mu \mu^\theta \alpha)$. Just as we did in case (A) we now consider the following two sub-cases:

(i) $\tau$ is cuspidal, i.e., $\left(\frac{\mu}{\mu^\theta}\right)^2 \neq 1$. In particular this implies that both $\tau_1$ and $\tau_2$ cannot fail to be cuspidal and hence (2.0.5) must have a pole of order 2 or 3 at $s = 1$. First, note that $L^S(s, \text{As}(\tau) \otimes \chi_0^{-1})$ cannot have a pole of order 2 or more, for if it did, $\chi \chi^\theta$ should appear at least twice in (2.0.10) which is not possible since $\tau$ is cuspidal. Hence, if the order of the pole is 3, $\chi_0 = 1$ and $\text{As}(\tau)$ contains the trivial representation. In fact, since

\[ L^S(s, \Pi \times \bar{\Pi}) = L^S(s, \pi \times \bar{\pi}, R) = L_F^S(s, \text{As}(\tau))L_E^S(s, \tau)\zeta_F(s)^2, \]

we see that the order of the pole at $s = 1$ is 3 if and only if $\text{As}(\tau)$ contains the trivial representation.

Now, if (2.0.5) has a pole of order 2 at $s = 1$ and $\chi_0 \neq 1$, then $\delta = \chi \chi^\theta$ and $L^S(s, \text{As}(\tau) \times \chi_0^{-1})$ should have a simple pole at $s = 1$. However, the latter condition implies that $\delta = \chi \chi^\theta$ appears in either $\tau_1'$ or $\tau_2'$ (and not both!) which in turn implies that either $\mu \mu^\theta = \mu \mu^\theta \alpha$ or $\mu \mu^\theta \alpha = \mu \mu^\theta \alpha$. In either case, we obtain $\tau_1'^\theta \simeq \tau \otimes \omega$ for some idele class character $\omega$ of $E$. Then, as explained in Remark 2.0.1, we may assume $\omega_0 \neq 1$. In particular, since $\pi$ is induced from a unique quadratic extension, this forces $\omega \omega_0^\theta = \delta$. Since $(\pi')^\theta \simeq \pi \otimes \chi \otimes \pi \otimes \omega \chi^\theta$, we may replace $\pi'$ by $(\pi')^\theta$ and $\chi$ by $\chi^\theta \omega$ in (2.0.5) to conclude that $\chi_0 \omega_0 = 1$. 

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(ii) \( \tau \) is not cuspidal, i.e., \( \left( \frac{\mu}{v^2} \right)^2 = 1 \). Let \( \eta \) be as in (2.0.6), then as observed before, \( \pi \) is induced from three distinct quadratic extensions of \( F \) corresponding to the characters \( \delta, \eta \) and \( \eta \delta \). Then \( \tau_1 = \text{cuspidal} \) if and only if \( \tau_2 \) is cuspidal and this precisely happens when \( \frac{\mu}{v^2} \) is not \( \tilde{\theta} \)-invariant, in particular, \( \eta^\theta \neq \eta, \eta \delta \) and \( \eta \eta^\theta \neq \delta \). Let us, for the moment suppose that both \( \tau_1 \) and \( \tau_2 \) are cuspidal, then \( \Pi \) is an isobaric sum of two cuspidal representations of \( GL_2(\mathbb{A}_F) \) and consequently \( L^S(s, \Pi \times \tilde{\Pi}) \) has a pole of order 2 at \( s = 1 \). Moreover (2.0.7) now holds with \( \delta^\theta = \delta \) and \( \delta_0 = 1 \), in particular, the right hand side of (2.0.7) is given by

\[
L_F^S(s, \chi_0)^2 L_F^S(s, \eta_0 \chi_0)^2 L_F^S(s, \eta \eta_0 \chi_\theta) L_F^S(s, \eta \delta \chi_\theta) L_F^S(s, \eta_\delta \chi_\theta) L_F^S(s, \eta \eta_\delta \chi_\theta)^2
\]

Since \( \eta^\theta \neq \eta \), the above expression has a pole of order 2 if and only if either \( \chi_0 = 1 \) or \( \chi_0 \eta_0 = 1 \). In either case we are done with the proof of our theorem.

So for the rest of the proof, let us suppose that \( \tau_1 \) (and hence \( \tau_2 \)) is not cuspidal. Then either \( \eta^\theta = \eta \) or \( \eta^\theta = \eta \delta \). If \( \eta^\theta = \eta \) – namely, the three extensions corresponding to \( \pi \) are all Galois over \( F \) – then

\[
\text{ord}_{s=1} L^S(s, \Pi \times \tilde{\Pi}) = \begin{cases} 
3 & \text{ if and only if } \eta_0 \neq 1 \text{ (and hence } = \delta_{E/F}) \\
4 & \text{ otherwise.}
\end{cases}
\]

If the order of the pole is 4, i.e., \( \eta_0 = 1 \), it follows from (2.0.11) that \( \chi_0 = 1 \). On the other hand, if the order of the pole is 2, i.e., \( \eta_0 = \delta_{E/F} \), then once again from (2.0.11) it follows that exactly one of the characters \( \chi_0, \chi_0 \eta_0, \chi \chi^\theta \delta \), is trivial. Observe that, if \( \delta = \chi \chi^\theta \), since \( \pi^\theta \) differs from \( \pi \) by an abelian twist, then we may proceed as explained in case (i) above. Finally, if \( \eta^\theta = \eta \delta \) – in other words, the extensions corresponding to \( \eta \) and \( \eta \delta \) are not Galois over \( F \) – then

\[
\text{ord}_{s=1} L^S(s, \Pi \times \tilde{\Pi}) = 3
\]

and it follows from (2.0.11) that exactly one of the characters \( \chi_0, \eta_0 \chi_0 \), is trivial. Thus as argued before, by adjusting \( \chi \), if necessary, we are done with the proof of the theorem.

\[
\square
\]

3. Automorphic representations of \( GL_2 \) and Dirichlet series

3.1. Fourier coefficients of cuspidal automorphic representations. Let \( \mathfrak{o} \) denote the ring of integers of \( F \). For \( v < \infty \) we let \( \mathfrak{o}_v \) denote the ring of integers of \( F_v \), \( p_v \) the unique prime ideal of \( \mathfrak{o}_v \), \( \varpi_v \) a choice of generator of \( p_v \) which is normalized so that \( |\varpi_v|^v = q_v^{-1} \), where \( q_v \) is the cardinality of \( \mathfrak{o}_v / p_v \). We will use either \( \mathfrak{o}^\times \) or \( U_v \) for the group of local units. The symbol \( \mathbb{A}_{F,f} \) will denote the ring of finite ad` eles and \( F_\infty \) will denote \( \prod_{v|\infty} F_v \) so that \( \mathbb{A}_F = F_\infty \times \mathbb{A}_{F,f} \). (We let \( x_\infty \) and \( x_f \) denote the corresponding components of \( x \in \mathbb{A}_F \).) For any character \( \chi \) of \( \mathbb{A}^\times \), we let \( \chi_f \) denote its restriction to \( \mathbb{A}_{F,f}^\times \). We use the notation \( Cl_F \) to denote the class group of \( F \), in other words, it denotes the group of fractional ideals of \( F \) modulo the principal ideals. For any ideal \( \mathfrak{a} \) of \( F \), we write \( \mathfrak{a}^\times \) to denote the set of non zero elements in \( \mathfrak{a} \). For any integral ideal \( \mathfrak{m} \) of \( F \), we write \( N(\mathfrak{m}) \), the norm of the ideal \( \mathfrak{m} \), to denote the index of \( \mathfrak{m} \) in \( \mathfrak{o} \).
Let $Z(\mathbb{A}_F)$ denote the center of $GL_2(\mathbb{A}_F)$. For each place $v$, we let $Z(F_v)$ denote the center of $GL_2(F_v)$. We fix $K = \prod_v K_v$ of $GL_2(\mathbb{A}_F)$, a maximal compact subgroup of $GL_2(\mathbb{A}_F)$, where $K_v = GL_2(\mathfrak{o}_v)$, if $v < \infty$, $K_v = O(2, \mathbb{R})$, if $v$ is real, and $K_v = U(2, \mathbb{C})$ if $v$ is complex. Let $N$ be the maximal unipotent radical of $GL_2$. We write $N(\mathbb{A}_F)$ (resp. $N(F_v)$) to denote the corresponding $\mathbb{A}_F$-points (resp. $F_v$-points).

We fix the usual additive character $\psi = \otimes_v \psi_v$ of $F \setminus \mathbb{A}_F$ whose conductor is the inverse different $\mathfrak{d}^{-1}$ of $F$. Namely, $\psi = \psi_0 \circ \text{tr}$, where $\text{tr}$ is the trace map from $\mathbb{A}_F$ to $\mathbb{A}_\mathbb{Q}$, and $\psi_0$ is the standard additive character of $\mathbb{Q} \setminus \mathbb{A}_\mathbb{Q}$ which is unramified when restricted to $\mathbb{Q}_p$, $p$ a prime, and is $c(x)$ when restricted to $\mathbb{R}$. Let $d \in \mathbb{A}_F^\times$ be such that $(d) = \prod_{v < \infty} \mathfrak{p}_v^{\text{ord}_v(d_v)} = \mathfrak{o}$. Let us now fix our choice of Haar measure on the idele class group: For each finite place $v$ of $F$, let $d^v x_v$ be the Haar measure on $F_v^\times$ such that the volume of $\mathfrak{o}_v^\times$ is one. For $v|\infty$, if $dx_v$ is the ordinary Lebesgue measure if $v$ is real and twice the ordinary Lebesgue measure if $v$ is complex, we let $d^v x_v = \frac{dx_v}{|x_v|_v}$. Then, our choice of local Haar measures yields a unique Haar measure $d^x$ on $\mathbb{A}^\times_F$ such that the volume of $\prod_{v < \infty} \mathfrak{o}_v^\times$ is one.

Let $\pi$ be a cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ with central character $\omega_\pi$. We write $V_\pi$ to denote the space of $\pi$. Note that $V_\pi$ is a function space on $GL_2(F) \setminus GL_2(\mathbb{A}_F)$. For each $v$, let $W(\pi_v, \psi_v)$ denote the $\psi_v$-Whittaker model of $\pi_v$. We let $W(\pi, \psi) = \bigotimes_v W(\pi_v, \psi_v)$ denote the corresponding global Whittaker model of $\pi$.

We let $c \subset \mathbb{A}_F^\times$ be the conductor of $\pi$. We write $c$ as $c = \prod_{v < \infty} \mathfrak{p}_v^{m_v}$, where $m_v = 0$ whenever $\pi_v$ is unramified and $m_v > 0$ otherwise. We let

$$K_{1,v}(\mathfrak{p}_v^{m_v}) = \{ g \in GL_2(\mathfrak{o}_v) : g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod \mathfrak{p}_v^{m_v} \},$$

and

$$K_{0,v}(\mathfrak{p}_v^{m_v}) = \{ g \in GL_2(\mathfrak{o}_v) : g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \mathfrak{p}_v^{m_v} \}.$$

We then set $K_1(c) = \prod_{v < \infty} K_{1,v}(\mathfrak{p}_v^{m_v})$ and $K_0(c) = \prod_{v < \infty} K_{0,v}(\mathfrak{p}_v^{m_v})$. The definition of $c$ is such that for each finite $v$, the dimension of the space of $K_{1,v}(\mathfrak{p}_v^{m_v})$-fixed vectors in $\pi_v$ is one. Moreover, $K_{0,v}(\mathfrak{p}_v^{m_v})$ acts on this one dimensional space by the central character $\omega_{\pi_v}$, where $\omega_{\pi_v}$ determines a character of $K_{0,v}(\mathfrak{p}_v^{m_v})$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \omega_{\pi_v}(d)$ if $m_v > 0$ and $\omega_{\pi_v}(g_v) = 1$ if $m_v = 0$. Observe that $c$ determines an integral ideal of $F$ which is also denoted as $c$.

For every $t \in \mathbb{A}_F^\times$, we denote by $(t)$ the fractional ideal associated with $t$. Let $h$ be the number of ideal classes in the ideal class group $Cl_F$ of $F$. We choose $h$ elements $t_1, \ldots, t_h$ in $\mathbb{A}_F^\times$ so that $(t_i)_\infty = 1$ and so that the fractional ideals $(t_i)$ form a complete set of representatives of such ideal classes. Set $g_i = \begin{pmatrix} t_i \\ 1 \end{pmatrix}$, then by strong approximation approximation theorem for $GL_2$ we have

$$GL_2(\mathbb{A}_F) = \prod_{i=1}^h GL_2(F)g_iG_\infty K_0(c).$$
Here $G_\infty = \prod_{v|\infty} GL_2(F_v)$. There is a unique intertwining map from $V_\pi$ onto $W(\pi, \psi)$ which we will denote as $\xi \mapsto W_\xi$. Every $\xi \in V_\pi$ has a Fourier expansion

$$\xi(g) = \sum_{\gamma \in F^x} W_\xi \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right) \;; \; \xi \in V_\pi, \; g \in GL_2(\mathbb{A}_F).$$

For each finite $v$, there is a unique $\phi_v \in V_{\pi_v}$ for which the corresponding $W_{\phi_v}$ is such that $W_{\phi_v} \left( \begin{pmatrix} d_v^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right) = 1$, and $W_{\phi_v}$ transforms via the central character $\omega_{\pi_v}$ for the action of $K_{0,v}(p_v^{m_v})$. For $v|\infty$, we pick $\phi_v \in V_{\pi_v}$ such that

$$\int_{F_v^x} W_{\phi_v} \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) |y|_{v}^{s-\frac{1}{2}} d^x y = L(s, \pi_v),$$

where $L(s, \pi_v)$ is the local $L$-function attached to $\pi_v$. (See [9],[6].) The function $\phi = \otimes_v \phi_v$ is called the new vector of $\pi$. If, for each place $v$ of $F$, $L(s, \pi_v)$ denotes the local $L$-function attached to $\pi_v$, let us form the completed $L$-function $\Lambda(s, \pi)$ by setting $\Lambda(s, \pi) = \prod_v L(s, \pi_v)$, then our choice of $\phi \in V_\pi$ is such that

$$N(\mathfrak{o})^{\frac{1}{2} - s} \int_{C(\mathfrak{o})} \phi \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) |y|^{s-\frac{1}{2}} d^x y = \Lambda(s, \pi).$$

Let $L(s, \pi)$ denote the finite part of $\Lambda(s, \pi)$, i.e., the product over the finite places, we will express $L(s, \pi)$ as a Dirichlet series. (It is worth pointing out that the notation $L(s, \pi)$ is often used to denote the completed $L$-function in the literature.) First note that the function $y \mapsto \phi_\xi \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right)$ is invariant under $\prod_v \mathfrak{o}_v^x$. Further, since for each $v < \infty$, the choice of the measure $d^x y_v$ is such that $\mathfrak{o}_v^x$ has unit volume, we see that

$$\Lambda(s, \pi) = N(\mathfrak{o})^{\frac{1}{2} - s} \int_{C(\mathfrak{o})} \phi \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) |y|^{s-\frac{1}{2}} d^x y,$$

where $C(\mathfrak{o}) = F^x \setminus \mathbb{A}_F^x / \prod_v \mathfrak{o}_v^x$.

Let $a(y)$ denote the matrix $\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$. Let us write

$$\phi \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) = \sum_{\gamma \in F^x} a_\phi(y, \gamma) W_{\phi, \infty}(a(\gamma \infty y_\infty)), $$

where $a_\phi(y, \gamma) = \prod_{v<\infty} W_{\phi_v}(a(\gamma y_v))$ and $W_{\phi, \infty} = \prod_{v|\infty} W_{\phi_v}$. Observe that $a_\phi(y, \gamma)$ only depends on the finite part $y_f$ of $y$. Moreover, for $v < \infty$, it is easy to see that $W_{\phi_v}(a \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \neq 0$ for $a \notin \mathfrak{o}_v^{-1}$, where $\mathfrak{o}_v^{-1} = (d_v^{-1}) \mathfrak{o}_v$ is the conductor of $\psi_v$. Hence we have

$$\phi(a(y)) = \sum_{\gamma \in ((y)^{-1} \mathfrak{o}_v^{-1})^x} a_\phi(y, \gamma) W_{\phi, \infty}(a(\gamma \infty y_\infty)), $$

where $(y)$ is the fractional ideal $(y) = \prod_{v<\infty} p_v^{ord_v(y_v)}$. Since $W_{\phi_v}(\cdot)$ is right invariant under $K_{1,v}(p_v^{m_v})$, it follows that

$a_\phi(y, \epsilon \gamma) = a_\phi(y, \gamma)$, $\forall \; \epsilon \in \mathfrak{o}_v^x$. 


Consider the equivalence relation $\sim$ in $(y)^{-1}d^{-1}$ which is given as follows: $\gamma_1 \sim \gamma_2$ if and only if $\gamma_1 = \epsilon \gamma_2$, for some $\epsilon \in o^\times$. Then the right hand side of (3.1.2) is given by

\[ (3.1.3) \sum_{\gamma \in \sim ((y)^{-1}d^{-1})^\times} a_\phi(y, \gamma) \sum_{\eta \in o^\times} W_{\phi, \infty}(a(\eta \gamma \infty y \infty)). \]

We have the exact sequence

\[ 1 \rightarrow o^\times \backslash \mathcal{F}_\infty \rightarrow C(o) \rightarrow \mathcal{C}_F \rightarrow 1, \]

and by fixing the representatives $t_1, \ldots, t_h$, we have chosen a section of the last map. Using this we see that

\[ \int_{C(o)} \phi((y 1)) |y|^{s-\frac{1}{2}} d^s y = \int_{o^\times \backslash \mathcal{F}_\infty} \sum_j \phi((t_j y 1)) |t_j y|^{s-\frac{1}{2}} d^s y; \]

which in turn by (3.1.3) yields

\[ \Lambda(s, \pi) = N(o)^{\frac{1}{2}-s} \sum_{j=1}^h \left( \sum_{\gamma \in \sim ((a_j)^{-1}d^{-1})^\times} a_\phi(t_j, \gamma) |t_j|^{s-\frac{1}{2}} \right) \]

\[ \times \int_{o^\times \backslash \mathcal{F}_\infty} \sum_{\eta \in o^\times} W_{\phi, \infty}(a(\eta \gamma \infty y \infty)) |y \infty|^{s-\frac{1}{2}} d^s y \infty. \]

Effecting the change $y \infty \mapsto \gamma \infty^{-1} y \infty$ and combining the integral with the sum over $\eta$ we get

\[ \Lambda(s, \pi) = N(o)^{\frac{1}{2}-s} \left( \sum_{j=1}^h \sum_{\gamma \in \sim ((a_j)^{-1}d^{-1})^\times} a_\phi(t_j, \gamma) |t_j|^{s-\frac{1}{2}} |\gamma \infty|^{s+\frac{1}{2}} \right) \]

\[ \times \int \mathcal{F}_\infty W_{\phi, \infty}(a(y \infty)) |y \infty|^{s-\frac{1}{2}} d^s y \infty. \]

We define the normalized Fourier coefficients (of $\pi$) as follows: Given an integral ideal $m$, there is a unique $k$, $1 \leq k \leq h$, such that $m = (\gamma) a_k o$ for some $\gamma$ in $F$, we then set

\[ \lambda_\pi(m) = \sqrt{N((\gamma)(a_k o))} a_\phi(t_k, \gamma). \]

Therefore, considering our choice of $\phi_v, v|\infty$, and noting that $|\gamma \infty| = N(\gamma)$, we see that (3.1.4) takes the form

\[ \Lambda(s, \pi) = \left( \sum_{m \subset o} \lambda_\pi(m) \frac{N(m)^s}{N(m)} \right) L(s, \pi \infty). \]

Here $L(s, \pi \infty) = \prod_{v|\infty} L(s, \pi_v)$. Therefore, we have expressed the finite $L$-function $L(s, \pi)$ as a Dirichlet series, i.e.,

\[ L(s, \pi) = \sum_{m \subset o} \frac{\lambda_\pi(m)}{N(m)^s}. \]
3.2. Hecke operators. Here, we recall the definition of the Hecke operators. (See [25, §2] and the references therein.) Although the discussion in [25] is for totally real fields, the facts about the Hecke operators, namely, (3.2.3) and (3.2.4) below, remains valid for any number field. Keeping the notation of the previous section, let \( W = G_\infty \times K_0(\mathfrak{c}) \). Let us put
\[
Y = Y(\mathfrak{c}) = GL_2(A_F) \cap (G_\infty \times \prod_{v < \infty} Y(p_v^{m_v}))
\]
with
\[
Y(p_v^{m_v}) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL_2(F_v) : c \in p_v^{m_v}; a, b, d \in \mathfrak{o}_v \right\}.
\]
Our definition of \( Y(p_v^{m_v}) \) is different from the one in [25], in the sense that the twisting by the local different is missing in our definition, instead we have normalized the Fourier coefficients \( \lambda_\pi(m) \) accordingly with a twist by the different ideal \( \mathfrak{o} \). For \( y \in Y \) write \( W_y W = \bigcup_j y_j W \) as a disjoint union of right \( W \)-cosets, and assume that \((y_j)_\infty = 1\) for each \( j \). For \( f \in V_\pi \) such that \( f(gk) = \omega_\pi(k)f(g) \), \( k \in K_0(\mathfrak{c}) \), we define a function \( f|_{W_y W} \), which is again in \( V_\pi \) having the same \( K_0(\mathfrak{c}) \)-transformation property as \( f \), by
\[
(f|_{W_y W})(x) = \sum_j \omega_\pi(y_j)^{-1} f(xy_j).
\]
This is independent of the choice of \( y_j \)'s. Here, we have extended the definition of \( \omega_\pi \) from \( K_0(\mathfrak{c}) \) to the finite part of \( Y \). In fact, if we choose the Haar measure \( dk = \prod_{v < \infty} dk_v \) on \( K_0(\mathfrak{c}) \) such that the volume of \( K_{0,v}(p_v^{m_v}) \) is one with respect to \( dk_v \) for all finite \( v \), then one can check that
\[
(f|_{W_y W})(x) = \int_{K_0(\mathfrak{c})} \omega_\pi^{-1}(k)f(xky)dk.
\]
For each nonzero integral ideal \( m \) of \( F \), we consider the operator \( T_\mathfrak{c}(m) \) acting on functions in \( V_\pi \) with the aforementioned \( K_0(\mathfrak{c}) \)-transformation property through
\[
T_\mathfrak{c}(m)f = N(m)^{-\frac{1}{2}} \sum_{i(det(y)) = m} f|_{W_y W},
\]
where the sum is taken over distinct double cosets \( W_y W \) with \( y \in Y \) such that \( i(det(y)) = m \). Further, if \( m \) is prime to \( \mathfrak{c} \), we let \( S_\mathfrak{c}(m) \) be the operator defined by
\[
S_\mathfrak{c}(m)f = f|_{W_\mathfrak{a} W},
\]
where \( \mathfrak{a} \) is any element of \( \mathfrak{A}_F \) such that \( i(\mathfrak{a}) = m \) (the choice of \( \mathfrak{a} \) is irrelevant since, for each \( v \) with \( p_v \) prime to \( \mathfrak{c} \), \( f \) is right \( K_v \)-invariant); if \( m \) is not prime to \( \mathfrak{c} \), we define \( S_\mathfrak{c}(m) \) to be zero. Then one has the well known relation (cf. [25, §2])
\[
T_\mathfrak{c}(m)T_\mathfrak{c}(n) = \sum_{\mathfrak{a} \supset m+n} S_\mathfrak{c}(\mathfrak{a})T_\mathfrak{c}(\mathfrak{a}^{-2}mn)
\]
and we have the formal Euler product
\[
\sum_a T_\mathfrak{c}(m)N \mathfrak{m}^{-s} = \prod_p (1 - T_\mathfrak{c}(p)Np^{-s} + S_\mathfrak{c}(p)Np^{-2s})^{-1}.
\]
It is known that the function \( \phi \) is a common eigenfunction for all the Hecke operators \( T_\mathfrak{c}(m) \) and \( S_\mathfrak{c}(m) \). In fact \( T_\mathfrak{c}(m)\phi = \lambda_\pi(m)\phi \) and \( S_\mathfrak{c}(m)\phi = \omega_\pi(m)\phi \) for all integral ideals \( m \).
that.

From (3.2.5), it follows that the coefficients 

\[ \lambda_{\pi}(m)\lambda_{\pi}(n) = \sum_{a \in \mathbb{Z}} \omega_{\pi}(a)\lambda_{\pi}(a^{-2}mn). \]

3.3. Dirichlet series. Let \( E/F \) be a quadratic extension of number fields. Suppose \( \pi = \otimes_{w} \omega_{w} \) is a unitary cuspidal representation of \( GL_{2}(\mathbb{A}_{E}) \) with central character \( \omega_{\pi} \); then one can attach two kinds of \( L \)-series to \( \pi \). First one is the standard \( L \)-series \( L(s, \pi) \) defined in §3.1 through the choice of a normalized new vector \( \phi_{\xi} \):

\[ L(s, \pi) = \sum_{m} \lambda_{\pi}(m)N(m)^{-s} \]

where \( m \) runs over all integral ideals of \( E \). This converges for sufficiently large \( \Re(s) \), and can be continued to an analytic function on the whole \( s \)-plane; further it follows from (3.2.4) that

\[ L(s, \pi) = \prod_{\wp}(1 - \lambda_{\pi}(\wp)N(\wp)^{-s} + \omega_{\pi}(\wp)N(\wp)^{-2s})^{-1}, \]

where the product is over prime ideals \( \wp \) of \( E \). A second one is the \( L \)-series studied by Asai[3]:

\[ L_{Asai}(s, \pi) = L(2s, \omega_{\pi}|_{F}) \sum_{n \in \mathbb{Z}} \frac{\lambda_{\pi}(no_{E})}{N(n)^s}. \]

The main point here is that the second Dirichlet series (3.3.1) admits an Euler product over primes \( p \) of \( F \). Namely, for every prime ideal \( \wp \) of \( E \), let us factor

\[ X^2 - \lambda_{\pi}(\wp)X + \omega_{\pi}(\wp) = (X - \alpha_{\wp})(X - \beta_{\wp}). \]

Lemma 3.3.1. The \( L \)-series \( L_{Asai}(s, \pi) \) is given by the Euler product

\[ L_{Asai}(s, \pi) = \prod_{p} P_{p}(N(p)^{-s}), \]

where

\[ P_{p}(X)^{-1} = \begin{cases} (1 - \alpha_{p}^{2}X)(1 - \beta_{p}^{2}X)(1 - \alpha_{p}\beta_{p}X) & \text{if } po_{E} = \wp^{2} \\ (1 - \alpha_{p}^{2}X)(1 - \beta_{p}^{2}X)(1 - \alpha_{p}\beta_{p}X^{2}) & \text{if } po_{E} = \wp \\ (1 - \alpha_{p}\alpha_{p'}X)(1 - \alpha_{p}\beta_{p'}X)(1 - \beta_{p}\alpha_{p'}X)(1 - \beta_{p}\beta_{p'}X) & \text{if } po_{E} = \wp \wp' \end{cases} \]

Proof. From (3.2.5), it follows that the coefficients \( \lambda_{\pi}(m) \) are multiplicative; then it follows that \( L_{Asai}(s, \pi) = \prod_{p} P_{p}(N(p)^{-s}) \), where the local factor \( P_{p}(N(p)^{-s}) \) is the product of the Euler factor of \( L(2s, \omega_{\pi}|_{F}) \) at \( p \) and \( \sum_{r \geq 0} \lambda_{\pi}(p^{r}o_{E})N(p)^{-rs} \). Now for any prime ideal \( \wp \) of \( E \), we have

\[ \sum_{r \geq 0} \lambda_{\pi}(\wp^{r})X^{r} = (1 - \lambda_{\pi}(\wp)X + \omega_{\pi}(\wp)X^{2})^{-1}. \]

Then if \( po_{E} = \wp \), it follows that

\[ P_{p}(N(p)^{-s})^{-1} = (1 - \omega_{\pi}(\wp)N(p)^{-2s})(1 - \lambda_{\pi}(\wp)N(p)^{-s} + \omega_{\pi}(\wp)N(p)^{-2s}) = (1 - \alpha_{\wp}\beta_{\wp}N(p)^{-2s})(1 - \alpha_{\wp}N(p)^{-s})(1 - \beta_{\wp}N(p)^{-s}). \]
Suppose $\mathfrak{p}o_E = \wp^2$; then for $r \geq 1$, by virtue of (3.2.5), we have the relation
\[
\lambda_\pi(\mathfrak{p}o_E)\lambda_\pi(\wp^r o_E) = \lambda_\pi(\wp^{r-1} o_E)\omega_\pi(\wp)^2 + \lambda_\pi(\wp^{r+1} o_E) + \omega_\pi(\wp)\lambda_\pi(\wp^r o_E)
\]
which in turn implies that
\[
(3.3.2) \quad \sum_{r \geq 0} \lambda_\pi(\wp^r o_E)X^r = \frac{1 + \omega_\pi(\wp)X}{1 - \lambda_\pi(\mathfrak{p}o_E)X + \omega_\pi(\wp)^2 X^2 + \omega_\pi(\wp)X}.
\]
We also have the relation $\lambda_\pi(\wp)^2 = \lambda_\pi(\wp^2) + \omega_\pi(\wp)$ which readily follows from (3.2.5); since $\alpha_\wp + \beta_\wp = \lambda_\pi(\wp)$, $\alpha_\wp\beta_\wp = \omega_\pi(\wp)$, we see that (3.3.2) may be rewritten as
\[
\sum_{r \geq 0} \lambda_\pi(\wp^r o_E)X^r = \frac{1 + \alpha_\wp\beta_\wp X}{(1 - \alpha_\wp^2 X)(1 - \beta_\wp X)}.
\]
Since the Euler factor of $L(2s, \omega_\pi|_F)$ at $\wp$ is $(1 - \alpha_\wp^2 \beta_\wp^2 N(\wp)^{-2s})^{-1}$, it follows that $P_\wp(N(\wp)^{-s})$ is given by the inverse of
\[
(1 - \alpha_\wp^2 N(\wp)^{-s})(1 - \beta_\wp^2 N(\wp)^{-s})(1 - \alpha_\wp\beta_\wp N(\wp)^{-s}).
\]
Finally, let us suppose that $\mathfrak{p}o_E = \wp\wp'$. The calculation here is lengthy but straightforward. Namely, by (3.2.5), for $r \geq 2$, we see that
\[
\lambda_\pi(\mathfrak{p}o_E)\lambda_\pi(\wp^r o_E) = \lambda_\pi(\wp^{r+1} o_E) + \omega_\pi(\mathfrak{p}o_E)\lambda_\pi(\wp^{r-1} o_E) + \omega_\pi(\wp)\lambda_\pi(\wp^r o_E)
\]
and that $\lambda_\pi(\wp^r)\lambda_\pi(\wp) = \lambda_\pi(\wp^{r+1}) + \omega_\wp(\wp)\lambda_\pi(\wp^{r-1})$, $\wp = \wp', \wp''$; then using these relations, it can be verified that
\[
\sum_{r \geq 0} \lambda_\pi(\wp^r o_E)N(\wp)^{-rs} = \frac{1 - \alpha_\wp\beta_\wp N(\wp)^{-2s}}{(1 - \alpha_\wp\alpha_\wp'\beta_\wp' N(\wp)^{-s})(1 - \beta_\wp\alpha_\wp' N(\wp)^{-s})(1 - \beta_\wp N(\wp)^{-s})}
\]
This completes the proof of our lemma.

\section{A Multiplicity One Type Result for $GL_2$}

In this section we state our main result and the proofs essentially follow from our results in §2 and §3, respectively.

\begin{theorem}
Let $E/F$ a quadratic extension of number fields as usual. Suppose $\pi$ and $\pi'$ are two unitary cuspidal representations of $GL_2(\mathbb{A}_E)$ whose central characters are such that $\omega_\pi|_C = \omega_\pi'|_{C_F}$. Let us also suppose that the Hecke eigenvalues of $\pi$ and $\pi'$ satisfy the condition $\lambda_\pi(\wp) = \lambda_{\pi'}(\wp)$ for almost all prime ideals $\wp$ of $F$. Then there exists an idèle class character $\nu$ of $E$ such that $\pi_\gamma \otimes \nu \simeq \pi'$, $\gamma \in \text{Gal}(E/F)$. Further, $\nu$ can be chosen so that $(\nu|_{C_F}) = 1$.
\end{theorem}

\begin{proof}
For any finite place $v$ of $F$ corresponding to a prime ideal $\wp$, it follows from Lemma 3.3.1 and the description of the local factors $L(s, As(\pi_v))$ that $L(s, As(\pi_v)) = P_\wp(N(\wp)^{-s})$. From our description of $P_\wp(N(\wp)^{-s})$ in the proof of Lemma 3.3.1 (in terms of the Fourier coefficients of $\pi$), it follows that $As(\pi_v) \simeq As(\pi'_v)$ for almost all finite places $v$ of $F$. Since $As(\pi)$ and $As(\pi')$ are isobaric automorphic representations [12], the usual multiplicity one theorem of Jacquet and Shalika [11, 10] imply that $As(\pi) \simeq As(\pi')$. Now our result follows from Theorem 2.0.2 and Theorem 2.0.4.
\end{proof}
Remark 4.0.1. In fact if one assumes that both $As(\pi)$ and $As(\pi')$ are cuspidal, then by [13] it is enough to require that $\lambda_\pi(p) = \lambda_{\pi'}(p)$ for all $p \in \mathcal{O}_F$ such that the norm $N(p)$ satisfies $N(p) \leq Y(\pi, \pi')$ for some finite constant $Y(\pi, \pi')$. See [4] for effective bounds on $Y(\pi, \pi')$.

The question was initially raised to the author in the context of Hilbert modular forms. For the sake of reference, we will include a version of Theorem 4.0.2 for Hilbert modular forms. Let $E/F$ be a quadratic extension of totally real fields. Let $d$ be the degree of $E/F$. Given $k = (k_1, \ldots, k_d)$ a $d$-tuple of positive integers, and a finite order Hecke character $\chi$ of $E$, let $S_k(\chi)$ be the collection of all cuspidal Hilbert modular newforms of weight $k$, which are common eigenfunctions for all the Hecke operators, and having central character $\chi$. (We refer the reader to [25, §2] for the definition and basic properties of Hilbert modular forms.) It is well known that $f \in S_k(\chi)$ corresponds to a cuspidal automorphic representation $\pi(f)$ of $GL_2(\mathbb{A}_E)$. Under this correspondence, the central character of $\pi(f)$ is $\chi| \cdot |^{k_0-1}$, where $k_0 = \max\{k_i - 1\}$. Then the following is a consequence of Theorem 4.0.2:

Theorem 4.0.3. Let $E/F$ be a quadratic extension of totally real fields. Let $\chi$ and $\chi'$ be finite order Hecke characters of $E$ such that $\chi|_{C_F} = \chi'|_{C_F}$. Suppose $f \in S_k(\chi), f' \in S_k(\chi')$, are normalized newforms, defined with respect to the field $E$, whose Fourier coefficients satisfy the relation

$$c(n|_{C_E}, f) = c(n|_{C_E}, f')$$

for all non zero integral ideals $n \subset \mathcal{O}_F$. Then there is a finite order Hecke character $\nu$ of $C_E$ such that $\nu|_{C_E} = 1$ and

$$c(\gamma(m), f) = \nu(m)c(m, f'), \forall m \subset \mathcal{O}_E, \text{ where } \gamma \in \text{Gal}(E/F).$$

References


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