ON MODULI SPACES OF REAL CURVES IN SYMPLECTIC MANIFOLDS

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Abstract

Given a symplectic manifold (X, ω), an almost complex structure J, and an antisymplectic involution ϕ, we study genus zero real J-holomorphic curves in X. There are two types of such curves, those that can be divided into two J-holomorphic discs and those that cannot. Moduli spaces of J-holomorphic discs are more studied in the literature; in this case, we develop and use some degeneration techniques to add to the previous results and get a better understanding of these moduli spaces. We also study the second case, for which the orientation problem is different and define (and calculate) some invariants using these moduli spaces. As shown in this thesis, these two cases are tied together and often need to be combined to get a fully well-defined theory.

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In the name of God.

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Contents

Chapter 1

Introduction

Let (X, ω, ϕ) be a symplectic manifold, which we will assume to be connected throughout this thesis, with a real structure ϕ, i.e a diffeomorphism $\phi\colon X \to X$ such that $\phi^2 = \mathrm{id}_X$ and $\phi^*\omega = -\omega$. Let $L = \mathrm{Fix}(\phi) \subset X$ be the fixed point locus of ϕ; L is a Lagrangian submanifold of (X, ω) which can be empty. In the simplest case of $(X, \omega) = (\mathbb{P}^1, \omega_{\mathrm{FS}})$, where ω_{FS} is the Fubini-Study symplectic form, there are involutions of both types. An almost complex structure J on TX is called (ω, ϕ)-compatible if $\phi^*J = -J$ and $\omega(\cdot, J\cdot)$ is a metric.

Fix a compatible almost complex structure J. Let $u\colon \mathbb{P}^1 \to X$ be an n-marked somewhere injective J-holomorphic sphere, i.e.

$$du + J \circ du \circ j = 0, \qquad u^{-1}(u(z)) = \{z\} \quad \text{for almost every } z \in \mathbb{P}^1,$$

where j is the complex structure of $\mathbb{P}^1$. We call such a J-holomorphic map real if its image (as a marked curve) is invariant under the action of ϕ. In this case, pulling back ϕ to $\mathbb{P}^1$, we get an involution on $\mathbb{P}^1$, which may or may not have fixed points and preserves the set of marked points.

There are two isomorphism classes of antisymplectic involutions on $\mathbb{P}^1$: those that have fixed points and those that do not. After a change of coordinates, an antisymplectic involution with fixed points can be written as

$$\tau\colon \mathbb{P}^1 \to \mathbb{P}^1, \qquad \tau([z,w]) = [\bar{w}, \bar{z}],$$

while a fixed point free involution can be written as

$$\tau\colon \mathbb{P}^1 \to \mathbb{P}^1, \qquad \tau([z,w]) = [\bar{w}, -\bar{z}].$$

We define $\mathcal{M}_l(X,A)^{\phi,\eta}$ to be the moduli space of degree A genus zero J-holomorphic curves $u\colon \mathbb{P}^1 \to X$ satisfying

$$u = \phi \circ u \circ \eta$$

with $l = \frac{n}{2}$ pairs of disjoint conjugate marked points. Similarly, we define $\mathcal{M}_{k,l}(X,\beta)^{\phi,\tau}$ to be the moduli space of degree A genus zero J-holomorphic curves $u\colon \mathbb{P}^1 \to X$ satisfying

$$u = \phi \circ u \circ \tau \tag{1.1}$$

with k real marked points and $l = \frac{n-k}{2}$ pairs of disjoint conjugate marked points. By [20, Appendix C], these moduli spaces have real virtual dimension

$$\dim^{\mathrm{vir}} \mathcal{M}_l(X,A)^{\phi,\eta}, \dim^{\mathrm{vir}} \mathcal{M}_{k,l}(X,A)^{\phi,\tau} = \dim_{\mathbb{C}} X + c_1(A) + n - 3.$$

Every n-marked J-holomorphic map $u\colon \mathbb{P}^1 \to X$ satisfying (1.1) corresponds to two J holomorphic discs

$$u\colon (D^2, S^1) \to (X, L)$$

with k boundary marked points and l interior marked points, with $k + 2l = n$, representing some relative homology classes $\beta, -\phi_*\beta \in H_2(X,L)$. We define $\mathcal{M}^{\mathrm{disc}}_{k,l}(X,L,\beta)$ to be the moduli space of such J-holomorphic discs. The two discs above need not be in the same homology class, but both have the same Maslov index and symplectic area; see [7, Definition 2.4.17]. In the presence of an involution, we define an equivalence relation $\sim$ on $H_2(X, L)$ by

$$\beta_1 \sim \beta_2 \;\Leftrightarrow\; \mu(\beta_1) = \mu(\beta_2),\; \omega(\beta_1) = \omega(\beta_2). \tag{1.2}$$

In the the definition of $\mathcal{M}_{k,l}^{\mathrm{disc}}(X, L, \beta)$, we often consider β to be an element of $H_2(X, L)/\sim$, instead of $H_2(X, L)$. For an arbitrary pair (X, L), we define the moduli space $\mathcal{M}_{k,l}^{\mathrm{disc}}(X, L, \beta)$ without any involution.

In all cases above, let $\overline{\mathcal{M}}_n(X, A)^{\phi,\eta}$, $\overline{\mathcal{M}}_n(X, A)^{\phi,\tau}$, and $\overline{\mathcal{M}}_{k,l}^{\mathrm{disc}}(X, L, \beta)$ be the stable map compactifications of $\mathcal{M}_n(X, A)^{\phi,\eta}$, $\mathcal{M}_n(X, A)^{\phi,\tau}$, and $\mathcal{M}_{k,l}^{\mathrm{disc}}(X, L, \beta)$; see [7, Section 7] for the definition. Let

$$\begin{aligned}
&\mathrm{ev}_i\colon \overline{\mathcal{M}}_l(X, A)^{\phi,\eta} \to X, &&\mathrm{ev}_i([u, \Sigma, (\xi_j = (z_j, \eta(z_j)))_{j=1}^{l}]) = u(z_i),\\
&\mathrm{ev}_i^B\colon \overline{\mathcal{M}}_{k,l}^{\mathrm{disc}}(X, L, \beta) \to L, &&\mathrm{ev}_i^B([u, \Sigma, (w_j)_{j=1}^{k}, (z_j)_{j=1}^{l})]) = u(w_i),\\
&\mathrm{ev}_i\colon \overline{\mathcal{M}}_{k,l}^{\mathrm{disc}}(X, L, \beta) \to X, &&\mathrm{ev}_i([u, \Sigma, (w_j)_{j=1}^{k}, (z_j)_{j=1}^{l})]) = u(z_i),
\end{aligned} \tag{1.3}$$

be the natural evaluation maps.

For the classic moduli space $\overline{\mathcal{M}}_n(X, A)$ of J-holomorphic spheres in homology class A, Gromov-Witten invariants are defined via integrals of the form

$$\langle \theta_1, \cdots, \theta_n \rangle_A = \int_{[\overline{\mathcal{M}}_n(X,A)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\theta_1) \wedge \cdots \wedge \mathrm{ev}_n^*(\theta_n), \tag{1.4}$$

where θ_i's are cohomology classes on X (see [6],[17],[24]). For these integrals to make sense and to be independent of J, $\overline{\mathcal{M}}_n(X, A)$ should have a virtually orientable fundamental cycle without real codimension one boundary.

One would like to define similar invariants for the moduli spaces and evaluation maps in (1.3). The existence of such invariants for $\overline{\mathcal{M}}^{\rm disc}_{k,l}(X,L,\beta)$ is predicted by physicists ([1], [13], [22], [32]), but there are obstacles to defining such invariants mathematically. In addition to the transversality issues (which are also present in the classical case), issues concerning orientability and codimension one boundary arise.

1.1 Open GW invariants

Open GW invariants, i.e. invariants arising from the moduli space $\overline{\mathcal{M}}^{\rm disc}_{k,l}(X,L,\beta)$, have been defined in a number of settings by Liu [18], Welschinger [26],[27],[28], Solomon [25], Fukaya [5], and Georgieva [9].

The moduli spaces $\overline{\mathcal{M}}^{\rm disc}_{k,l}(X,L,\beta)$ have two types of codimension one boundary; see Figure 1.1. The first type, called disc bubbling, consists of maps from two discs with a boundary point in common. This boundary breaks into unions of components isomorphic to

$$\mathcal{M}^{\rm disc}_{1+k_1,l_1}(X,L,\beta_1)\times_{(\mathrm{ev}^B_1,\mathrm{ev}^B_1)}\mathcal{M}^{\rm disc}_{1+k_2,l_2}(X,L,\beta_2)/G, \tag{1.5}$$

where

$$k_1+k_2=k,\quad l_1+l_2=l,\quad \beta_1+\beta_2=\beta,\quad G=\begin{cases}\mathbb{Z}_2, & \text{if } k,l=0,\ \beta_1=\beta_2;\\ \{1\}, & \text{otherwise.}\end{cases}$$

The second type, called sphere bubbling, appears only if $k=0$ and β lies in the image of the natural homomorphism $j\colon H_2(X)\to H_2(X,L)$. It consists of maps from $\mathbb{P}^1$

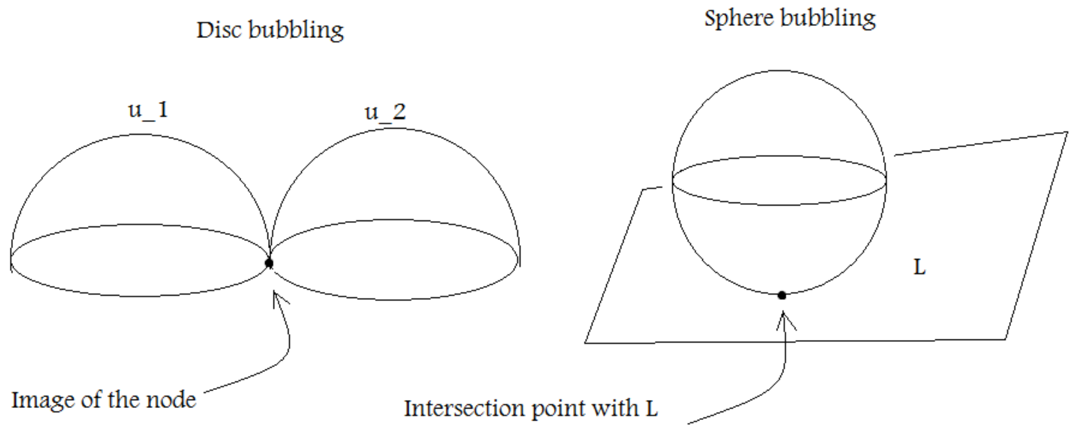


Figure 1.1: The codimension one boundary in $\overline{\mathcal{M}}_{k,l}^{\mathrm{disc}}(X,L,\beta)$

taking an extra marked point to L. This boundary is isomorphic to

$$\bigsqcup_{\tilde{\beta}\in j^{-1}(\beta)} \mathcal{M}_{1+l}(X,\tilde{\beta}) \times_{\mathrm{ev}_1} L. \tag{1.6}$$

The boundary problem is present in nearly all cases. It has been overcome in a number of cases by either adding other terms to compensate for the effect of the boundary ([30], [31], [5]) or by gluing boundary components to each other to get moduli spaces without boundary ([25], [9]). None of these methods can address the issue of sphere bubbling; we return to this issue in Section 1.3.

Whereas moduli spaces of closed curves have a canonical orientation induced by J, $\overline{\mathcal{M}}_{k,l}^{\mathrm{disc}}(X,L,\beta)$ is not necessarily orientable. Moreover, if it is orientable, there is no canonical orientation. If L has a spin structure, then $\overline{\mathcal{M}}_{k,l}^{\mathrm{disc}}(X,L,\beta)$ is orientable and a choice of spin structure canonically determines an orientation on $\overline{\mathcal{M}}_{k,l}^{\mathrm{disc}}(X,L,\beta)$; see [7, Section 8].

In [25], the disc bubbling problem is resolved by using the involution ϕ on X to identify the disc bubbling boundary components and thus define a moduli space without such boundary. If the sphere bubbling does not happen, e.g. when $\partial\beta \neq 0 \in$

$H_1(X, L)$, the resulting moduli space is orientable and gives rise to some invariants of (X, ω, ϕ).

Open GW counts for (X, L), when X is a Calabi-Yau threefold and $L \cong S^3$, are defined in [5]. These counts depend on the choice of almost-complex structure via a wall-crossing formula, due to the sphere bubbling issue. They are a priori real numbers, which are predicted in [5] and shown in this thesis to be rational; see [5, Conjecture 8.1] and Corollary 1.2 below.

We use degeneration techniques to get a better understanding of moduli spaces of J-holomorphic discs and to compute open GW invariants. If the Lagrangian L is S^n or $\mathbb{RP}^n$, we degenerate the symplectic manifold X to a nodal singular symplectic manifold, $X_+ \cup_D X_-$, where D is the intersection divisor, through a family of symplectic manifolds over a small disk, $\pi \colon \mathcal{X} \to \Delta$. We then relate the moduli spaces in the smooth fibers to some fiber product of the moduli spaces in the singular fiber. The benefit of this approach is that we can transfer some of the obstacles mentioned above to better known symplectic manifolds (e.g. projective spaces) independent of the original manifold. As an application, we obtain the following result.

Theorem 1.1. *Let (X^{2n}, ω) be a symplectic manifold with $n \geq 3$ and $c_1(TX) = 0$. If $L \subset X$ is a Lagrangian submanifold diffeomorphic to S^n and $E \in \mathbb{R}^+$, there is an open subset U_E of the set $\mathcal{J}$ of all (ω, L)-compatible almost-complex structures such that the moduli space $\overline{\mathcal{M}}^{\mathrm{disc}}(X, L, J, \beta)$ is empty whenever $\omega(\beta) < E$ and $J \in U_E$.*

This result is also stated in [29, Corollary 4.3]. Its proof in [29] involves degenerating the almost-complex structure to a singular one obtained by stretching a neighborhood of the Lagrangian and studying the behavior of the moduli space in the limit. This stretching surgery appears in Symplectic Field Theory [4] and can be performed near any Lagrangian manifold in any symplectic manifold, but the result is a non-compact manifold. By contrast, our techniques work only if there is a Hamiltonian S^1-action in T^*L, but we get closed symplectic manifolds in the end. We show that as we move toward certain exotic almost complex structures, the sphere bubbling happens for all J-holomorphic discs and they all disappear.

Corollary 1.2. *The disc counts for (X, L), where (X, ω) is a Calabi-Yau threefold and $L \subset X$ is a Lagrangian submanifold diffeomorphic to S^3, defined by [5] are rational numbers.*

This corollary, which confirms [5, Conjecture 8.1], follows immediately from Theorem 1.1, since the wall crossing changes the disc counts defined in [5] by rational numbers; see [5, Section 6].

Corollary 1.3. *Let (X^{2n}, ω) be a symplectic manifold with $n \geq 3$ and $c_1(TX) = 0$. If $L \subset X$ is a Lagrangian submanifold diffeomorphic to S^n, then L is not displacable, i.e. there exists no Hamiltonian isotopy $\psi_t \colon X \to X$ such that $\psi_1(X) \cap X = \emptyset$.*

This corollary is a special case of [7, Theorem H], but our argument avoids the technical issues that are the focus of [7]. Since there are no J-holomorphic discs in (X, L), there is no difficulty in defining the Floer homology groups for Lagrangians L_1, L_2, with either $L_1 = L_2$ or $L_1 \pitchfork L_2$, as described in [7, Section 1.1], or showing that they are preserved when either Lagrangian is deformed by a Hamiltonian isotopy. Thus, if $\psi_t \colon X \to X$ is any Hamiltonian isotopy such that $L \pitchfork \psi_1(L)$, then

$$HF^*(L, \psi_1(L)) \cong HF^*(L, L) \cong H^*(L),$$

which implies that L is not displacable.

Let $\pi \colon \mathcal{X} \to \Delta$ be a family of symplectic manifolds over a disk $\Delta \subset \mathbb{C}$ obtained from the symplectic sum construction for $X_+ \cup_D X_-$; see the paragraph preceding Theorem 1.1. An antisymplectic involution ϕ on X such that $L = \mathrm{Fix}(\phi)$ is S^n or $\mathbb{RP}^n$ then induces an antisymplectic involution $\phi_{\mathcal{X}}$ on $\mathcal{X}$ covering the standard conjugation on Δ such that $\mathrm{Fix}(\phi_{\mathcal{X}}) \cap X_+ = \emptyset$; see Corollary 2.9. In particular, Proposition 1.5 below applies to (X_+, ϕ_+). On the other hand, (X_-, ϕ_-) is a symplectic manifold with a real structure independent of X. For example, if $L \cong \mathbb{RP}^n$, then (X_-, ϕ_-) is symplectomorphic to $(\mathbb{P}^n, \omega_{FS}, \tau_n)$, where ω_{FS} is some multiple of Fubini-Study metric and τ_n is the standard complex conjugation; if $L \cong S^n$, then (X_-, ϕ_-) is symplectomorphic to $(Q^n, \omega_{FS}, \tau_{n+1})$, where $Q^n \subset \mathbb{P}^{n+1}$ is a quadratic hypersurface given by a real equation.

By contrast with the spherical case of Theorem 1.1, non-trivial open GW invariants do exist when L is diffeomorphic to the real projective plane. For example, the odd-degree open invariants of the quintic threefold computed in [23] are not zero. In [23], equivariant localization technique is used to reduce the computation in the open case to the closed case. Our approach is different, but similar in flavor: we use degeneration to reduce many computations in the open case to the closed relative case.

Theorem 1.4. *Let (X^6, ω, ϕ) be a symplectic manifold with $c_1(TX) = 0$ and $L \cong \mathbb{RP}^3$. For any equivalence class $\beta \in H_2(X, L)/\sim$ with $\partial\beta \neq 0 \in H^1(L)$, the open GW invariants N_β^{disc} are a universal linear combination of the relative GW invariants of a pair (X_+, D), where X_+ is a symplectic 6-fold with $c_1(X_+) = 0$ canonically constructed from X and D is a smooth divisor diffeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.*

This is proved in Section 5.2, where we derive an explicit formula relating the open GW invariants of X and relative invariants of (X_+, D). Although we state these theorems for Calabi-Yau manifolds, our degeneration technique can be used for any manifold X, as long as L is S^n, $\mathbb{RP}^n$, or some other special Lens space; see Chapter 2.

1.2 Real GW invariants

The moduli spaces $\overline{\mathcal{M}}_n(X, A)^{\phi,\eta}$ have mostly been ignored in the literature. As we show, the codimension one boundary consists of maps from a wedge of two spheres taking node to L. The restriction of each map to the two spheres determines elements of $\mathcal{M}_{1+n}(X, A/2)$, where $A \in H_2(X)/\sim$, that differ by the involution

$$\begin{aligned} &\phi_{\mathcal{M}} \colon \mathcal{M}_{1+n}(X, A/2) \times_{\mathrm{ev}_1} L \to \mathcal{M}_{1+n}(X, A/2) \times_{\mathrm{ev}_1} L, \\ &\phi_{\mathcal{M}}[u(z), (z_0, \cdots, z_l)] = [\phi \circ u \circ c, (-\bar{z}_0, \cdots, -\bar{z}_n)], \end{aligned} \tag{1.7}$$

where $c \colon \mathbb{P}^1 \to \mathbb{P}^1$, $c(z) = -\bar{z}$. Thus, the codimension one boundary breaks into unions of components isomorphic to

$$\mathcal{M}_{1+n}(X, A/2) \times_{\mathrm{ev}_1} L/\mathbb{Z}_2. \tag{1.8}$$

In particular, if $\mathrm{Fix}(\phi) = \emptyset$, there are no codimension boundary components, and we obtain the following result.

Proposition 1.5. *If (X, ω, ϕ) is a symplectic manifold with a real structure ϕ and* $\mathrm{Fix}(\phi) = \emptyset$, *$\overline{\mathcal{M}}_n(X, A)^{\phi,\eta}$ has a topology with respect to which it is compact and Hausdorff. It has a Kuranishi structure without boundary of virtual real dimension*

$$d = c_1(A) + \dim_{\mathbb{C}} X - 3 + 2n.$$

Thus, it determines an element of $H_d(\overline{\mathcal{M}}_n(X, A)^{\phi,\eta}, \mathcal{O})$, where $\mathcal{O}$ is the orientation bundle.

This proposition is proved in Chapter 3. In order to define invariants, we also need to consider the orientation problem, which has not been studied before. A real structure on a vector bundle $E \to X$ is an anticomplex linear involution $\phi_E \colon E \to E$ covering ϕ. A real square root of any complex line bundle $E \to X$ with real structure ϕ_E is a complex line bundle $E' \to X$ with real structure $\phi_{E'}$ such that

$$(E, \phi_E) \cong (E' \otimes E', \phi_{E'} \otimes \phi_{E'}).$$

The involution ϕ on X canonically lifts to an involution ϕ_{K_X} on the complex line bundle $K_X = \Lambda_{\mathbb{C}}^{\mathrm{top}} T^*X$.

Theorem 1.6. *Let (X, ω, ϕ) be a symplectic manifold with a real structure. If (K_X, ϕ_{K_X}) admits a real square root, all moduli spaces $\overline{\mathcal{M}}_n(X, A)^{\phi,\eta}$ are orientable. Moreover, a choice of a real square root canonically determines an orientation on $\overline{\mathcal{M}}_n(X, A)^{\phi,\eta}$.*

This theorem is proved in Section 3.3.

Remark 1.7. If $L \to \mathbb{P}^1$ a holomorphic line bundle with a complex antilinear lift $\tilde{\eta}$ of $\eta \colon \mathbb{P}^1 \to \mathbb{P}^1$, for all $k \in \mathbb{Z}$ there is a decomposition

$$H^0(L \otimes (T\mathbb{P}^1)^{\otimes k}) = H^0_+(L \otimes (T\mathbb{P}^1)^{\otimes k}) \oplus H^0_-(L \otimes (T\mathbb{P}^1)^{\otimes k})$$

into the ± 1 eigenspaces of the endomorphism

$$H^0(L \otimes (T\mathbb{P}^1)^{\otimes k}) \to H^0(L \otimes (T\mathbb{P}^1)^{\otimes k}), \quad \xi \to \tilde{\eta} \circ \xi \circ \eta;$$

the two eigenspaces are interchanged by J. Since the action of η on $\mathbb{P}^1$ has no fixed points and $H^0(L \otimes (T\mathbb{P}^1)^{\otimes k})$ is nonzero for k large enough, the zeros of every element of $H^0_+(L \otimes (T\mathbb{P}^1)^{\otimes k})$ come in pairs and thus $\deg L$ is even. Hence, if $\overline{\mathcal{M}}_n(X, A)^{\phi,\eta}$ is non-empty, then $2|K_X(A)$. Thus, if K_X has a real square root, then $4|K_X(A)$ whenever $\mathcal{M}_n(X, A)^{\phi,\eta}$ is non-empty.

Remark 1.8. If (X, ω, ϕ) is a Kahler manifold with a complex conjugation ϕ and $E' \to X$ is a holomorphic bundle, then $E' \otimes \overline{\phi^* E'}$ is a holomorphic line bundle with a real structure. Hence, if $E \to X$ is a holomorphic line bundle, $E = 2E'$, and $\overline{\phi^* E'} = E'$, then E admits a real structure. Since $\overline{\phi^* K_X} = K_X$, it follows that K_X admits a real square root if $4|K_X$.

Conjecture 1.9. *If* $4|K_X(A)$*, then* $\overline{\mathcal{M}}_n(X,A)^{\phi,\eta}$ *is orientable.*

An example with $\overline{\mathcal{M}}_n(X,A)^{\phi,\eta}$ non-orientable is described in Section 3.3. However, we are not aware of any example with X simply connected and $\overline{\mathcal{M}}_n(X,A)^{\phi,\eta}$ not orientable.

These invariants in some cases are computed in Chapter 6 and compared to similar disc invariants. If $c_1(TX) = 0$, then $\dim^{\mathrm{vir}} \overline{\mathcal{M}}(X,A)^{\phi,\eta} = 0$; in this case we define N_A^{real} to be the virtual count

$$N_A^{\mathrm{real}}(X) = \#[\overline{\mathcal{M}}(X,A)^{\phi,\eta}]^{\mathrm{vir}} \in \mathbb{Q}.$$

1.3 Mixed GW invariants

If the sphere bubbling is present, we cannot define disc invariants. In this case, instead of the moduli space $\mathcal{M}_{0,l}^{\mathrm{disc}}(X,\beta)$, we consider its $\mathbb{Z}_2$-quotient $\mathcal{M}_{0,l}(X,A)^{\phi,\tau}$ (or simply $\mathcal{M}_l(X,A)^{\phi,\tau}$). As described in Sections 1.1 and 1.2, the codimension one boundary corresponding to sphere bubbling in $\overline{\mathcal{M}}_l(X,A)^{\phi,\tau}$ is the same as the codimension one boundary of $\overline{\mathcal{M}}_l(X,A)^{\phi,\eta}$. By attaching $\overline{\mathcal{M}}_l(X,A)^{\phi,\tau}$ and $\overline{\mathcal{M}}_l(X,A)^{\phi,\eta}$ along their common boundary, we obtain a moduli space $\overline{\mathcal{M}}_l(X,A)^{\phi}$ whose codimension one boundary corresponds to disc bubbling in $\overline{\mathcal{M}}_{0,l}^{\mathrm{disc}}(X,\beta)$. We then use the method of [25] to identify the boundary components of $\overline{\mathcal{M}}_l(X,A)^{\phi}$ and get a moduli space $\widetilde{\mathcal{M}}_l(X,A)^{\phi}$ without boundary. If $L \cong S^3$ and X is a real symplectic Calabi-Yau threefold, then $\widetilde{\mathcal{M}}(X,A)^{\phi}$ is zero-dimensional and orientable. Therefore, we can define mixed real GW invariants of (X,ϕ) by

$$N_A^{\phi}(X) = \#[\widetilde{\mathcal{M}}(X,A)^{\phi}]^{\mathrm{vir}}, \qquad A \in H_2(X)/\sim .$$

By applying our degeneration technique, we prove the following statement in Section 5.

Theorem 1.10. *Let* (X^6, ω, ϕ) *be a real symplectic manifold with* $c_1(TX) = 0$*. If* $L \cong S^3$*, then*

$$N_A^{\phi}(X) = N_A^{\text{real}}(X_+).$$

In Chapter 2, we describe our degeneration setting, reviewing the symplectic cut and sum constructions along the way. We also show that ever antisymplectic involution ϕ is "standard" in a properly chosen Weinstein neighborhood of Fix(ϕ); see Lemma 2.8. In Chapter 3, we investigate the boundary and orientation problems for moduli spaces of real curves without fixed point and define some new invariants. In Chapter 4, we first review the definition of open GW invariants, then discuss some examples for Calabi-Yau threefolds, and finally introduce the notion of relative open invariants. In Chapter 5, we construct moduli spaces of discs over a family of symplectic manifolds obtained by the symplectic sum, paying special attention to the moduli space over singular fiber, and then prove Theorems 1.4 and 1.10. Finally, in Chapter 6, we compute some real and open GW invariants of $\mathbb{P}^3$ and compare them with each other.

Chapter 2

A fibration corresponding to Lens spaces

The symplectic cut procedure, introduced in [14], is a surgery technique for symplectic manifolds by means of which we can decompose a given symplectic manifold into two pieces, each again a symplectic manifold. There is an inverse operation, the symplectic sum, that glues two manifolds into one.

Let $\Delta \subset \mathbb{C}$ denote a disk centered at the origin and $\Delta^* = \Delta \setminus 0$. If $\pi\colon \mathcal{X} \to \Delta$ is any map and $\lambda \in \Delta$, let $\mathcal{X}_\lambda \equiv \pi^{-1}(\lambda)$ be the fiber over λ. A symplectic fibration is a pair $(\pi\colon \mathcal{X} \to \Delta, \omega_{\mathcal{X}})$ such that π is surjective, $(\mathcal{X}, \omega_{\mathcal{X}})$ is a symplectic manifold, X_λ is a symplectic submanifold of $(\mathcal{X}, \omega_{\mathcal{X}})$ for every $\lambda \in \Delta^*$, and $\mathcal{X}_0$ is a union of symplectic submanifolds of $(\mathcal{X}, \omega_{\mathcal{X}})$ meeting along smooth symplectic divisors. A Lagrangian subfibration of $(\pi\colon \mathcal{X} \to \Delta, \omega_{\mathcal{X}})$ is a submanifold $\mathcal{L} \subset \mathcal{X}$ disjoint from the singular locus of $\mathcal{X}_0$ such that $\pi(\mathcal{L}) = \Delta$ and $\mathcal{L}_\lambda \subset \mathcal{X}_\lambda$ is a Lagrangian submanifold for every $\lambda \in \Delta$. Thus, $\mathcal{L} \cong L_0 \times \Delta$ as fibrations over Δ and $(\mathcal{X}_\lambda, L_\lambda)$ is symplectically isotopic to $(\mathcal{X}_{\lambda'}, L_{\lambda'})$ for all $\lambda, \lambda' \in \Delta^*$.

An admissible almost complex structure on a symplectic fibration $(\pi\colon \mathcal{X} \to \Delta, \omega_{\mathcal{X}})$ is an $\omega_{\mathcal{X}}$-compatible almost complex structure on $\mathcal{X}$ which preserves $\ker d\pi$, restricts to an almost complex structure on the singular locus D of $\mathcal{X}_0$, and satisfies

$$N_{J_{\mathcal{X}}}(u,v)\in T_xD \qquad \forall\, u\in T_xD,\ v\in T_x\mathcal{X}_0,\ x\in D,$$

where $N_{J_{\mathcal{X}}}$ is the Nijenhuis tensor of $J_{\mathcal{X}}$. We denote the set of all admissible almost complex structures on $\mathcal{X}$ by $\mathcal{J}_{\mathcal{X}}$. A real structure on a symplectic fibration $(\pi:\mathcal{X}\to\Delta,\omega_{\mathcal{X}})$ is an anti-symplectic involution $\phi_{\mathcal{X}}:\mathcal{X}\to\mathcal{X}$ covering the standard complex conjugation on Δ. A $\phi_{\mathcal{X}}$-compatible admissible almost complex structure on $(\pi:\mathcal{X}\to\Delta,\omega_{\mathcal{X}})$ is an element $J_{\mathcal{X}}\in\mathcal{J}_{\mathcal{X}}$ such that $\phi_{\mathcal{X}}^*J_{\mathcal{X}}=-J_{\mathcal{X}}$. We denote the set of such almost complex structures by $\mathcal{J}_{\phi_{\mathcal{X}}}$.

A Lens space is the quotient of S^n by a free $\mathbb{Z}_k$-action by isometries. Every fixed-point free map $S^n\to S^n$ is homotopic to the antipodal map, which has degree -1 if n is even. Thus, if $\mathbb{Z}_k$ acts freely on S^n with n even, then $k\le 2$. If $n=2m-1$, a free action of $\mathbb{Z}_k$ on $S^n\subset\mathbb{C}^m$ by isometries is generated by a map of the form

$$S^n\to S^n, \qquad (z_1,\ldots,z_m)\to(\xi_1z_1,\ldots,\xi_mz_m),$$

for some primitive k-th roots $\xi_1,\ldots,\xi_m$ of 1. Any such action extends to an action on

$$Q^n\equiv\Big\{[z_0,\ldots,z_{n+1}]\in\mathbb{P}^{n+1}:\ z_0^2=\sum_{j=1}^{n+1}z_j^2\Big\}\supset Q^n_{\mathbb{R}}\equiv Q^n\cap\mathbb{RP}^n\cong S^n,$$

$$[z_0,\ldots,z_{n+1}]\to\begin{cases}[z_0,-z_1,\ldots,-z_{n+1}], & \text{if } k=2,\\ [z_0,\xi_1^{\mathbb{R}}z_1+\xi_1^{i\mathbb{R}}z_2,\xi_1^{\mathbb{R}}z_2-\xi_1^{i\mathbb{R}}z_1,\ldots,\xi_m^{\mathbb{R}}z_n+\xi_m^{i\mathbb{R}}z_{n+1},\xi_m^{\mathbb{R}}z_{n+1}-\xi_m^{i\mathbb{R}}z_n], & \text{if } n=2m-1,\end{cases}$$

where $\xi_j^{\mathbb{R}}$ and $\xi_j^{i\mathbb{R}}$ are the real and imaginary parts of ξ_j, respectively. This extension preserves the divisor

$$D_n\equiv\Big\{[z_0,\ldots,z_{n+1}]\in Q^n:\ z_0=0\Big\}\cong Q^{n-1}.$$

We call a Lens space $S^n/\mathbb{Z}_k$ archetypal if $k=1,2$ or if the induced action of $\mathbb{Z}_k$ on Q^n is free. Furthermore, $Q^n_{\mathbb{R}}$ is the fixed-point locus of the restriction τ^Q_n to Q^n of the standard involution

$$\tau_{n+1}\colon \mathbb{P}^{n+1}\to\mathbb{P}^{n+1}, \qquad [z_0,\ldots,z_{n+1}]\to[\bar{z}_0,\ldots,\bar{z}_{n+1}].$$

This restriction induces an anti-symplectic involution on the quotient $Q^n/\mathbb{Z}_k$, which we still denote by τ^Q_n.

Proposition 2.1. *Let (X,ω) be a symplectic n-fold with a Lagrangian L diffeomorphic to an archetypal Lens space $S^n/\mathbb{Z}_k$. There exists a symplectic fibration $\pi\colon\mathcal{X}\to\Delta$ with a Lagrangian subfibration $\mathcal{L}$ and $\mathcal{X}_0=X_-\cup_D X_+$, where X_- and X_+ are symplectic manifolds and $D=X_-\cap X_+$, so that*

1. *$(\mathcal{X}_\lambda,\omega_{\mathcal{X}}|_{X_\lambda},L_\lambda)$ is symplectically isotopic to (X,ω,L) for every $\lambda\in\Delta^*$;*

2. *$L_0\subset X_-$ and $(X_-,\omega_{\mathcal{X}}|_{X_-},D,L_0)$ is symplectomorphic to $(Q^n/\mathbb{Z}_k,\omega_{\mathrm{FS}},D_n/\mathbb{Z}_k,Q^n_{\mathbb{R}}/\mathbb{Z}_k)$;*

3. *$c_1(TX_+)=\frac{2k-\delta_{k,2}-n}{k}\mathrm{PD}_{X_+}[D]$ if $c_1(TX)=0$.*

Moreover, the space $\mathcal{J}_{\mathcal{X}}$ is non-empty and path-connected.

If in addition $L=\mathrm{Fix}(\phi)$ for a real structure ϕ on (X,ω), the above symplectic fibration can be chosen so that it admits a real structure $\phi_{\mathcal{X}}$ with $\mathcal{L}=\mathrm{Fix}(\phi_{\mathcal{X}})$, $(X_\lambda,\omega_{\mathcal{X}}|_{X_\lambda},\phi|_{X_\lambda})$ symplectically isotopic to (X,ω,ϕ) for every $\lambda\in\Delta^*$, and $(X_-,\omega_{\mathcal{X}}|_{X_-},D,\phi_{\mathcal{X}}|_{X_-})$ symplectomorphic to $(Q^n/\mathbb{Z}_k,\omega_{\mathrm{FS}},D_n/\mathbb{Z}_k,\tau^Q_n)$. In this case, the space $\mathcal{J}_{\phi_{\mathcal{X}}}$ is non-empty and path-connected.

Remark 2.2. If the action of $\mathbb{Z}_k$ on Q^n is not free, then $Q^n/\mathbb{Z}_k$ may be an orbifold and we get a fibration $\mathcal{X}$ which is singular along D. By resolving the singularities we get a fibration similar to that of Proposition 2.1, but in this thesis we are only interested in the $k=1,2$ cases.

Remark 2.3. Let $\mathcal{SM}$ be the category of all projective varieties (of any dimension). Let $\mathcal{SM}^+$ be the free abelian group generated by $\mathcal{SM}$ and $\mathcal{RS} \subset \mathcal{SM}^+$ be the set of all double point relations

$$[X] - [X_-] - [X_+] + [\mathbb{P}(N_D^{X_\pm} \oplus \mathbb{C})],$$

where X is the symplectic sum of $(X_\pm, D)$ and $\mathbb{P}(N_D^{X_\pm} \oplus \mathbb{C})$ is a $\mathbb{P}^1$ bundle over D. By [15, Corollary 3], $\mathcal{SM}^+/\mathcal{RS}$ is generated by the product of projective spaces. Thus, theoretically a problem can be reduced to one for projective spaces if we know how things change in a symplectic sum/cut.

In Section 2.1, we review the symplectic cut procedure and apply it to a canonical S^1-action on T^*S^n. In Section 2.2, we outline the symplectic sum procedure, apply it to the result of the symplectic cut of the previous section, and build the symplectic fibration of Proposition 2.1. Finally in Section 2.3, we show that a real structure on the starting manifold induces an involution on the symplectic fibration of Proposition 2.1.

2.1 Symplectic cut

Let (X^{2n}, ω) be a symplectic manifold and $V^{2n-1} \subset X$ be a smooth orientable submanifold of X with a free Hamiltonian S^1-action on some open neighborhood U of V. Let $h\colon U \to \mathbb{R}$ be its moment map. Assume that $a \in \mathbb{R}$ is a regular value for h and that $V = V_a = h^{-1}(a) \subset X$. Since V_a is then invariant under the S^1-action, we can construct the quotient space $D = V/S^1$. It inherits a symplectic structure from X and has real dimension $\dim_{\mathbb{R}} X - 2$; see [21, Theorem 1]. We construct a new symplectic manifold $(X_{\mathrm{cut}}, \omega_{\mathrm{cut}})$ by cutting X along V_a and contracting the two boundaries with respect to the S^1-action in such a way that X_{cut} contains two copies of D as symplectic divisors with dual normal bundles.

More precisely, extend the S^1-action on $U \subset X$ to $(U\times\mathbb{C}, \omega\oplus\omega_0)$ by multiplication by $e^{\pm i\theta}$ in the second factor. This is a Hamiltonian action with moment map

$$h_\pm \colon U \times \mathbb{C} \to \mathbb{R}, \qquad (x,z) \to h(x) \mp \frac{|z|^2}{2},$$

for which $a \in \mathbb{R}$ is still a regular value. Let $V_\pm = h_\pm^{-1}(a)$ and $U_\pm = V_\pm/S^1$. The smooth open manifold $U_\pm$ inherits a symplectic structure $\omega_\pm$ from $\omega \oplus \omega_0$. The symplectic divisors

$$D = D_\pm = (V_\pm \cap (U \times \{0\}))/S^1$$

are identical to the symplectic manifold $D = h^{-1}(a)/S^1$ and $N_D^{U_-} \cong (N_D^{U_+})^*$. Define X_{cut} to be the closed symplectic manifold obtained by gluing the open charts $U_\pm$ and $X \setminus V$ via the symplectic gluing map

$$U_{\pm(h-a)>0} \subset X \setminus V \to U_\pm \setminus D, \quad x \to (x, \sqrt{\pm(h(x)-a)}) \in V_\pm \xrightarrow{\text{proj}} U_\pm, \tag{2.1}$$

on the overlap. If V seperates X into two connected components, then X_{cut} is a union of two symplectic manifolds $(X_\pm, \omega_\pm)$ with $X_\pm$ containing $U_\pm$.

Suppose $L \cong S^n$ or $\mathbb{RP}^n$. There is a metric g on TL with respect to which all geodesics are circles of fixed length. The metric g on TL induces a metric g^{-1} on T^*L, where T^*L is the cotangent bundle with its canonical symplectic form ω_L. Let $(x^1, \cdots, x^n, y_1, \cdots, y_n)$ be local coordinates on T^*L obtained by the representation $\sum y_i dx^i$ of 1-forms. Consider the length function $h(x,y) = \sqrt{y_i y_j g^{ij}}$ on T^*L associated to g^{-1}. This function is smooth away from the zero section L_0 of T^*L. Let φ_t be the Hamiltonian flow corresponding to h on $T^*L \setminus L_0$. Every trajectory of φ_t is dual with respect to g to $\gamma' \subset TL$ for some geodesic γ in L. Therefore, φ_t is a free Hamiltonian S^1-action on $T^*L \setminus L_0$. Applying the symplectic cut procedure to this S^1-action in a neighborhood of $V_a = h^{-1}(a)$, $a > 0$, we get two symplectic manifolds

$(T^*L)_+$ and $(T^*L)_-$. Since $L = \{h \equiv 0\} \subset (h < a)$, we get a copy of L inside $(T^*L)_-$, which is disjoint from the symplectic divisor $D \cong h^{-1}(a)/S^1$. Up to a scalar factor, the symplectic geometry of $((T^*L)_-, w_-)$ only depends on L.

If $L \subset X$ is a Lagrangian manifold, by Weinstein neighborhood theorem, a neighborhood $U_L \subset X$ of L is symplectomorphic to a neighborhood of $L_0 \subset T^*L$. Therefore, if L is a Lens space, there is an S^1-action on $U_L \setminus L$ (for U_L small enough). Let $X_\pm$ be the symplectic manifolds obtained by symplectic cut with respect to this action.

If $L \cong S^n/\mathbb{Z}_k$ is an archetypal Lens space other than S^n and $\mathbb{RP}^n$, the isometric action of $\mathbb{Z}_k$ on S^n extends to T^*S^n and maps the S^1-orbits to the S^1-orbits. Thus, it extends to an action of $\mathbb{Z}_k$ on $(T^*S^n)_\pm$. By definition, this action is free and we define $(T^*L)_\pm$ to be quotient of T^*S^n with respect to this action.

2.2 Symplectic sum

In this section, we review the symplectic sum construction, following [12, Section 2]. Throughout this section $(X_\pm, D)$ are two symplectic manifolds each containing a copy of a symplectic divisor D so that their normal bundles $N_D^{X_\pm}$ are dual to each other.

The symplectic sum construction involves gluing three open symplectic charts:

$$\mathcal{X}_\pm = (X_\pm \setminus D) \times \Delta, \quad \mathcal{X}_{\text{neck}} = \{(p, x, y) \in N_D^{X_+} \oplus N_D^{X_-} | \quad |x|, |y| \leq 1, |xy| < \delta\},$$

where $\Delta \subset \mathbb{C}$ is a small disk, $\delta \in \mathbb{R}^+$ is sufficiently small, and $|\cdot|$ denotes a Hermitian norm on $N_D^{X_+}$ and the dual Hermitian norm on $N_D^{X_-} \cong (N_D^{X_+})^*$. The symplectic structure on $\mathcal{X}_\pm$ is isomorphic to $\omega_\pm \oplus \omega_0$.

Given a complex line bundle $\pi\colon E \to D$, fix a hermitian metric on E. Let

$$\rho\colon E \to \mathbb{R}, \quad \rho(x) = \frac{1}{2}|x|^2, \qquad \rho^*\colon E^* \to \mathbb{R}, \quad \rho^*(y) = \frac{1}{2}|y|^2.$$

A Hermittian connection in E defines a 1-form α on $E \backslash D$ with $\alpha(\partial\theta) = 1$. This is the pull-back of the connection form to the circle bundle viewed as a principal S^1-bundle. On the total space of E, we define a symplectic form by

$$\omega_E = \pi^* \omega_D + d(\rho\alpha).$$

It extends across the zero section and is S^1-invariant. The dual bundle E^* inherits a dual connection α^*. Hence, we get a symplectic form on $\pi \colon E \oplus E^* \to D$,

$$\begin{aligned} \omega_{\text{neck}} &= \pi^* \omega_D + d(\rho \wedge \alpha) + d(\rho^* \wedge \alpha^*) \\ &= \pi^* \omega_D + (\rho - \rho^*) \pi^* F + d\rho \wedge \alpha + d\rho^* \wedge \alpha^*, \end{aligned}$$

where F is the curvature 2-form of α. This space admits an S^1-action given by

$$(p, x, y) \to (p, e^{i\theta} x, e^{-i\theta} y)$$

with moment map $\rho^* - \rho$. There is also a natural S^1-invariant map

$$\lambda \colon E \oplus E^* \to \mathbb{C}, \qquad (p, x, y) \to xy \in \mathbb{C}.$$

Putting $E = N_D^{X_+}$, we obtain a symplectic form ω_{neck} on $\mathcal{X}_{\text{neck}}$.

Let $\mathcal{X}$ be the smooth manifold obtained by gluing the three charts by the diffeomorphisms

$$\psi_\pm \colon \mathcal{X}_{\text{neck}} \setminus N_D^{X_\mp} \to \mathcal{X}_\pm, \quad (p, x_+, x_-) \to ((p, x_\pm), x_+ x_-). \tag{2.2}$$

The map $\lambda \colon \mathcal{X}_{\text{neck}} \to \mathbb{C}$ extends to the whole of $\mathcal{X}$ and gives $\mathcal{X}$ the structure of a fibration over Δ such that the fiber over zero is $X_- \cup_D X_+$ and the other fibers are smooth manifolds.

Remark 2.4. If the manifolds $X_\pm$ are obtained from the symplectic cut procedure on (X,ω) along $V = h^{-1}(a) \subset U \subset X$, then $\mathcal{X}_\pm \cong X_{\pm(h-a)>0} \times \Delta$ and $\mathcal{X}_{\mathrm{neck}} \cong \tilde{h}^{-1}(a)/S^1$, where

$$\tilde{h}\colon U \times \mathbb{C} \times \mathbb{C} \to \mathbb{R}, \quad \tilde{h}(p,x,y) = h(p) - \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2,$$

is the moment map for the S^1-action $(p,x,y) \to (p, e^{i\theta}x, e^{-i\theta}y)$ and ω_{neck} is the symplectic structure induced from $\omega \oplus \omega_0 \oplus \omega_0$ via symplectic cut.

We next define a symplectic structure on $\mathcal{X}$. By the Symplectic Neighborhood Theorem [19, Chapter 3], a neighborhood of D in $X_\pm$ is symplectomorphic to the disc bundle of radius $\epsilon \leq 1$ in $N_D^{X_\pm}$. We can assume $\epsilon = 1$ (by some re-scaling). Let $\omega_0 = r dr d\theta$ be the symplectic form on $\mathbb{C}$. In the overlap region, where $1-\delta < |x| < 1$ and $|y| < \delta$,

$$\omega_{\mathrm{neck}} = \omega_{X_+} + d(\rho^*\alpha^*) \quad \text{and} \quad \psi_+^*(\omega_{X_+} \oplus \omega_0) = \omega_{X_+} + \lambda^*\omega_0,$$

because of the symplectic neighborhood identification. We have

$$\lambda^*(2r dr d\theta) = d(|\lambda|^2 \lambda^* d\theta) = 4d(\rho\rho^*\lambda^* d\theta) = 4d(\rho\rho^*(\alpha + \alpha^*)).$$

We can smoothly merge $\lambda^*\omega_0$ into $d(\rho^*\eta^*)$ by replacing $2\rho^*\rho(\alpha+\alpha^*)$ by

$$\eta(2\rho^*\rho(\alpha+\alpha^*)) + (1-\eta)\rho^*\alpha^*,$$

where $\eta(|x|)$ is a cutoff function such that $\eta(|x|) = 1$ if $|x| \geq 1$, $\eta(x) = 0$ if $|x| \leq 1-\delta$, and $d\eta < 2/\delta$. If δ is sufficiently small, the closed two-form

$$\omega_{X_+} + d(\eta(2\rho^*\rho(\alpha+\alpha^*)) + (1-\eta)\rho^*\alpha^*)$$

is non-degenerate; see [12, Section 2]. We can do the same procedure for the other overlap, thus obtaining a symplectic form $\omega_{\mathcal{X}}$ on $\mathcal{X}$.

Suppose $c_1(TX) = 0$. Let $A \in H_2(X_+)$ and $s = A \cdot D$. After multiplying A by some scalar, there is $B \in H_2(X_-)$ such that $B \cdot D = s$. We can glue $A\#B$ and deform it into a homology class C in the smooth fibers $\mathcal{X}_\lambda$. By [12, Lemma 2.4],

$$0 = K_{\mathcal{X}_\lambda}(C) = K_{X_-}(B) + K_{X_+}(A) + 2s.$$

In each case, K_{X_-} is equal to $\frac{1-(k_0+n)}{k} \cdot [D]$, where $k_0 = 1 + \delta_{k,2}$ is the order of the branching of $Q^n \to X_-$ along Q^{n-1}, since

$$K_{Q^n} = \pi^* K_{X_-} + (k_0 - 1)[Q^{n-1}].$$

and since $K_{Q^n} = -n[Q^{n-1}]$, this confirms (3) in Proposition 2.1.

Remark 2.5. If $k = 2$, then $L = \mathbb{RP}^n$ and $(X_-, \omega_-) \cong (\mathbb{P}^n, \omega_{\mathrm{FS}})$. Moreover, if $n = 3$, then $c_1(TX_+) = 0$.

The statement of Proposition 2.1 concerning $\mathcal{J}_{\mathcal{X}}$ follows from [12, Lemma 2.3] and [11, Theorem A.2].

2.3 Involution on the symplectic cut and sum

It remains to prove the claim of Proposition 2.1 concerning antisymplectic involutions. We call an antisymplectic involution ϕ and an S^1-action $e^{i\theta}\colon U \to U$ with moment map $h\colon U \to \mathbb{R}$ compatible if $h = h \circ \phi$. Since the Hamiltonian flow for $h \circ \phi$ is $\phi \circ e^{-i\theta} \circ \phi$, $\phi \circ e^{i\theta} = e^{-i\theta} \circ \phi$ for a ϕ-compatible S^1-action.

If $D \subset X$ is a symplectic submanifold preserved by an involution ϕ on X, the differential $d\phi$ induces a linear map

$$\phi_*\colon N_D^X \to N_D^X, \qquad v \to d\phi(v) + T_{\phi(p)}D \quad \forall v \in T_pX, p \in D,$$

covering $\phi\colon D \to D$.

Lemma 2.6. *Let (X, ω, ϕ) be a symplectic manifold with a real structure, $h\colon U \to S^1$ be the moment map for a free Hamiltonian S^1-action on an open subset $U \subset X$, and $a \in \mathbb{R}$ be a regular value of h so that the hypersurface $V_a \equiv h^{-1}(a)$ is non-empty. Let $X_{\rm cut}$ be the corresponding symplectic manifold obtained from (X, ω) by symplectically cutting along V_a as in Section 2.1 and let $D_\pm \cong D \subset X_{\rm cut}$ be the corresponding divisors. If the S^1-action is compatible with ϕ, there is a real structure $\phi_{\rm cut}$ on $X_{\rm cut}$ preserving $D_\pm$ such that the canonical projection map $X \to X_{\rm cut}/D_+ \sim D_-$ intertwines the involution ϕ and ϕ_{cut} and*

$$(N_{D_+}^{X_{\rm cut}} \otimes N_{D_-}^{X_{\rm cut}}, (\phi_{\rm cut})_* \otimes (\phi_{\rm cut})_*) \cong (D \times \mathbb{C}, \phi_{\rm cut} \times c),$$

where c is the standard complex conjugation on $\mathbb{C}$. If in addition $\mathrm{Fix}(\phi) \cap V_a = \emptyset$, *then* $\mathrm{Fix}(\phi_{\rm cut}) \cap D_\pm = \emptyset$.

Proof. We continue with the notation of the symplectic cut construction in Section 2.1. We define $\phi_{\rm cut}$ on $X_{\rm cut}$ by

$$\phi_{\rm cut}\colon X \setminus V \to X \setminus V, \quad x \to \phi(x), \qquad \phi_\pm\colon U_\pm \to U_\pm, \quad [x, z] \to [\phi(x), \bar{z}].$$

Since the moment maps $h_\pm$ are invariant with respect to the involution $(x, z) \to (\phi(x), \bar{z})$ on $U \times \mathbb{C}$ and $\phi \circ e^{i\theta} = e^{-i\theta} \circ \phi$, the second map above is well-defined. Since it is induced by an antisymplectic map on $U \times \mathbb{C}$, it is antisymplectic.

Suppose ϕ preserves an orbit $S^1 \cdot p \subset V_a$. then $\phi(p) = e^{ia} \cdot p$ for some $e^{ia} \in S^1$ and

$$\phi(e^{i\theta} \cdot p) = e^{i(a-2\theta)} \cdot e^{i\theta} \cdot p \qquad \forall e^{i\theta} \in S^1.$$

Thus, $e^{ia/2} \cdot p \in \mathrm{Fix}(\phi) \cap V_a$. This implies the last claim. □

Remark 2.7. If V_a seperates X into two connected components, then ϕ_{cut} restricted to $X_\pm$ is an involution $\phi_\pm$ agreeing on the common divisor D.

Lemma 2.8. *Let (X, ω, ϕ) be a symplectic manifold with a real structure and $L = \mathrm{Fix}(\phi)$. Then there exist a neighborhood $N(L) \subset T^*L$ of the zero section, a neighborhood $U \subset X$ of L, and a diffeomorphism*

$$\psi\colon (N(L), L) \to (U, L) \qquad \text{s.t.} \qquad \psi^*\omega = \omega_L, \quad \psi^{-1} \circ \phi \circ \psi = \tau_L,$$

where τ_L and ω_L are the canonical antisymplectic involution and symplectic form on T^*L, respectively.

Proof. The proof is a modification of the proofs of [19, Theorem 3.33] and [19, Lemma 3.14]. Let J be an (ω, ϕ)-compatible almost complex structure on X and denote by g_J the associated metric. Let

$$\Phi_q\colon T_q^*L \to T_qL, \qquad g_J(\Phi_q(v^*), v) = v^*(v) \quad \forall v^* \in T_q^*L,\ v \in T_qL,$$

be the isomorphism induced by the metric g_J. Define

$$\psi\colon T^*L \to X \quad \text{by} \quad \psi(q, v^*) = \exp_q(J_q\Phi_q(v^*)).$$

Since ϕ is an isometry of g_J,

$$\phi(\exp_q(J_q u)) = \exp_q(D\phi(J_q u)) = \exp_q(-J_q D\phi(u)) = \exp_q(-J_q u) \quad \forall v \in T_qL.$$

Therefore, $\psi^{-1}\circ\phi\circ\psi(q,v^*) = (q,-v^*) = \tau_L(q,v^*)$. By the proof of [19, Theorem 3.33], $\psi^*\omega|_L = \omega_L$.

Define $\omega_1 = \psi^*\omega$ and $\omega_0 = \omega_L$; then ω_0 and ω_1 are two symplectic forms on $N(L)$ such that $\tau_L^*\omega_i = -\omega_i$. By [19, Lemma 3.14], there is a path of symplectomorphisms φ_t such that $\varphi_1^*\omega_1 = \omega_0$. We show that φ_t can be chosen to commute with τ_L, i.e.

$$\tau_L \circ \varphi_t = \varphi_t \circ \tau_L. \tag{2.3}$$

If σ is as in [19, (3.7)], $d\sigma = \omega_1 - \omega_0$ and $\alpha \equiv \frac{\sigma+\tau_L^*\sigma}{2}$ is a τ_L-invariant closed 1-form. Replacing σ by $\sigma - \frac{\alpha}{2}$, we can assume that $\tau_L^*\sigma = -\sigma$. This implies that the path of symplectomorphisms φ_t given by the vector field X_t such that

$$\sigma = \iota_{X_t}(t\omega_1 + (1-t)\omega_0)$$

satisfies (2.3). □

Corollary 2.9. *Let (X,ω,ϕ) be a symplectic manifold with a real structure and $L =$ $\mathrm{Fix}(\phi)$. If L is an archetypal Lens space, there is a symplectic cutting of X into symplectic manifolds $(X_\pm,\omega_\pm)$ with antisymplectic involutions $\phi_\pm$ and $\phi_\pm$-invariant symplectic divisor D so that* $\mathrm{Fix}(\phi_+) = \emptyset$, $\mathrm{Fix}(\phi_-) = L$, *and there are isomorphisms*

$$(X_-,\omega_-,D,L) \cong (Q^n/\mathbb{Z}_k, \omega_{\mathrm{FS}}, D_n/\mathbb{Z}_k, Q^n_{\mathbb{R}}/\mathbb{Z}_k),$$
$$(N_D^{X_+}\otimes N_D^{X_-}, (\phi_+)_*\otimes(\phi_-)_*) \cong (D\times\mathbb{C}, \phi_\pm\times c).$$

Proof. If L is a Lens space and U is a Weinstein neighborhood as in Lemma 2.8, then $\phi|_U \cong \tau_L$ is compatible with the associated S^1-action of L described in the second half of Section 2.1 and therefore descends to the symplectic cut by Lemma 2.6. □

We next obtain a similar statement for ϕ-invariant symplectic submanifolds.

Lemma 2.10. *Let (X, ω, ϕ) be a symplectic manifold with a real structure and $D \subset X$ a symplectic divisor preserved by ϕ. Then there exist a neighborhood $N(D) \subset N_D^X$ of the zero section, a neighborhood $U \subset X$ of D, and a diffeomorphism*

$$\psi\colon (N(D), D) \to (U, D) \qquad \text{s.t.} \qquad \psi^{-1} \circ \phi \circ \psi\colon N(D) \to N(D),$$

is equal to ϕ_*.

Proof. The proof is a modification of the proof of [19, Theorem 3.30]. Let J be an (ω, ϕ)-compatible almost complex structure on X and denote by g_J the associated metric. There is an isomorphism $N_D^X \cong TD^\omega$, where TD^ω is the orthogonal complement of TD in $TX|_D$. Let $\exp\colon TD^\omega \to X$ be the exponential map associated to g_J. Since ϕ is an isometry with respect to g_J,

$$\phi\big(\exp(v)\big) = \exp\big(\phi_*(v)\big) \qquad \forall\, v \in TD^\omega.$$

The restriction ψ of exp to some neighborhood $N(D)$ of $D \subset TD^\omega$ is a diffeomorphism onto an open subset $U \subset X$. □

Lemma 2.11. *Let $(X_\pm, \omega_\pm, \phi_\pm)$ be symplectic manifolds with real structures and $D \subset X_\pm$ be a common symplectic divisor preserved by $\phi_\pm$ such that $\phi_-|_D = \phi_+|_D$. If*

$$(N_D^{X_+} \otimes N_D^{X_-}, (\phi_+)_* \otimes (\phi_-)_*) \cong (D \times \mathbb{C}, \phi_\pm \times c), \tag{2.4}$$

where c is the standard complex conjugation on $\mathbb{C}$, the corresponding symplectic sum fibration $\mathcal{X} \to \Delta$ as in Section 2.2 can be constructed so that it admits an antisymplectic involution $\phi_{\mathcal{X}}$ such that $\phi_{\mathcal{X}}|_{X_\pm} = \phi_\pm$.

Proof. For the purposes of the symplectic sum construction, we identify neighborhoods of D in $X_\pm$ and in $N_D^{X_\pm}$ as in Lemma 2.10 and use an isomorphism as in 2.4.

With notation as in Section 2.2, the involutions $\phi_\pm$ on $X_\pm$ extends to $\mathcal{X}$ by

$$\mathcal{X}_\pm \to \mathcal{X}_\pm, \qquad (p,z) \to (\phi_\pm(p), \bar{z}),$$
$$\mathcal{X}_{\mathrm{neck}} \to \mathcal{X}_{\mathrm{neck}}, \qquad (p,x,y) \to (\phi_\pm(p), (\phi_+)_* x, (\phi_-)_* y).$$

By (2.2) and (2.4), these involutions agree on the overlaps and are intertwined by λ with the conjugation on Δ. For this involution to be compatible with the symplectic structure on $\mathcal{X}$, we choose the bump function η used in the merging procedure to be symmetric with respect to the involution. □

The statement of Proposition 2.1 concerning $\mathcal{J}_{\phi_{\mathcal{X}}}$ follows from the proofs of [12, Lemma 2.3] and [11, Theorem A.2], since each step in the proofs is compatible with the involution.

A (D,ϕ)-compatible almost complex structure on $\mathcal{X}$ can be constructed by viewing $X_\pm$ as symplectic cuts of $X = X_\lambda$ for some $\lambda \in \Delta^*$. Start from an almost complex structure J on X which is compatible with the involution and the S^1-action on U. We know that $\mathcal{X}_{\mathrm{neck}} = \tilde{h}^{-1}(a)/S^1$, where

$$\tilde{h}\colon U \times \mathbb{C} \times \mathbb{C} \to \mathbb{R}, \qquad \tilde{h}(p,x,y) = h(p) - \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2,$$

is the moment map. The almost complex structure $J \oplus j \oplus j$ on $U \times \mathbb{C} \times \mathbb{C}$, where j is the standard complex structure on $\mathbb{C}$, induces an almost complex structure J_{neck} on $\mathcal{X}_{\mathrm{neck}}$, which has the required compatibility properties. There is also a natural extension of J to an almost complex structure $J_\pm$ on $\mathcal{X}_\pm$. Merging the corresponding metrics, $\omega_{\mathrm{neck}}(\cdot, J_{\mathrm{neck}}\cdot)$ and $\omega_\pm(\cdot, J_\pm\cdot)$, away from D and applying the polarization procedure of [11, Appendix], we get an almost complex structure on the total space.

Chapter 3

Moduli spaces of real curves without fixed points

In this chapter, we study the moduli space of real curves of genus zero without real points. As before, let

$$\eta, \tau \colon \mathbb{P}^1 \to \mathbb{P}^1, \qquad \eta(z) = \frac{-1}{\bar{z}}, \quad \tau(z) = \frac{1}{\bar{z}}, \tag{3.1}$$

be the standard representatives for the two isomorphism classes of antiholomorphic involutions on $\mathbb{P}^1$. Denote by G_η the set of Mobius transformations (automorphisms of $\mathbb{P}^1$), $\rho(z) = \dfrac{az+b}{cz+d}$, commuting with η. Similarly, let G_τ be the set of Mobius transformations commuting with τ and preserving each of the two discs in the complement of $\mathrm{Fix}(\tau) \cong S^1$.

There is an exact sequence

$$\{1\} \to U(1) \to U(2) \to G_\eta \to \{1\}.$$

Therefore, $G_\eta \cong U(2)/U(1) \cong S^3$ is a compact orientable Lie group. It acts freely and transitively on the sphere bundle $S(T\mathbb{P}^1)$ of $T\mathbb{P}^1$. The orientation on $S(T\mathbb{P}^1)$ as the boundary of the disc bundle $D(T\mathbb{P}^1)$ with its complex orientation thus induces an orientation on G_η. Similarly, G_τ acts freely and transitively on

$$T(\mathrm{Int}D^2) \cong (\mathrm{Int}D^2) \times S^1$$

and thus inherits an orientation from $D^2 \times S^1$.

Remark 3.1. Unlike G_η, G_τ is a non-compact Lie group. As a manifold of holomorphic maps from $(D^2, \partial D^2)$ to itself, G_η admits a natural compactification as $D^2 \times S^1$, which is a manifold with boundary $S^1 \times S^1$. As described in Chapter 1, the moduli spaces $\overline{\mathcal{M}}_l(X, A)^{\phi,\eta}$ and $\overline{\mathcal{M}}_{k,l}(X, A)^{\phi,\tau}$ exhibit a similar pattern: if $\mathrm{Fix}(\phi) \neq \emptyset$, then the latter moduli space has many more boundary components than the former moduli space.

There are many symplectic manifolds (X, ω) admitting antisymplectic involutions without fixed points:

- **Odd-dimensional projective spaces**. The involutions $\eta, \tau : \mathbb{P}^1 \to \mathbb{P}^1$ are special cases of the antiholomorphic involutions $\eta_{2m-1}, \eta_{2m-1} : \mathbb{P}^{2m-1} \to \mathbb{P}^{2m-1}$, where

$$\eta_{2m-1}\big([z_1, z_2, \ldots, z_{2m-1}, z_{2m}]\big) = \big([-\bar{z}_2, \bar{z}_1, \ldots, -\bar{z}_{2m}, \bar{z}_{2m-1}]\big), \tag{3.2}$$

$$\tau_{2m-1}\big([z_1, z_2, \ldots, z_{2m-1}, z_{2m}]\big) = \big([\bar{z}_2, \bar{z}_1, \ldots, \bar{z}_{2m}, \bar{z}_{2m-1}]\big). \tag{3.3}$$

  We note that $\mathrm{Fix}(\eta_{2m-1}) = \emptyset$ and $\mathrm{Fix}(\tau_{2m-1}) \cong \mathbb{RP}^{2m-1}$.

- **Symplectic manifolds obtained from symplectic cut surgery**. Let (X, ω, ϕ) be an arbitrary symplectic manifold with an antisymplectic involution so that $L = \mathrm{Fix}(\phi)$ is diffeomorphic to a Lens space $S^n/\mathbb{Z}_k$. By Proposition 2.1, there exists a symplectic manifold (X_+, ω_+) with an antisymplectic involution

ϕ_+ so that $(X\backslash L, \omega) \cong (X_+\backslash D, \omega_+)$ for a symplectic divisor $D \subset X_+$, $\phi \cong \phi_+$ outside of small neighborhoods of L and D, but $\mathrm{Fix}(\phi_+) = \emptyset$.

- **Lagrangian torus fibrations**. A large class of symplectic manifolds with SYZ fibrations and antisymplectic involutions is constructed in [3, Section 1.4]. In many cases, these involutions have no fixed points.

3.1 Moduli spaces of real curves

Let (X, ω, ϕ) be as before. An almost complex structure J compatible with ω is called real if $\phi^* J = -J$. Denote the set of such almost complex structures by $\mathcal{J}_\phi$. For any $J \in \mathcal{J}_\phi$, we call a J-holomorphic curve $u\colon \mathbb{P}^1 \to X$ real if the image of u is invariant under the involution ϕ. If u is somewhere injective, then pulling back ϕ to $\mathbb{P}^1$ via u, we get an induced involution σ_u on $\mathbb{P}^1$. If σ_u has any fixed points, then the Lagrangian submanifold $L = \mathrm{Fix}(\phi)$ is non-empty. In this case, we can divide u into two J-holomorphic discs with boundary on L. In this chapter, we focus primarily on the case when σ_u has no fixed points and therefore in suitable coordinates $\sigma_u = \eta$, with η as in (3.1), and thus $\mathrm{Im}(u) \subset X \setminus L$. This is the case for any u if ϕ has no fixed points.

Using the same techniques as in the proof of [20, Theorem 3.1.5], but preserving ϕ-invariance, it is straight-forward to establish the following generic regularity statement.

Lemma 3.2. *Let (X, ω, ϕ) be a symplectic manifold with a real structure. For a generic almost complex structure $J \in \mathcal{J}_\phi$, $\mathcal{M}_n(X, A)^{\phi,\eta}$ is a smooth manifold of real dimension $c_1(A) + 2n + \dim_{\mathbb{C}} X - 3$.*

The involution ϕ on X induces an involution $\tilde{\phi}$ on the moduli space $\mathcal{M}_{2n}(X,A)$ of all degree A $2n$-marked somewhere injective J-holomorphic spheres:

$$\tilde{\phi}\big([u, z_1, z_2, \ldots, z_{2n-1}, z_{2n}]\big) = [\phi \circ u \circ \eta, \eta(z_2), \eta(z_1), \ldots, \eta(z_{2n}), \eta(z_{2n-1})].$$

The fixed point locus of $\tilde{\phi}$ is precisely $\mathcal{M}_n(X,A)^{\phi,\eta}$. Therefore, intuitively, $\mathcal{M}_n(X,A)^{\phi,\eta}$ has half the dimension of $\mathcal{M}_{2n}(X,A)$.

Remark 3.3. For every J-holomorphic sphere $u_0 \colon \mathbb{P}^1 \to X$, there exists at most one antiholomorphic involution η_{u_0} such that $\mathrm{Fix}(\eta_{u_0}) = \emptyset$ and $u_0 = \phi \circ u_0 \circ \eta_{u_0}$.

Let $\overline{\mathcal{M}}_n(X,A)^{\phi,\eta}$ denote the stable map compactification of $\mathcal{M}_n(X,A)^{\phi,\eta}$. This is a closed subset of $\mathcal{M}_{2n}(X,A)$ consisting of maps $[u, z_1, \ldots, z_{2n}]$ with the property that there exists an antiholomorphic involution η_u on the domain Σ_u of u such that

$$|\mathrm{Fix}(\eta_u)| \leq 1, \qquad u = \phi \circ u \circ \eta_u, \qquad \phi_u(z_2) = z_1, \ \ldots, \ \phi_u(z_{2n}) = z_{2n-1}.$$

Thus, there are two possible cases for $\eta_u \colon \Sigma_u \to \Sigma_u$:

1. $\Sigma = \Sigma_0 \cup \bigcup_i (\Sigma_i \sqcup \Sigma_{\bar{i}})$, $\eta_u : \Sigma_0 \to \Sigma_0$ is an antiholomorphic involution without fixed points, and $\eta_u : \Sigma_i \to \Sigma_{\bar{i}}$ is an antiholomorphic map with inverse $\eta_u \colon \Sigma_{\bar{i}} \to \Sigma_i$;

2. $\Sigma = \bigcup_i (\Sigma_i \cup \Sigma_{\bar{i}})$, $\eta_u : \Sigma_i \to \Sigma_{\bar{i}}$ is an antiholomorphic map with inverse $\eta_u \colon \Sigma_{\bar{i}} \to \Sigma_i$.

In the second case, η_u fixes a node of Σ_u. Since it must be mapped by u to $\mathrm{Fix}(\phi)$, $\overline{\mathcal{M}}_n(X,A)^{\phi,\eta}$ contains no such elements if $\mathrm{Fix}(\phi) = \emptyset$.

Lemma 3.4. *Let (X, ω, ϕ) be a symplectic manifold with a real structure. If* $\mathrm{Fix}(\phi) =$ $\emptyset$*, all boundary strata of* $\overline{\mathcal{M}}_n(X,A)^{\phi,\eta}$ *are of virtual codimension at least two.*

Proof. The virtual codimension of a boundary stratum of $\overline{\mathcal{M}}_n(X, A)^{\phi,\eta}$ is the number of nodes in the domains of the elements of the stratum. Since $\overline{\mathcal{M}}_n(X, A)^{\phi,\eta}$ contains no elements of the second type above, its boundary strata have codimension at least two. □

Thus, $\overline{\mathcal{M}}_n(X, A)^{\phi,\eta}$ is a moduli space without codimension one boundary in this case, and there is a hope of defining GW-type invariants directly from $\overline{\mathcal{M}}_n(X, A)^{\phi,\eta}$. In order to do this, it remains to study the orientation problem.

Remark 3.5. If $\mathrm{Fix}(\phi) \neq \emptyset$, $\partial\overline{\mathcal{M}}_n(X, A)^{\phi,\eta}$ need not be empty, but it also appears as the boundary of $\overline{\mathcal{M}}_{0,n}(X, A)^{\phi,\tau}$. After studying the relevant orientations, we identify the two moduli spaces along their common boundaries and obtain a new moduli space which (after some other modifications special to J-holomorphic discs) has no boundary. As noted in [23, Section 1.5], the moduli spaces $\overline{\mathcal{M}}_{0,n}(X, A)^{\phi,\tau}$ and $\overline{\mathcal{M}}_n(X, A)^{\phi,\eta}$ often need to be combined in order to get well-defined invariants. We explain this in more detail in Chapters 4, 5, and 6.

Remark 3.6. If $\tilde{u}\colon \mathbb{P}^1 \to X$ is any J-holomorphic map such that $\tilde{u} = \phi \circ \tilde{u} \circ \eta$, then $\tilde{u} = u \circ \pi_r$ for some degree r branched covering $\pi_r\colon \mathbb{P}^1 \to \mathbb{P}^1$ such that $\pi_r \circ \eta = \eta \circ \pi_r$ and somewhere injective J-holomorphic map $u : \mathbb{P}^1 \to X$ such that $u = \phi \circ u \circ \eta$. The commutative condition on π_r implies that the zeros and poles of π_r (viewed as a rational function on $\mathbb{C}$) are interchanged by η and the degree r is odd.

In Section 3.2, we construct a Kuranishi structure for the moduli space $\overline{\mathcal{M}}_n(X, A)^{\phi,\eta}$. In Section 3.3, we discuss the orientation problem.

3.2 Kuranishi structure on the moduli of real curves

Let (X, ω, ϕ) be as before and fix some $J \in \mathcal{J}_\phi$. In this section, we briefly explain how to construct a Kuranishi structure for the moduli space $\mathcal{M}_n(X, A)^{\phi,\eta}$. Such a construction for $\overline{\mathcal{M}}_{k,l}(X, A)^{\phi,\tau}$ is described in [25, Section 7]; we only describe the necessary adjustments. For simplicity, we ignore the marked points until the end of this construction.

For $(u, (z_i, \overline{z_i})_{i=1}^n) \in \mathcal{M}_n(X, A)^{\phi,\eta}$, let

$$E_u \equiv u^*TX \to \mathbb{P}^1, \quad E_u^{0,1} \equiv (T^*\mathbb{P}^1)^{0,1} \otimes_{\mathbb{C}} E_u.$$

There are commutative diagrams

$$\begin{array}{ccc} E_u & \xrightarrow{T_\phi} & E_u \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{P}^1 & \xrightarrow{\eta} & \mathbb{P}^1 \end{array} \qquad\qquad \begin{array}{ccc} E_u^{0,1} & \xrightarrow{T_\phi^1} & E_u^{0,1} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{P}^1 & \xrightarrow{\mathrm{id}} & \mathbb{P}^1 \end{array} \tag{3.4}$$

where $T_\phi v = d\phi(v)$ and $T_\phi^1 \alpha = d\phi \circ \alpha \circ d\eta$. The deformation theory of $\mathcal{M}_0(X, A)^{\phi,\eta}$ is described by the linearization of the Cauchy-Riemann operator,

$$L_{J,u} \colon W^{k,p}(E_u) \to W^{k-1,p}(E_u^{0,1}), \quad p > 2, k \geq 1; \tag{3.5}$$

see [20, Chapter 3] for a similar situation. If ∇ is the Levi-Civita connection of the metric $\omega(\cdot, J\cdot)$, $L_{J,u}$ can be written as

$$L_{J,u}(\xi) = \frac{1}{2}(\nabla\xi + J\nabla\xi \circ j) - \frac{1}{2}J(\nabla_\xi J)\partial_J(u).$$

These all fit into a commutative diagram

$$\begin{array}{ccc} W^{k,p}(E_u) & \xrightarrow{L_{J,u}} & W^{k-1,p}(E_u^{0,1}) \\ \downarrow{\scriptstyle \tilde{T}_\phi} & & \downarrow{\scriptstyle \tilde{T}^1_\phi} \\ W^{k,p}(E_u) & \xrightarrow{L_{J,u}} & W^{k-1,p}(E_u^{0,1}) \end{array}$$

where $\{\tilde{T}_\phi \xi\}(z) = T_\phi(\xi(\eta(z)))$ and $\{\tilde{T}^1_\phi \alpha\}(z) = T^1_\phi(\alpha(z))$. Let

$$\begin{aligned} W^{k,p}(E_u)_{\mathbb{R}} &= \{\xi \in W^{k,p}(E_u) \mid \tilde{T}_\phi(\xi) = \xi\}, \\ W^{k-1,p}(E_u^{0,1})_{\mathbb{R}} &= \{\alpha \in W^{k-1,p}(E_u^{0,1}) \mid \tilde{T}^1_\phi(\alpha) = \alpha\} \end{aligned} \tag{3.6}$$

denote the spaces of real sections. Let $H^0(E_u)_{\mathbb{R}}$ and $H^1(E_u)_{\mathbb{R}}$ be the kernel and cokernel, respectively, of the restricted operator

$$L_{J,u} \colon W^{k,p}(E_u)_{\mathbb{R}} \to W^{k-1,p}(E_u 0, 1)_{\mathbb{R}}.$$

If $H^1(E_u)_{\mathbb{R}} = 0$, then $\mathcal{M}_n(X, A)^{\phi,\eta}$ is a manifold near u of real dimension

$$\dim_{\mathbb{R}} H^0(E_u)_{\mathbb{R}} - \dim G_\eta = \mathrm{ind}_{\mathbb{R}}(L_{J,u}) - 3 = c_1(A) + \dim_{\mathbb{C}} X - 3; \tag{3.7}$$

see [20, Theorem C.1.10]. Each pair of conjugate marked points increases the dimension by two and we get the dimension formula in Lemma 3.2.

If $H^1(E_u)_{\mathbb{R}} \neq 0$, we construct a Kuranishi chart around u. For this aim, we choose finite-dimensional complex subspaces $\mathcal{E}_u \subset W^{k,p-1}(E_u^{0,1})$ such that

1. every $\xi \in \mathcal{E}_u$ is smooth and supported away from the boundary and marked points;
2. $\tilde{T}^1_\phi(\mathcal{E}_u) = \mathcal{E}_u$;
3. $L_{J,u}$ modulo $\mathcal{E}_u$ is surjective.

We then choose our Kuranishi neighborhood to be $V(u) = [\bar{\partial}^{-1}(\mathcal{E}_u)]_{\mathbb{R}}$ (modulo G_η), which is a smooth manifold of dimension

$$c_1(A) + \dim_{\mathbb{C}} X - 3 + 2n + \dim_{\mathbb{C}}(\mathcal{E}_u).$$

The obstruction bundle $\mathcal{E}(u)$ at each $f \in V(u)$ is obtained by parallel translation of $\mathcal{E}_u$ with respect to the induced metric of J. We thus get a Kuranishi neighborhood $(V(u), \mathcal{E}(u))$. The Kuranishi map in this case is just the Cauchy-Riemann operator $f \to \bar{\partial}(f)$.

In order to construct Kuranishi charts for u in the boundary of $\overline{\mathcal{M}_n}(X, A)^{\phi,\eta}$, we need gluing theorems as in [7, Chapter 7]. The gluing theorems are identical to those for J-holomorphic discs; we thus omit the details and refer the reader to [7].

If $L = \text{Fix}(\phi)$ is non-empty, $\overline{\mathcal{M}_n}(X, A)^{\phi,\eta}$ might have non-empty boundary. A boundary curve is of the form $(u, \Sigma = \Sigma_1 \cup_q \Sigma_2)$, where $\Sigma_i = \mathbb{P}^1$, $\eta\colon \Sigma_1 \to \Sigma_2$, and $u(q) \in L$. After a suitable reparametrization, we may assume $q = 0 \in \mathbb{P}^1$ and $\eta(z) = -\overline{w}$. For real parameters $\epsilon \neq 0$, we can glue the domain into a family of smooth curves

$$\Sigma_\epsilon = \{(z, w) \in \mathbb{C} : zw = \epsilon\}.$$

For $\epsilon \in \mathbb{R}$, Σ_ϵ inherits a complex conjugation from η:

$$\eta^\epsilon\colon \Sigma_\epsilon \to \Sigma_\epsilon, \qquad \eta_\epsilon(z, w) = (-\overline{w}, -\overline{z}).$$

For $\epsilon < 0$, η_ϵ has an S^1 set of fixed points and for $\epsilon > 0$, η_ϵ is fixed point free.

By smoothing in one direction (ϵ positive), we get real curves without fixed points in $\overline{\mathcal{M}}(X, A)^{\phi,\eta}$; by smoothing in the other direction (ϵ negative), we get real curves with fixed points in $\overline{\mathcal{M}}(X, A)^{\phi,\tau}$. Thus, $\overline{\mathcal{M}}(X, A)^{\phi,\eta}$ and $\overline{\mathcal{M}}(X, A)^{\phi,\tau}$ have a boundary in common; this boundary is described in (1.8). We identify the common boundary and glue the two moduli spaces to get a new moduli space whose only boundary component comes from the real sphere bubbling (disc bubbling). We define $\overline{\mathcal{M}}_{0,n}(X,A)^{\phi}$ to be the resulted space. We come back to this moduli space later. We summarize these arguments in the following proposition.

Proposition 3.7. *Let (X,ω,ϕ) be a symplectic manifold with a real structure. The moduli space $\overline{\mathcal{M}}_l(X,A)^{\phi}$ has a topology with respect to which it is compact and Hausdorff. It has a Kuranishi structure with boundary of virtual real dimension $c_1(A)+\dim_{\mathbb{C}} X-3+2l$. Moreover, the boundary strata can be expressed in terms of fiber products of the form*

$$\mathcal{M}_{1,l_1}(X,A_1)^{\phi,\tau} \times_{(\mathrm{ev}_1^B,\mathrm{ev}_1^B)} \mathcal{M}_{1,l_2}(X,A_2)^{\phi,\tau}/G,$$

where

$$l_1+l_2=l, \quad A_1+A_2=A, \quad G=\begin{cases}\mathbb{Z}_2, & \text{if } l=0,\ A_1=A_2;\\ \{1\}, & \text{otherwise.}\end{cases}$$

3.3 Orientation

In the orientation problem for $\mathcal{M}_n(X,A)^{\phi,\eta}$, it is sufficient to consider the case $n=0$ because any pair of marked points $(z_i,\overline{z_i})$ increases the tangent space by $T_{z_i}\mathbb{P}^1$, which has a canonical orientation. Let $\mathcal{P}_0(X,A)^{\phi,\eta}$ be the moduli space of parametrized J-holomorphic maps (before quotienting by G_η) so that

$$\mathcal{M}_0(X,A)^{\phi,\eta}=\mathcal{P}_0(X,A)^{\phi,\eta}/G_\eta.$$

In fact, $\mathcal{P}_0(X,A)^{\phi,\eta}$ is a principal S^3-bundle over moduli space. In order to put an orientation on $\mathcal{M}_0(X,A)^{\phi,\eta}$, it is enough to orient $\mathcal{P}_0(X,A)^{\phi,\eta}$. For this, we need to orient the determinant of the index bundle

$$\Lambda^{\mathrm{top}} H^0(E)_{\mathbb{R}} \otimes \Lambda^{\mathrm{top}} (H^1(E)_{\mathbb{R}})^*,$$

where $E = u^*TX$ and $H^0(E)_{\mathbb{R}}$ and $H^1(E)_{\mathbb{R}}$ are the kernel and cokernel of a real Cauchy-Riemann operator. Recall that E admits an anticomplex linear involution T_ϕ; see the left diagram in (3.4).

Definition 3.8. Let $E \to \mathbb{P}^1$ be a complex vector bundle with a real structure ϕ covering η. We call a trivialization of E over $\mathbb{C}^*$,

$$\begin{array}{ccc} E & \xrightarrow{\psi} & \mathbb{C}^* \times \mathbb{C}^m \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{C}^* & \xrightarrow{\mathrm{id}} & \mathbb{C}^* \end{array}$$

admissible if the involution $\phi_\psi(z) = \psi_{\eta(z)} \circ \phi \circ \psi_z^{-1}$ coincides with the standard involution $C := (z, v) \to (\eta(z), \bar{v})$. Admissible trivializations ψ and ψ' of (E, ϕ) over $\mathbb{C}^*$ are called homotopic if there is a family of such trivializations ψ_t, $t \in [0, 1]$, such that $\psi_0 = \psi$ and $\psi_1 = \psi'$.

Lemma 3.9. *For every complex vector bundle $E \to \mathbb{P}^1$ with a real structure ϕ covering η, there are two homotopy classes of admissible trivializations over $\mathbb{C}^*$. Moreover, for every admissible trivialization ψ and every map of the form*

$$R_{(\epsilon_i)} \colon \mathbb{C}^* \times \mathbb{C}^m \to \mathbb{C}^* \times \mathbb{C}^m, \quad R_{(\epsilon_i)}(z, v) = (\epsilon_1 v_1, \cdots, \epsilon_m v_m), \quad \epsilon_i = \pm 1,$$

$R \circ \psi$ is another admissible trivialization which is in same homotopy class as ψ if and only if $\prod \epsilon_i = 1$.

Proof. **Existence.** As a complex vector bundle, E is trivial over $\mathbb{C}^*$. Therefore, we can fix a trivialization $\psi\colon E \to \mathbb{C}^* \times \mathbb{C}^m$. The involution ϕ then corresponds to a map

$$\phi_\psi\colon \mathbb{C}^* \to \mathrm{GL}(2m, \mathbb{R})$$

whose image lies in the set of anticomplex linear matrices. We find a change of trivialization matrix

$$A\colon \mathbb{C}^* \to \mathrm{GL}(m, \mathbb{C}) \qquad \text{s.t.} \quad A_{\eta(z)} \circ \phi_\psi \circ A_z^{-1} = C. \tag{3.8}$$

Let $B_\psi(z) = C \circ \phi_\psi(z) \in \mathrm{GL}(m, \mathbb{C})$. Composing on the left by C, we can rewrite (3.8) to get the equivalent equation

$$\overline{A_{\eta(z)}} \circ B_\psi \circ A_z^{-1} = \mathbb{I}_m. \tag{3.9}$$

Since ϕ_ψ is an involution, $\overline{B_\psi(\eta(z))} B_\psi(z) = \mathbb{I}_m$. For $z \in \mathbb{H} \setminus \{0\}$, where $\mathbb{H}$ is the closed upper half plane, let $\alpha(z) \in \mathrm{GL}(m, \mathbb{C})$ be a family of matrices such that

$$\alpha(r) = \begin{cases} \mathbb{I}_m & \text{if } r \in \mathbb{R}^+; \\ \overline{B_\psi(\eta(r))} & \text{if } r \in \mathbb{R}^-. \end{cases}$$

Next define

$$A(z) = \begin{cases} \alpha(z) B_\psi(z) & \text{if } z \in \mathbb{H} - \{0\}; \\ \overline{\alpha(\eta(z))} & \text{if } z \in \overline{\mathbb{H}} - \{0\}. \end{cases}$$

It is easy to check that A is well-defined and satisfies (3.9).

Homotopy classes of admissible trivializations. If ψ is an admissible trivialization, any other admissible trivialization is of the form $A \circ \psi$, where

$$A\colon \mathbb{C}^* \to \mathrm{GL}(m, \mathbb{C}) \qquad \text{and} \quad \overline{A(\eta(z))}A(z)^{-1} = \mathbb{I}_m. \tag{3.10}$$

The question is whether A is homotopic to identity through a family A_t of matrices satisfying the same equation as (3.10). Let

$$G = \left\{\gamma\colon [0,1] \to \mathrm{GL}(m, \mathbb{C}) \mid \gamma(0) = \overline{\gamma(1)}\right\}, \quad G_0 = \{\gamma \in G : \gamma(0) = \mathbb{I}_m\};$$

the set G is a group under pointwise multiplication, while G_0 is its subgroup.

The restriction of A to the upper semi-circle, $\{z = e^{i\pi t} \mid t \in [0,1]\}$, determines an element of G. In fact, the space of A's satisfying (3.9) is homotopic to G. The map

$$G \to \mathrm{GL}(m, \mathbb{C}), \quad \gamma \to \gamma(0),$$

is a fiber bundle with fiber G_0. From the associated long exact sequence,

$$\cdots \to \pi_1(\mathrm{GL}(m, \mathbb{C})) \to \pi_0(G_0) \cong \pi_1(\mathrm{GL}(m, \mathbb{C})) \to \pi_0(G) \to \pi_0(\mathrm{GL}(m, \mathbb{C})) \to 0$$

we conclude that $\pi_0(G) = \mathbb{Z}/2\mathbb{Z}$.[1] Therefore, there are two homotopy classes of trivializations.

The remaining claim of the lemma is checked by chasing the maps in the long exact sequence. □

Lemma 3.10. *Let $E \to \mathbb{P}^1$ be a complex vector bundle with a real structure ϕ lifting η. Every admissible trivialization of (E, ϕ) over $\mathbb{C}^* \subset \mathbb{P}^1$ canonically determines an ori-*

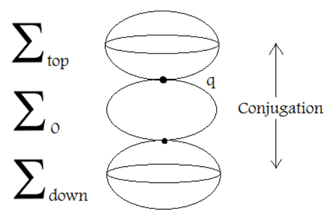
[1]The homomorphism $\pi_1(\mathrm{GL}(m, \mathbb{C})) \to \pi_0(G_0) \cong \pi_1(\mathrm{GL}(m, \mathbb{C})) \cong \mathbb{Z}$ is multiplication by -2 for the following reason. The loop $\gamma : [0,1] \to \mathrm{GL}(m, \mathbb{C})$ of the diagonal matrices with the first entry $e^{2i\pi t}$ and the remaining entries 1 generates $\pi_1(\mathrm{GL}(m, \mathbb{C}))$. It lifts to the path $s \to \gamma_s$ in G given by $\gamma_s(t) = \gamma((1-2s)t)$; the end point of this path is an element of $\pi_1(\mathrm{GL}(m, \mathbb{C}))$ homotopic to -2γ.

entation of $\Lambda^{\text{top}} H^0(E)_{\mathbb{R}} \otimes \Lambda^{\text{top}}(H^1(E)_{\mathbb{R}})^*$. The two orientations given by two different admissible trivializations coincide if and only if they are in the same homotopy class.

Proof. The proof is analogous to that of [7, Proposition 8.1.4]. Contracting each of the two circles

$$C_{0,r} = \{z \in \mathbb{C}^* \mid |z| = r\} \quad \text{and} \quad C_{\infty,r} = \left\{z \in \mathbb{C}^* \mid |z| = \frac{1}{r}\right\},$$

to a point, we obtain a nodal curve $\Sigma = \Sigma_{\text{top}} \cup \Sigma_0 \cup \Sigma_{\text{down}}$ (picture below) with an induced fixed point free involution η_Σ. We denote the quotient map by $\pi\colon \mathbb{P}^1 \to \Sigma$. Denote by q and $\eta_\Sigma(q)$ the nodal points of Σ. We may assume that q and $\eta_\Sigma(q)$ are respectively 0 and ∞ in $\Sigma_0 \cong \mathbb{P}^1$.



Via the given trivialization, the bundle (E, ϕ) descends to a bundle $(\tilde{E}, \tilde{\phi})$ over Σ so that

$$\tilde{E} \mid_{\Sigma_0} \cong \mathbb{P}^1 \times \mathbb{C}^m$$

and the involution $\tilde{\phi} \mid_{\Sigma_0}$ sends (z, v) to $(\eta_\Sigma(z), \bar{v})$. Over $\Sigma_{\text{top}} \cup \Sigma_{\text{down}}$, $\tilde{\phi}$ is an anti-complex linear map of the form

$$\tilde{\phi}\colon \tilde{E} \mid_{\Sigma_{\text{top}}} \to \tilde{E} \mid_{\Sigma_{\text{down}}}.$$

A section of $(\tilde{E}, \tilde{\phi})$ is of the form $\xi = (\xi_{\text{top}}, \xi_0, \xi_{\text{down}})$, with matching conditions at the nodes. A section ξ is real if and only if

$$\xi_{\text{down}}(\eta_\Sigma(z)) = \tilde{\phi}(\xi_{\text{top}}(z)) \quad \text{and} \quad \xi_0 \in \Gamma(\tilde{E} \mid_{\Sigma_0})_{\mathbb{R}}.$$

Therefore, it is determined by an arbitrary section of $\tilde{E}\mid_{\Sigma_{\text{top}}}$ and a real section of $\tilde{E}\mid_{\Sigma_0}$ which match at q.

The matching condition at the nodes gives a short exact sequence

$$0 \to W^{1,p}(\tilde{E}) \to W^{1,p}(\tilde{E}\mid_{\Sigma_{\text{top}}}) \oplus W^{1,p}(\tilde{E}\mid_{\Sigma_0}) \oplus W^{1,p}(\tilde{E}\mid_{\Sigma_{\text{down}}}) \to \mathbb{C}^m_q \oplus \mathbb{C}^m_{\eta_\Sigma(q)} \to 0.$$

The associated index of the pair $(\tilde{E}, \tilde{\phi})$ is given by the Cauchy-Riemann operator

$$\bar{\partial}_{\tilde{E}} \colon W^{1,p}(\tilde{E})_{\mathbb{R}} \to L^p(\Lambda^{0,1} \otimes \tilde{E}\mid_{\Sigma_{\text{top}}}) \oplus L^p(\Lambda^{0,1} \otimes \tilde{E}\mid_{\Sigma_0}) \oplus L^p(\Lambda^{0,1} \otimes \tilde{E}\mid_{\Sigma_{\text{down}}}).$$

This index bundle is equivalent to

$$\text{ind}_{\mathbb{R}}\bar{\partial}(W^{1,p}(\tilde{E}\mid_{\Sigma_{\text{top}}}) \oplus W^{1,p}(\tilde{E}\mid_{\Sigma_0}) \oplus W^{1,p}(\tilde{E}\mid_{\Sigma_{\text{down}}})) \otimes (\text{ind}_{\mathbb{R}}(\mathbb{C}^m_q \oplus \mathbb{C}^m_{\eta_\Sigma(q)}))^*.$$

Therefore, by orienting the index bundle of the middle and right terms in the short exact sequence, we get an orientation of the left-hand side. Over Σ_0, we have the trivial complex bundle with the canonical complex conjugation and the index bundle is canonically isomorphic (after deforming the Cauchy-Riemann operator) to

$$\Lambda^{\text{top}} H^0(\mathbb{P}^1 \times \mathbb{C}^m)_{\mathbb{R}} = \Lambda^{\text{top}}_{\mathbb{R}} \mathbb{R}^m \subset \Lambda^{\text{top}}_{\mathbb{C}} \mathbb{C}^m.$$

It inherits an orientation from the choice of trivialization. From the discussion of real sections above, we know that there is a canonical isomorphism

$$\text{ind}_{\mathbb{R}}(\tilde{E}\mid_{\Sigma_{\text{top}} \cup \Sigma_{\text{down}}}) \cong \text{ind}_{\mathbb{C}}(\tilde{E}\mid_{\Sigma_{\text{top}}}).$$

Since $\text{ind}_{\mathbb{C}}(\tilde{E}\mid_{\Sigma_{\text{top}}})$ carries an orientation induced by its complex structure, the above isomorphism gives an orientation of the left-hand side of the equation. Similarly, the right-hand side of the short exact sequence has a canonical orientation, being a complex vector space. □

By Lemma 3.10, a systematic way of orienting u^*TX over $\mathbb{C}^* \subset \mathbb{P}^1$ would orient the moduli space $\mathcal{P}_n(X,A)^{\phi,\eta}$. Let $K_X = \Lambda^{\text{top}}_{\mathbb{C}} T^*X$ be the canonical complex line bundle over X. It inherits an involution $K_\phi\colon K_X \to K_X$ (covering ϕ) from T_ϕ. Therefore, it is a complex line bundle with an involution. Any admissible trivialization of $u^*TX \mid_{\mathbb{C}^*}$ canonically induces an admissible trivialization of $u^*K_X \mid_{\mathbb{C}^*}$ and changing the homotopy class of admissible trivialization of the former changes the homotopy class of the induced admissible trivialization. We can therefore reduce the orientation problem to the problem of finding a canonical way of admissibly trivializing u^*K_X. This is an easier problem because K_X is just a line bundle and has less structure than TX.

Let $(L,\phi_L) \to (X,\phi)$ be any complex line bundle over X with an anticomplex linear involution ϕ_L covering ϕ. The line bundle $L^{\otimes 2}$ inherits an involution from the one on L by

$$\phi_{L^{\otimes 2}}(v_1 \otimes v_2) = \phi_L(v_1) \otimes \phi_L(v_2).$$

Every admissible trivialization of $u^*L \mid_{\mathbb{C}^*}$ induces an admissible trivialization of $u^*L^{\otimes 2} \mid_{\mathbb{C}^*}$. However, changing the homotopy class of trivialization of L does not change the homotopy class of the induced trivialization on $L^{\otimes 2}$, since changing the trivialization of L by the complex linear map R_{-1} of Lemma 3.9 changes the homotopy class of admissible trivialization of $L^{\otimes 2}$ by $R_{-1} \otimes R_{-1} = \text{id}$.

Corollary 3.11. *The complex line bundle $(L^{\otimes 2}, \phi_L \otimes_{\mathbb{C}} \phi_L)$ as above has a canonical admissible trivialization.*

Proof of Theorem 1.6. If K_X has a real square root, then by Corollary 3.11 there is a canonical choice of admissible trivialization. Therefore, $\mathcal{P}_n(X,A)^{\phi,\eta}$ is orientable. Moreover, a choice of trivialization is canonically determined by the choice of a real square root for K_X. Therefore, if K_X has a real square root, then $\mathcal{M}_n(X,A)^{\phi,\eta}$ is orientable. $\square$

In this thesis, we are interested in the following examples in which K_X has a square root. In Chapter 6 we consider $(\mathbb{P}^3, \eta_3)$; since $4|K_{\mathbb{P}^3}$, it has a real square root. If K_X is trivial as a complex line bundle (i.e X is a symplectic Calabi-Yau manifold), then K_X has a real square root; moreover, in this case we can fix an admissible trivialization of K_X itself over X (independent of any map u) and thus determine an orientation of moduli space $\mathcal{M}_n(X,A)^{\phi,\eta}$.

As illustrated by the two examples below, there are cases where the index bundle is not orientable. The first example is similar to the non-orientable example of [7, Section 8.1.2].

Example 3.12. Let $E = \mathbb{C} \times \mathbb{P}^1 \times S^1 \to \mathbb{P}^1 \times S^1$. Define a family of involutions,

$$\phi_s \colon E\,|_{\mathbb{P}^1\times\{s\}} \to E\,|_{\mathbb{P}^1\times\{s\}}, \quad \phi_s(z,v) = (\eta(z), \overline{e^{2\pi i s}v}) \quad \forall s \in S^1.$$

The real line bundle $F \to S^1$ given by $F_s = H^0(E\,|_{\mathbb{P}^1\times\{s\}})_{\mathbb{R}}$ is then not orientable.

Example 3.13. Let $X = \mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$, $A = \{\mathrm{pt}\} \times \mathbb{P}^1 \times \{\mathrm{pt}\} \in H_2(X)$,

$$\begin{aligned} &\phi \colon X \to X, \qquad \phi(s,t,z,w) = (s, -t, -\frac{1}{\bar{z}}, \overline{e^{2\pi i s}w}), \\ &Y = \{(s,t,w) \in \mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{P}^1 : (-t, \overline{e^{2\pi i s}w}) = (t,w)\}. \end{aligned} \tag{3.11}$$

The space Y is a union of two Klien bottles with double cover

$$\mathbb{R} \cup \{\infty\} \times \mathbb{R}/\mathbb{Z} \times \{0, 1/2\} \to Y, \quad (a,s,t) \to (2s, t, ae^{2\pi i s}).$$

Let $\pi \colon X \to \mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{P}^1$ be the projection to the first and third factors. Since

$$f : \mathcal{M}(X,A)^{\phi,\eta} \to Y, \qquad [u] \to \pi(\mathrm{Im}(u)),$$

is well-defined and is a diffeomorphism, it follows that $\mathcal{M}(X,A)^{\phi,\eta}$ is not orientable. If $\gamma \subset \mathcal{M}(X,A)^{\phi,\eta}$ is the preimage of the map $S^1 \to Y$, $s \to (s,0,0)$,

$$\gamma^*\det(T\mathcal{M}(X,A)^{\phi,\eta}) = \Lambda^{\mathrm{top}} H^0_{\mathbb{R}}(\gamma^* TX) \otimes (\Lambda^{\mathrm{top}}\mathrm{Lie}(G_\eta))^* = \mathbb{R} \otimes F,$$

where F is the unorientable line bundle in Example 3.12.

Chapter 4

J-holomorphic discs and open GW invariants

Throughout this section (X, ω, ϕ, L) denotes a symplectic manifold equipped with an antisymplectic involution ϕ whose fixed-point set is a Lagrangian L. We assume that L is orientable and spin and fix an orientation and a spin structure σ on L. In this case, open invariants are defined in [25] using perturbed Cauchy-Riemann equations. In Section 4.1 below, we review the construction of these invariants in the language of Kuranishi structures. In Section 4.2, we outline the construction of a relative version of open invarinats.

4.1 Review of open GW invariants

The involution ϕ on X induces an involution on $\mathcal{M}_{k,l}^{\mathrm{disc}}(X, L, \beta)$,

$$\begin{aligned} &\tau_{\mathcal{M}} \colon \mathcal{M}_{k,l}^{\mathrm{disc}}(X, L, \beta) \to \mathcal{M}_{k,l}^{\mathrm{disc}}(X, L, \beta), \\ &\tau_{\mathcal{M}}([u, \vec{z}, \vec{w}]) = [\phi \circ u \circ c, (\overline{z}_1, \overline{z}_{k-1}, \cdots, \overline{z}_2), (\bar{w}_1, \cdots, \bar{w}_l)]. \end{aligned} \tag{4.1}$$

where $c(z) = \bar{z}$. It naturally extends to maps with bubble domain, inducing an involution on $\overline{\mathcal{M}}_{k,l}^{\text{disc}}(X, L, \beta)$. We call a Kuranishi structure on $\overline{\mathcal{M}}_{k,l}^{\text{disc}}(X, L, \beta)$ $\tau_{\mathcal{M}}$-invariant if $\tau_{\mathcal{M}}$ extends to a map on Kuranishi neighborhoods and multisections. For $\beta \in H_2(X, L)/\sim$, there is an ètale double covering

$$\overline{\mathcal{M}}_{k,l}^{\text{disc}}(X, L, \beta) \to \overline{\mathcal{M}}_{k,l}(X, 2\beta)^{\phi,\tau},$$

with $\tau_{\mathcal{M}}$ acting on the fibers. If $k = l = 0$, $\dim_{\mathbb{C}} X = 3$, and $4|\mu(\beta)$, the action of $\tau_{\mathcal{M}}$ is orientation-preserving; see [8, Theorem 1.4] Therefore, an orientation on $\overline{\mathcal{M}}^{\text{disc}}(X, L, \beta)$ descends to an orientation of $\overline{\mathcal{M}}(X, 2\beta)^{\phi,\tau}$.

Proposition 4.1 ([8],[7, Chapter 7])**.** *Let (X, ω, ϕ) be a symplectic manifold with a real structure. The moduli space $\overline{\mathcal{M}}_{k,l}^{\text{disc}}(X, L, \beta)$ has a topology with respect to which it is compact and Hausdorff. It has a $\tau_{\mathcal{M}}$-invariant oriented Kuranishi structure with boundary and with virtual real dimension*

$$\dim^{\text{vir}}(\overline{\mathcal{M}}_{k,l}^{\text{disc}}(X, L, \beta)) = \dim_{\mathbb{C}} X + \mu(\beta) + k + 2l - 3.$$

The codimension one boundary components of $\overline{\mathcal{M}}_{k,l}^{\text{disc}}(X, L, \beta)$ are described by (1.5) and (1.6), in a way that respects the Kuranishi structures. A spin structure σ on L determines an orientation of $\overline{\mathcal{M}}_{k,l}^{\text{disc}}(X, L, \beta)$.

Let $\overline{\mathcal{M}}_{k,l}^{\text{disc}}(X, L, \beta)^{\sigma}$ denote the moduli space equipped with the orientation induced by σ. We are interested primarily in manifolds of real dimension six. Since the tangent bundle of every orientable manifold L of dimension three is trivial, L is automatically spin. A choice of trivialization of the tangent bundle of L determines an orientation on $\mathcal{M}_{k,l}^{\text{disc}}(X, L, \beta)$. Therefore, in this case by a spin structure we simply mean a choice of trivialization of TL.

If (X, L) has vanishing Maslov class and real dimension six, then

$$\overline{\mathcal{M}}^{\text{disc}}(X, L, \beta) \equiv \overline{\mathcal{M}}^{\text{disc}}_{0,0}(X, L, \beta)$$

has virtual dimension zero. We would like to define invariants by counting the number of elements in $\overline{\mathcal{M}}^{\text{disc}}(X, L, \beta)$. Fix a choice of $\tau_{\mathcal{M}}$-invariant multisection $\mathfrak{s}$ (see [25, Section 7]) whose zero locus is close to that of the Kuranishi map. Let $[\overline{\mathcal{M}}^{\text{disc}}(X, L, \beta)^{\sigma}]^{\mathfrak{s}}$ be the virtual fundamental class determined by the multisection $\mathfrak{s}$. Since the moduli space is zero-dimensional, its degree is a rational number, which we denote by $N^{\text{disc}}_{\beta,J,\mathfrak{s}}$; a priory it depends on J and $\mathfrak{s}$.

Given two different choices of $(J_i, \mathfrak{s}_i)$, $i = 0, 1$, let $\{J_t \in \mathcal{J}_{\phi}\}$, $t \in [0, 1]$, be a path of almost-complex structures joining J_0 and J_1. Let

$$\pi\colon \overline{\mathcal{M}}^{\text{disc}}(X, L, \{J_t\}, \beta) \equiv \coprod_{t\in[0,1]} \overline{\mathcal{M}}^{\text{disc}}(X, L, J_t, \beta) \to [0, 1],$$

be the projection map. There is an analogue of Proposition 4.1 for $\overline{\mathcal{M}}^{\text{disc}}(X, L, \{J_t\}, \beta)$. In fact, $\partial^1\overline{\mathcal{M}}^{\text{disc}}(X, L, \{J_t\}, \beta)$ is a union of $\overline{\mathcal{M}}^{\text{disc}}(X, L, J_i, \beta)$ and the boundary terms of the form (1.5) and (1.6).

Choose a $\tau_{\mathcal{M}}$-invariant multisection $\mathfrak{s}$ for $\overline{\mathcal{M}}^{\text{disc}}(X, L, \{J_t\}, \beta)$ such that $\mathfrak{s}|_{\pi^{-1}(i)} = \mathfrak{s}_i$ and let $[\overline{\mathcal{M}}^{\text{disc}}(X, L, \{J_t\}, \beta)^{\sigma}]^{\mathfrak{s}}$ be the one-dimensional fundamental chain of $\mathfrak{s}$. Then,

$$\partial[\overline{\mathcal{M}}^{\text{disc}}(X, L, \{J_t\}, \beta)^{\sigma}]^{\mathfrak{s}} = [\partial\overline{\mathcal{M}}^{\text{disc}}(X, L, \{J_t\}, \beta)^{\sigma}]^{\mathfrak{s}}$$

and

$$\begin{aligned} N^{\text{disc}}_{\beta,J_1,\mathfrak{s}_1} - N^{\text{disc}}_{\beta,J_0,\mathfrak{s}_0} &= \#[\mathcal{M}_1(X, \{J_t\}, \tilde{\beta}) \times_{\text{ev}_1} L]^{\mathfrak{s}} \\ &+ \#[\mathcal{M}^{\text{disc}}_{1,0}(X, L, \{J_t\}, \beta_1)^{\sigma} \times_{(\text{ev}^B_1,\text{ev}^B_1)} \mathcal{M}^{\text{disc}}_{1,0}(X, L, \{J_t\}, \beta_2)^{\sigma}]^{\mathfrak{s}}. \end{aligned} \tag{4.2}$$

We would like to see if the right-hand side of this equation vanishes. For this, define an involution $\tau_{\mathfrak{glue}}$ on $\mathcal{M}^{\text{disc}}_{1,0}(X, L, \{J_t\}, \beta_1) \times_{(\text{ev}_1^B, \text{ev}_1^B)} \mathcal{M}^{\text{disc}}_{1,0}(X, L, \{J_t\}, \beta_2)$ by

$$(u_1, u_2) \to (u_1, \tau_{\mathcal{M}}(u_2)).$$

Remark 4.2. Every

$$u_1 \cup u_2 \in \mathcal{M}^{\text{disc}}_{1,0}(X, L, \{J_t\}, \beta_1) \times_{(\text{ev}_1^B, \text{ev}_1^B)} \mathcal{M}^{\text{disc}}_{1,0}(X, L, \{J_t\}, \beta_2)$$

is also an element

$$u_2 \cup u_1 \in \mathcal{M}^{\text{disc}}_{1,0}(X, L, \{J_t\}, \beta_2) \times_{(\text{ev}_1^B, \text{ev}_1^B)} \mathcal{M}^{\text{disc}}_{1,0}(X, L, \{J_t\}, \beta_1).$$

Therefore, $\tau_{\mathfrak{glue}}$ is not well-defined by the above. In order to avoid this ambiguity, we will assume that β is odd ($\beta \neq 2\beta'$) and therefore β_i are different and we can fix the class that we flip. Moreover, if we assume $H_2(L) = \mathbb{Z}_2$, then for $\beta = \beta_1 + \beta_2$ with $\partial\beta \neq 0$, there is a unique one with $\partial\beta_i \neq 0$ and we can decide to always flip this one. Note that in this case, all boundary strata are of disc bubbling type. For a general β, we will instead consider $\overline{\mathcal{M}}(X, 2\beta)^{\phi,\tau}$ and then there is a well-defined flip independent of the ordering the terms.

Proposition 4.3 ([8, Theorem 4.9])**.** *Let (X, ω, ϕ) be a symplectic manifold of real dimension six with a real structure. Suppose $L = \text{Fix}(\phi)$ is orientable and $c_1(TX) = 0$. Then $\tau_{\mathfrak{glue}}$ is an orientation-reversing involution.*

Corollary 4.4. *If β is odd, the terms on the right-hand side of(4.2) come in pairs with opposite signs. Therefore, the right-hand side of (4.2) is zero, and the numbers $N^{\text{disc}}_{\beta,J,\mathfrak{s}}$ are independent of J and of the multisection $\mathfrak{s}$.*

With the same argument, if β is even, the terms corresponding to disc bubbling cancel out and we get

$$N^{\mathrm{disc}}_{\beta,J_1,\mathfrak{s}_1} - N^{\mathrm{disc}}_{\beta,J_0,\mathfrak{s}_0} = \#[\mathcal{M}_1(X,\{J_t\},\tilde{\beta}) \times_{\mathrm{ev}_1} L]^{\mathfrak{s}}. \tag{4.3}$$

Remark 4.5. There is a mistake in the statement of [8, Theorem 4.9], which claims that some other map, τ_{clop}, is also orientation-reversing, which is not true. That would imply that disc invariants of even degree are well-defined, but as we mentioned before, we also need to add the contribution of $\overline{\mathcal{M}}(X,2\beta)^{\phi,c}$ to get well-defined invariants.

For (X,ω,L,ϕ) as before and $\partial\beta \neq 0$, the numbers $N^{\mathrm{disc}}_{\beta} = N^{\mathrm{disc}}_{\beta,J,\mathfrak{s}}$ defined above are called open Gromov-Witten invariants of (X,L) in the class $\beta \in H_2(X,L)/\sim$. These numbers are independent of the choices of real almost-complex structure J, of ϕ-compatible Kuranishi structure and multisection $\mathfrak{s}$, and of isotopy class of antisymplectic involution fixing L. In a similar fashion, one can define open GW invariants for other symplectic manifolds and with marked points.

4.2 Relative open GW invariants

Let (X,ω,ϕ) be as before and $D \subset X$ be a smooth symplectic divisor invariant under ϕ such that $L \cap D = \emptyset$. The definition of relative open GW invariants is a combination of the definitions of open GW invariants and of ordinary relative GW invariants. In Section 5.3, we outline the construction of Kuranishi structures for the compactified relative open moduli spaces. We use these relative moduli spaces to derive a sum formula, as done in [12] for closed GW invariants, to relate the open GW invariants of (X,L) defined in Section 4.1 and the ordinary relative GW invariants of (X_+,D).

Definition 4.6. An almost-complex structure $J \in \mathcal{J}_\phi$ is said to be compatible with D if J preserves TD and

$$N_J(\xi, v) \in T_x D \qquad \forall v \in T_x D,\ \xi \in T_x X,\ x \in D.$$

whereN_J is the nijenhuis tensor of J.

A J-holomorphic map $u\colon (\Sigma, \partial\Sigma) \to (X, L)$ is called regular if it has no components mapped into D. If $J \in \mathcal{J}_\phi$ is D-compatible, a regular J-holomorphic map intersects D in a finite set $(p_1, \cdots, p_k)$ of points with positive multiplicities $(s_1, \cdots, s_k)$, just as in the holomorphic situation and $s = s_1 + \cdots + s_k = [u] \cdot D$; see [33, Section 2]. The vector $\rho = (s_1, \cdots, s_k)$ is called the intersection pattern. Since $L \cap D = \emptyset$, all intersection points are interior points. In relative open GW theory, we are interested in the moduli space $\mathcal{M}^{\text{disc}}_{k,l}(X, L, D, \rho, \beta)$, whose elements are $[(\Sigma, \partial\Sigma), u, \vec{z}, \vec{w}, \vec{\xi}]$, where

- $u\colon (\Sigma, \partial\Sigma) \to (X, L)$ is a regular genus zero J-holomorphic map,
- $\vec{z}$ and $\vec{w}$ are tuples of k boundary and l interior marked points, respectively,
- $[u] = \beta \in H_2(X, L)$ and $u^{-1}(D) = \sum s_i \xi_i$, i.e. $\vec{\xi}$ is the set of ordered marked points corresponding to intersection points with D with contact of order s_i at $u(\xi_i)$,

such that the marked map $(u, \Sigma, \partial\Sigma, \vec{w}, \vec{z}, \vec{\xi})$ is stable.

We next describe a suitable compactification of this moduli space, denoted by $\overline{\mathcal{M}}^{\text{disc}}_{k,l}(X, L, D, \rho, \beta)$, and an orientable closed virtual cycle with which to define GW invariants.

The limiting maps in the stable compactification of this moduli space might not be regular and might have several components mapping into D. Since $L \cap D = \emptyset$, all the components of a limiting curve which are mapped into D are maps from closed curves attached to other components away from the boundary. So the definitions of relative stable maps in [11] and [16] readily extend to this case.

The normal bundle N_D^X of D in X is a complex line bundle with an inner product and a compatible connection induced by the Riemannian connection on X. Define

$$Y_D = \mathbb{P}(N_D^X \oplus \mathbb{C}).$$

The bundle map $\iota\colon (N_D^X) \to Y_D$ defined by $\iota(x) = [x, 1]$ on each fiber is an embedding onto the complement of the infinity section $D_\infty \subset Y_D$. There is a $\mathbb{C}^*$-action on Y_D which comes from scalar multiplication on N_D^X. Over each point of D, we can identify the fiber of Y_D with $\mathbb{P}^1$ and give it the Kähler structure (ω_ϵ, j) of the 2-sphere of radius ϵ. Then $\iota\colon \mathbb{C} \to Y_D$ is a holomorphic map with $\iota^*\omega_\epsilon = d\psi_\epsilon \wedge d\theta$, where

$$\psi_\epsilon(r) = \frac{2\epsilon^2 r^2}{1 + r^2}.$$

This construction globalizes by interpreting r as the norm on the fibers of N_D^X, replacing $d\theta$ by the connection 1-form α of N_D^X, and including the curvature F of that connection. Thus,

$$\iota^*\omega_\epsilon = \pi^*\omega_D + d(\psi_\epsilon \wedge \alpha)$$

is a closed form which is nondegenerate for small ϵ and its restriction to each fiber of N_D^X agrees with the volume form on the 2-sphere of radius ϵ. Furthermore, at each point $p \in N_D^X$, the connection identifies $T_p N_D^X$ with the fiber of $N_D^X \oplus TD$ at $\pi(p)$, and thus induces a complex structure on Y_D. Note that we have two copies of D inside Y_D, corresponding to the zero section and the section at infinity, which we denote by D_0 and D_∞, respectively.

Let $X[n]$ be the singular space obtained by attaching n copies of Y_D to X in such a way that the divisor D_0 of the i-th copy is attached to the divisor D_∞ of the $(i+1)$-

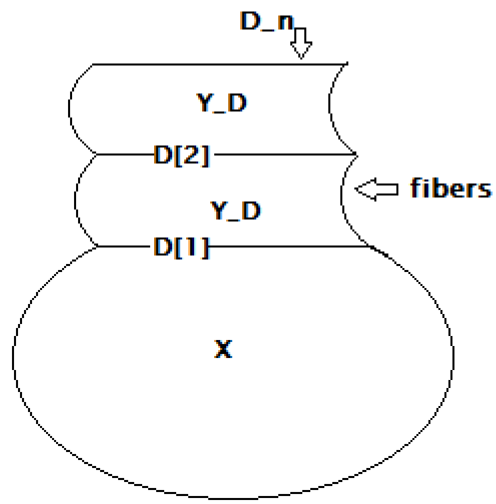


Figure 4.1: The singular manifold $X[n]$.

th copy; see Figure 4.1. Similar to Chapter 2, $X[n]$ can be realized as the singular central fiber of a symplectic fibration $\pi\colon \mathcal{X}[n] \to \Delta^n$ whose generic fiber is a smooth symplectic manifold isotopic to X.

Let D_n be the last copy of D_0 in the sequence and $D[i]$, for $i = 1, \cdots, n$, be the i-th copy of D_∞ in the sequence. The space $X[n]$ contains a copy of L which lies in X itself and is disjoint from all Y_D. There is an action of $G_n = (\mathbb{C}^*)^n$ on $X[n]$ which comes from the $\mathbb{C}^*$ action on each copy of Y_D.

Definition 4.7. A stable relative genus zero bordered J-holomorphic map to $X[n]$ is a tuple $[u, (\Sigma, \partial\Sigma), \vec{z}, \vec{w}, \vec{\xi}]$, where

1. $\Sigma = \Sigma_0 \cup \cdots \cup \Sigma_n$ is a connected bordered nodal curve of arithmetic genus zero, Σ_i, $i \geq 1$, is a closed curve (not necessarily connected), and $\partial\Sigma \cong S^1$,

2. $\vec{z}, \vec{w}, \vec{\xi}$ are tuples of distinct smooth points on Σ, $\vec{z}$ is a tuple of boundary marked points in an anticlockwise order, $\vec{w}$ is a tuple of interior marked points, and $\vec{\xi}$ is a tuple of marked points on the last layer Σ_n,

3. $u\colon \Sigma_0 \to X$ is a regular J-holomorphic map into X and $u\colon \Sigma_i \to Y_D[i]$ are regular J_{Y_D}-holomorphic maps,

4. $u^{-1}(D[i]) = \{\xi_{i,1}, \cdots, \xi_{i,j_i}\}$ is discrete for $i = 1, \cdots, n$, each $\xi_{i,j}$ is a node of Σ connecting Σ_{i-1} and Σ_i, and $u\mid_{\Sigma_{i-1}}$ and $u\mid_{\Sigma_i}$ have same contact orders with $D[i]$,

such that the automorphism group of $f = [u, (\Sigma, \partial\Sigma), \vec{w}, \vec{z}, \vec{\xi}]$

$$\mathfrak{Aut}(f) = \left\{ (h, \sigma) \mid \sigma \in G_n,\ h \in \mathfrak{Aut}(\Sigma, \vec{z}, \vec{w}, \vec{\xi})\ \ \sigma \circ u = u \circ h \right\},$$

is finite.

Let $\overline{\mathcal{M}}^{\text{disc}}_{k,l}(X, L, D, \rho, \beta)$ denote the set of the equivalence classes of all bordered stable relative maps in class β and with intersection pattern ρ. This moduli space is Hausdorff and compact. In Section 5.3, we outline the construction of a virtual fundamental class for this space.

Let

$$\text{ev} = (\text{ev}^B_{\vec{z}}, \text{ev}_{\vec{w}}, \text{ev}_{\vec{\xi}})\colon \overline{\mathcal{M}}^{\text{disc}}_{k,l}(X, L, D, \rho, \beta) \to L^k \times X^l \times D^{l(\rho)},$$

where $l(\rho) = m$ if $\rho = (s_1, \cdots, s_m)$, be the total evaluation map. Open relative GW invariants are obtained by integrating pull-backs under ev of differential forms on $\overline{\mathcal{M}}^{\text{disc}}_{k,l}(X, L, D, \rho, \beta)$. As in the absolute case, we encounter issues concerning codimension one boundaries and orientation. If L is the fixed point set of an antisymplectic involution compatible with D, we can use the same technique as in Section 4.1 to define relative open invariants. The antisymplectic involution on X extends to $X[n]$ and induces an involution on $\overline{\mathcal{M}}^{\text{disc}}_{k,l}(X, L, D, \rho, \beta)$. Therefore, we can get a cancellation of boundary terms as in Proposition 4.3.

Remark 4.8. We also need to consider relative invariants with disconnected domains,

$$(\Sigma, \partial\Sigma) = (\Sigma_0, \partial\Sigma_0) \cup \Sigma_1 \cup \cdots \cup \Sigma_k,$$

where Σ_i $(1 \leq i \leq k)$ has no boundary and $\partial\Sigma_0 \sim S^1$. We fix homology classes $\beta_0 \in H_2(X, L)/\sim$ and $\beta_i \in H_2(X)/\sim$, for $1 \leq i \leq k$, with $\beta = \beta_0 + \sum \beta_i$. With Γ denoting the above topological data, let $\overline{\mathcal{M}}^{\text{disc}}_{k,l}(X, L, D, \rho, \Gamma)$ be the moduli space of relative maps $u\colon (\Sigma, \partial\Sigma) \to (X[n], D_n)$ so that over each component u has the given topological type.

Example 4.9. Let $(X, \omega, \phi, L, D) = (\mathbb{CP}^3, \omega_{\text{FS}}, \tau_3, \mathbb{RP}^3, Q)$, where $Q = Q^2$ is the quadratic hypersurface with the real defining equation

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0.$$

Since $H_2(\mathbb{P}^3, \mathbb{RP}^3) \cong \frac{1}{2}\mathbb{Z}$, we write the elements of $H_2(X, L)$ by $[\frac{d}{2}]$. Note that $\mu([\frac{d}{2}]) = 4d \equiv 0 \mod 4$. Let $\rho_0 = (1, \cdots, 1)$, d be odd, and

$$\text{ev}_\xi\colon \overline{\mathcal{M}}^{\text{disc}}(\mathbb{P}^3, \mathbb{RP}^3, Q, \rho_0, [\frac{d}{2}]) \to Q^d$$

be the evaluation map at the contact points. Consider the incidence condition

$$\gamma := \{p_1, \tau_3(p_1)\} \times \cdots \times \{p_d, \tau_3(p_d)\} \subset Q^d,$$

where $p_1, \cdots, p_d$ are d general points in Q. Then $\overline{\mathcal{M}}^{\text{disc}}(\mathbb{P}^3, \mathbb{RP}^3, Q, \rho_0, [\frac{d}{2}]) \times_{\text{ev}} \gamma$ has virtual dimension zero and virtually counts the number of holomorphic discs intersecting Q at certain fixed points determined by γ. For d odd, the rational number

$$\alpha_d^{\text{rel,disc}} = \frac{1}{2^d} \#[\overline{\mathcal{M}}^{\text{disc}}(\mathbb{P}^3, \mathbb{RP}^3, Q, \rho_0, [\frac{d}{2}]) \times_{\text{ev}} \gamma]^{\text{vir}} \tag{4.4}$$

is well-defined; for d even, we need extra terms to make it invariant. These open relative GW invariants appear as coefficients in the proof of Theorem 1.4 and are the only open relative invariants we need later. Note that we chose γ symmetric with respect to the involution in order to make $\overline{\mathcal{M}}^{\mathrm{disc}}(\mathbb{P}^3, \mathbb{RP}^3, Q, \rho_0, [\frac{d}{2}]) \times_{\mathrm{ev}} \gamma$ closed under the involutions $\tau_{\mathfrak{glue}}$ and $\tau_{\mathcal{M}}$.

For a non-connected domain $\Sigma = (\Sigma_0, \partial\Sigma_0) \cup \bigcup_{i=1}^{k} \Sigma_k$, let Γ be a topological type as before with $\beta_0 \in H_2(\mathbb{CP}^3, \mathbb{RP}^3)$ and $\beta_i \in H_2(\mathbb{CP}^3)$, for $i = 1, \cdots, k$. Let $d_i = \mu(\beta_i)/4 \in \mathbb{Z}$ and

$$\gamma_\Gamma = \gamma_0 \times \gamma_1 \times \cdots \times \gamma_k, \tag{4.5}$$

where $\gamma_0 = \gamma$ as before and $\gamma_i = \{q_{i1}\} \times \cdots \times \{q_{id_i}\}$ is a single point in Q^{d_i}, whenever $i \geq 1$. We define

$$\alpha_\Gamma^{\mathrm{rel,disc}} = \frac{1}{2^{d_0}} \#[\overline{\mathcal{M}}^{\mathrm{disc}}(\mathbb{P}^3, \mathbb{RP}^3, Q, \rho_0, \Gamma) \times_{\mathrm{ev}} \gamma_\Gamma]^{\mathrm{vir}}. \tag{4.6}$$

These numbers are invariant under deformations of almost complex structure and γ_i's and are generalizations of the above invariants to non-connected domains. Since

$$\overline{\mathcal{M}}^{\mathrm{disc}}(\mathbb{P}^3, \mathbb{RP}^3, Q, \rho_0, \Gamma) = \overline{\mathcal{M}}^{\mathrm{disc}}(\mathbb{P}^3, \mathbb{RP}^3, Q, \rho_0, [\frac{d_0}{2}]) \times \prod_{i=1}^{k} \overline{\mathcal{M}}(\mathbb{P}^3, Q, \rho_0, [d_i]),$$

we find that

$$\alpha_\Gamma^{\mathrm{rel,disc}} = \alpha_{d_0}^{\mathrm{rel,disc}} \times \prod_{i=1}^{k} \alpha_{d_i}^{\mathrm{rel}}.$$

This shows that disconnected invariants reduce to connected ones.

Since $\mathbb{P}^3$ is Fano and $\rho_0 = (1, \cdots, 1)$, $\alpha_d^{\mathrm{rel,disc}} = \frac{1}{2^d} N_d^{\mathrm{disc}}$, where N_d^{disc} is the number of degree d disks in $\mathbb{P}^3$ passing through d pairs of conjugate points. This number can be computed by equivariant localization; see Chapter 6.

Chapter 5

Degeneration of moduli spaces

In this chapter, we build a cobordism between moduli spaces of holomorphic discs in a smooth fiber and the fiber product of moduli spaces of relative maps in the singular fiber in the fibration $\pi : \mathcal{X} \to \Delta$ constructed in Chapter 2. Using this cobordism, we prove Theorems 1.1, 1.4 and 1.10.

5.1 Proof of Theorem 1.1

Given (X, ω, L) as in the statement of Theorem 1.1, let $\pi\colon \mathcal{X} \to \Delta$ be the associated fibration constructed in Chapter 2. Let $\mathcal{J}^l_{\mathcal{X}}$ be the set of compatible almost-complex structures J of class C^l on $\mathcal{X}$, given by Proposition 2.1. Restricted to X_+, any such J is D-compatible in the sense of Definition 4.6. Let $\mathcal{M}^{\mathrm{reg}}(X_+, A)$ be the moduli space of D-regular degree A genus zero somewhere injective $J|_{X_+}$-holomorphic curves. By [20, Theorem 3.1.5] and Proposition 2.1

$$\dim^{\mathrm{vir}}_{\mathbb{R}}(\mathcal{M}^{\mathrm{reg}}(X_+, A)) = 2(n-3) + -2(n-2)\ [A] \cdot [D],$$

which is a negative number if $[A] \cdot [D] > 0$ and $n > 2$.

Lemma 5.1. *There is a dense subset* $\mathcal{J}_{\mathcal{X}}^{l,\mathrm{reg}} \subset \mathcal{J}_{\mathcal{X}}^{l}$ *such that* $\mathcal{M}^{\mathrm{reg}}(X_+, A) = \emptyset$ *for every* $J \in \mathcal{J}_{\mathcal{X}}^{l,\mathrm{reg}}$ *and every* $A \in H_2(X_+, \mathbb{Z})$ *with* $[A] \cdot [D] > 0$.

Proof. Let

$$\mathcal{M}^{\mathrm{reg}}(X_+, \mathcal{J}_{\mathcal{X}}^{l}, A) = \coprod_{J \in \mathcal{J}_{\mathcal{X}}^{l}} \mathcal{M}^{\mathrm{reg}}(X_+, J, A)$$

be the universal moduli space. By [20, Chapter 6], the linearization map

$$D_{J,u} \colon W^{k,p}(\mathbb{P}^1, u^*TX_+) \oplus T_J\mathcal{J}_{\mathcal{X}}^{l} \to W^{k-1,p}(\mathbb{P}^1, u^*TX_+ \otimes_J \Lambda_J^{0,1} T\mathbb{P}^1),$$
$$D_{J,u}(\xi, Y) = L_{J,u}(\xi) + \frac{1}{2} Y(u) du \circ j,$$

of the Cauchy-Riemann operator is surjective for every $(u, J) \in \mathcal{M}^{\mathrm{reg}}(X_+, \mathcal{J}_{\mathcal{X}}^{l}, A)$. The projection map $\pi \colon \mathcal{M}^{\mathrm{reg}}(X_+, \mathcal{J}_{\mathcal{X}}^{l}, A) \to \mathcal{J}_{\mathcal{X}}^{l}$ is Fredholm, and the kernel and cokernel of $d\pi$ are isomorphic to the kernel and cokernel of $L_{J,u}$. By the Sard-Smale theorem [20, Theorem A.5.1], the set of regular values of π is of the second category, provided $l - 1 \geq 0, \mathrm{ind}(L_{J,u})$. On the other hand, $J \in \mathcal{J}_{\mathcal{X}}^{l}$ being a regular value for $d\pi$ means that $L_{J,u}$ is surjective, and so $\mathcal{M}^{\mathrm{reg}}(X_+, J, A)$ is a negative-dimensional smooth moduli space and therefore empty. Taking intersection over all curve classes $A \in H_2(X_+, \mathbb{Z})$, we find a dense set of almost-complex structures for which all the moduli spaces $\mathcal{M}^{\mathrm{reg}}(X_+, J, A)$ with $[A] \cdot [D] > 0$ are empty. □

Given $J \in \mathcal{J}^{l,\mathrm{reg}}$ and $\lambda \in \Delta$, let $J_\lambda = J|_{X_\lambda}$ as before. Suppose $E > 0$, $\lambda_i \in \Delta^*$ is a sequence converging to 0 and $[u_i]$ is a sequence of J_{λ_i}-holomorphic discs such that $\omega_{\mathcal{X}}([u_i]) < E$. This sequence is a sequence of J-holomorphic discs in a compact subset of $\mathcal{X}$ with a uniform energy bound. By the Gromov Compactness Theorem, there is a J_0-holomorphic map $u_0 \colon B \to X_0$ and a sequence of orientation-preserving diffeomorphisms ψ of the domains of u_i such that a subsequence of $u_i \circ \psi_i$ converges to u_0. Furthermore, every component of u_0 has image in either X_- or X_+ and least one component maps to X_+ intersecting D in a nonempty discrete set. The last claim holds for the following reason. Each u_{λ_i} intersects L, so u_0 has non-empty intersection with $L \subset X_-$, which means there is a nonzero irreducible component u_- of u_0 mapped into X_-. We know D is obtained by a symplectic cut along some hypersurface $V_a = h^{-1}(a) \subset X$. Consider the contact hypersurfaces $V_{a+\epsilon}$ (for $\epsilon > 0$ small) in X. After performing symplectic cut along V_a, we get copies of $V_{a+\epsilon}$ in X_+ which are the boundaries of a tubular neighborhood of $D \subset X_+$. Each u_{λ_i} has a non-empty intersection with $V_{a+\epsilon}$, because the symplectic form inside the neighborhood of L surrounded by $V_{a+\epsilon}$ is exact and so there are no J_{λ_i}-holomorphic disc completely inside $V_{a+\epsilon}$. Thus, the limit curve u_0 has a non-empty intersection with $V_{a+\epsilon} \subset X_+$, and so there are some irreducible components of u_0 mapped into X_+ and not contained in $D \subset X_+$. Since the domain of u_0 is connected, a component of u_0 mapped into X_+ and not contained in D intersects D. Since $J_0|_{X_+}$ is D-compatible, this component intersects D at finitely many points. However, by Lemma 5.1, $\mathcal{M}^{\mathrm{reg}}(X_+, J, A) = \emptyset$ if $J \in \mathcal{J}_{\mathcal{X}}^{l,\mathrm{reg}}$ and $A \cdot D > 0$. Thus, $\overline{\mathcal{M}}^{\mathrm{disc}}(X, L, J_{\lambda_i}, \beta) = \emptyset$ for all $\lambda \in \Delta^*$ small and $\beta \in H_2(X, L)$ such that $\omega(\beta) < E$. Once again, by the Gromov Compactness Theorem this also holds for some neighborhood U_E of $J_\lambda \in \mathcal{J}_{X_\lambda}$. Since $X \cong X_\lambda$, this finishes the proof of Theorem 1.1.

5.2 Proof of Theorems 1.4 and 1.10

Let (X, ω, ϕ) be a symplectic manifold with a real structure such that $c_1(TX) = 0$ and $L \cong S^3$ or $\mathbb{RP}^3$. Let $\pi : \mathcal{X} \to \Delta$ be the associated fiberation of Chapter 2 and $\mathcal{Y} = \pi^{-1}([0,1]) \subset \mathcal{X}$. Each fiber of $\mathcal{Y} \to [0,1]$ is invariant under the induced involution $\phi_{\mathcal{X}}$. Fix some compatible J on $\mathcal{X}$ and define

$$\overline{\mathcal{M}}^{\mathrm{disc}}(\mathcal{Y}, L, \{J_t\}_{t\in(0,1]}, \beta) = \bigcup_{t\in(0,1]} \overline{\mathcal{M}}^{\mathrm{disc}}(X_t, L, J_t, \beta).$$

Let $\overline{\mathcal{M}}^{\text{disc}}(\mathcal{Y}, L, \{J_t\}_{t\in I}, \beta)$, $I = [0,1]$, be the relative stable map compactification of $\overline{\mathcal{M}}^{\text{disc}}(\mathcal{Y}, L, \{J_t\}_{t\in (0,1]}, \beta)$, similar to [12], [16], and Section 4.2, including maps to the fiber over zero.

Every element (u, Σ) of $\overline{\mathcal{M}}^{\text{disc}}(\mathcal{Y}, L, \{J_t\}_{t\in I}, \beta)$ in X_0 belongs to a fiber product of relative moduli spaces over X_- and X_+ with matching conditions on D,

$$(u, \Sigma) \in \overline{\mathcal{M}}^{\text{disc}}(X_-, L, D, \rho, \Gamma_-) \times_{(\text{ev}_{\xi^-}, \text{ev}_{\xi^+})} \overline{\mathcal{M}}(X_+, D, \rho, \Gamma_+), \tag{5.1}$$

where $\overline{\mathcal{M}}^{\text{disc}}(X_-, L, D, \rho, \Gamma_-)$ and $\overline{\mathcal{M}}(X_+, D, \rho, \Gamma_+)$ are the relative moduli spaces with the same intersection pattern ρ, $\xi^\pm$ are contact points with D, and $\Gamma_\pm$ encodes the data corresponding to the topological types of the domain and image.

There is a fiber-wise involution $\tau_{\mathcal{M}}$ on $\overline{\mathcal{M}}^{\text{disc}}(\mathcal{Y}, L, \{J_t\}_{t\in[0,1]}, \beta)$ as before. Fix a spin structure σ on L. For a tuple $\rho = (s_1, \cdots, s_k)$, $|\rho| = \prod s_i$.

Proposition 5.2. *Let (X^6, ω, ϕ) be a symplectic manifold with $c_1(TX) = 0$. The moduli space $\overline{\mathcal{M}}^{\text{disc}}(\mathcal{Y}, L, \{J_t\}_{t\in I}, \beta)$ has a topology with respect to which it is compact and Hausdorff. It has a $\tau_{\mathcal{M}}$-invariant oriented Kuranishi structure of virtual dimension 1 with respect to which the projection π is smooth. The codimension one boundary components correspond to the moduli spaces of the form (1.5), (1.6), the fiber over $\lambda = 1$, and (5.1) via a gluing map*

$$\begin{array}{ccc}
\mathcal{M}(X_0, \rho, \Gamma_-, \Gamma_+) & \xrightarrow{\iota_{(\Gamma_-, \Gamma_+)}} & \partial\overline{\mathcal{M}}^{\text{disc}}(\mathcal{Y}, L, \{J_t\}_{t\in I}, \beta) \\
{\scriptstyle |\rho|-\text{covering}}\big\downarrow{\scriptstyle \pi} & & \\
\mathcal{M}^{\text{disc}}(X_-, L, D, \rho, \Gamma_-) \times_{(\text{ev}_{\xi^-}, \text{ev}_{\xi^+})} \mathcal{M}(X_+, D, \rho, \Gamma_+) & &
\end{array}$$

which is compatible with the Kuranishi structures and the orientation induced by the spin structure σ.

We use this proposition to prove Theorems 1.4 and 1.10. In Section 5.3 below, we describe the Kuranishi structure and the covering space $\mathcal{M}(X_0, \rho, \Gamma_-, \Gamma_+)$.

For $A \in H_2(X, \mathbb{Z})$, let

$$\overline{\mathcal{M}}(\mathcal{Y}, \{J_t\}_{t\in I}, A)^{\phi} = \overline{\mathcal{M}}(\mathcal{Y}, \{J_t\}_{t\in I}, A)^{\phi,\tau} \cup \overline{\mathcal{M}}(\mathcal{Y}, \{J_t\}_{t\in I}, A)^{\phi,\eta},$$

with the common boundaries on the left-hand side identified. There is a Kuranishi structure on $\overline{\mathcal{M}}(\mathcal{Y}, \{J_t\}_{t\in I}, A)^{\phi,\eta}$ similar to one on the $\overline{\mathcal{M}}^{\mathrm{disc}}(\mathcal{Y}, L, \{J_t\}_{t\in I}, \beta)$, given in Proposition 5.2, and these two Kuranishi structures give a Kuranishi structure on $\overline{\mathcal{M}}(\mathcal{Y}, \{J_t\}_{t\in I}, A)^{\phi}$. Fix one such Kuranishi structure and multisection $\mathfrak{s}$ on $\overline{\mathcal{M}}(\mathcal{Y}, \{J_t\}_{t\in I}, A)^{\phi}$.

Since X is a symplectic Calabi-Yau 3-fold and TL is trivial, $\overline{\mathcal{M}}(\mathcal{Y}, \{J_t\}_{t\in I}, A)^{\phi,\eta}$ and $\overline{\mathcal{M}}(\mathcal{Y}, \{J_t\}_{t\in I}, A)^{\phi,\tau}$ are orientable; see Sections 3.3 and 4.1. The orientation on $\overline{\mathcal{M}}(\mathcal{Y}, \{J_t\}_{t\in I}, A)^{\phi,\tau}$ depends on the choice of trivialization of TL, while the orientation on $\overline{\mathcal{M}}(\mathcal{Y}, \{J_t\}_{t\in I}, A)^{\phi,\eta}$ depends on the choice of admissible trivialization of K_X. Both induce orientations on L.

Lemma 5.3. *For c corresponding to either η or τ and the gluing parameter $\epsilon \in \mathbb{R}^{\geq 0}$, the gluing map*

$$(\mathcal{M}_1(\mathcal{Y}, \{J_t\}_{t\in I}, \tilde{\beta}) \times_{\mathrm{ev}_1} L) \times \mathbb{R}^{\geq 0} \to \overline{\mathcal{M}}(\mathcal{Y}, \{J_t\}_{t\in I}, A)^{\phi,\tau \text{ or } \eta}$$

is orientation-preserving, provided the Lagrangian on the left-hand side is oriented by the chosen trivialization of TL in the $c = \tau$ case and of $(K_X|_L)_{\mathbb{R}}$ in the $c = \eta$ case.

Proof. A curve in the common boundary of these two moduli spaces is of the form $f = [u, \Sigma = \mathbb{P}^1_{\mathrm{top}} \cup_q \mathbb{P}^1_{\mathrm{down}}]$, with the involution c over Σ having one fixed point, $\mathrm{Fix}(c) = q$, the node q. We replace each such f, with the unstable map

$$\tilde{f} = [u, \Sigma' = \mathbb{P}^1_{\mathrm{top}} \cup \mathbb{P}^1_0 \cup \mathbb{P}^1_{\mathrm{down}}],$$

taking u to be constant $u(q)$ over the central part $\mathbb{P}^1_0$. The automorphism group of $\tilde{f}$ is S^1. We can view $\tilde{f}$ as an element of $\partial\overline{\mathcal{M}}(\mathcal{Y}, \{J_t\}_{t\in I}, A)^{\phi,\tau}$ by extending involution to $\mathbb{P}^1_0$ via $c\,|_{\mathbb{P}^1_0} = \tau$ and as an element of $\partial\overline{\mathcal{M}}(\mathcal{Y}, \{J_t\}_{t\in I}, A)^{\phi,\eta}$ by extending involution to $\mathbb{P}^1_0$ via $c\,|_{\mathbb{P}^1_0}=\eta$. The claim is then obtained following [7, Section 8.3] and [7, Section 7.4.1]). □

Lemma 5.3 implies that if the induced orientation on $\Lambda^{\text{top}}T^*L$ and $(K_X|_L)_{\mathbb{R}} \cong \Lambda^{\text{top}}T^*L$ are reverse of each other, the induced orientation on the common boundary $(\mathcal{M}_1(\mathcal{Y}, \{J_t\}_{t\in I}, \tilde{\beta}) \times_{\text{ev}_1} L)$ of $\partial\overline{\mathcal{M}}(\mathcal{Y}, \{J_t\}_{t\in I}, A)^{\phi,\tau}$ and $\partial\overline{\mathcal{M}}(\mathcal{Y}, \{J_t\}_{t\in I}, A)^{\phi,\eta}$ are reverse of each other and $\overline{\mathcal{M}}(\mathcal{Y}, \{J_t\}_{t\in I}, A)^{\phi}$ is oriented.

Proof of Theorem 1.10. By Proposition 2.1, $c_1(TX_+) = -PD(D)$. Therefore, for every curve class $B \in H_2(X_+)$, $B \cdot D > 0$,

$$\dim^{\text{vir}}(\overline{\mathcal{M}}(X_+, D, \rho, \Gamma_+)) \leq c_1(TX_+)(B) < 0,$$

whenever the image curves of type Γ_+ are in homology class B. Thus, the zero locus of a multisection for $\overline{\mathcal{M}}(X_+, D, \rho, \Gamma_+)^{\phi}$ is empty, unless Γ_+ contains no homology class with positive intersection with D. Therefore, the only non-trivial component in the fiber over zero is $\overline{\mathcal{M}}(X_+, A)^{\phi,c}$. We conclude that

$$\begin{aligned} &N^{\phi}_A = N^{\phi}_{(A,J|_{\pi^{-1}(1)},\mathfrak{s}|_{\pi^{-1}(1)})} = N^{\text{real}}_A(X_+) \\ &+ \sum_{A_1+A_2=A} \#[\overline{\mathcal{M}}_{1,0}(\mathcal{Y}, L, \{J_t\}_{t\in(0,1)}, A_1)^{\phi,\tau} \times_{(\text{ev}_1,\text{ev}_1)} \overline{\mathcal{M}}_{1,0}(\mathcal{Y}, L, \{J_t\}_{t\in(0,1)}, A_2)^{\phi,\tau}]^{\mathfrak{s}}. \end{aligned}$$

The last term above is zero by Proposition 4.3, which establishes Theorem 1.10. □

We now turn to the proof of Theorem 1.4. By Proposition 5.2,

$$\begin{aligned} N^{\text{disc}}_{(\beta,J|_{\pi^{-1}(1)},\mathfrak{s}|_{\pi^{-1}(1)})} = &\sum_{(\Gamma_-,\Gamma_+),\rho} \frac{1}{\mathfrak{Aut}(\Gamma_-,\Gamma_+)} \#[\mathcal{M}(\mathcal{Y}_0, \rho, \Gamma_-, \Gamma_+)]^{\mathfrak{s}} \\ &- \#[\overline{\mathcal{M}}^{\text{disc}}_{1,0}(\mathcal{Y}, L, \{J_t\}_{t\in\mathring{I}}, \beta_1) \times_{(\text{ev}_1,\text{ev}_1)} \overline{\mathcal{M}}^{\text{disc}}_{1,0}(\mathcal{Y}, L, \{J_t\}_{t\in\mathring{I}}, \beta_2)]^{\mathfrak{s}}, \end{aligned}$$

where $\mathring{I} = (0,1)$ and $\mathfrak{Aut}(\Gamma_-, \Gamma_+)$ is the finite automorphism group of (Γ_-, Γ_+) configuration. By Proposition 4.3, the last term above is zero. Therefore,

$$N_\beta^{\text{disc}} = \sum_{(\Gamma_-,\Gamma_+),\rho} \frac{1}{\mathfrak{Aut}(\Gamma_-, \Gamma_+)} \#[\mathcal{M}(\mathcal{Y}_0, \rho, \Gamma_-, \Gamma_+)]^{\mathfrak{s}}. \tag{5.2}$$

The sum on the left-hand side of (5.2) corresponds to the boundary terms coming from the central fiber, in a similar way to [16, Theorem 3.15] or [12, Theorem 12.3]. Therefore, (5.2) is an open version of the symplectic sum formula.

If $\rho = (s_1, \cdots, s_k)$

$$\dim^{\text{vir}}(\overline{\mathcal{M}}(X_+, D, \rho, \Gamma_+)) = k - \sum s_i.$$

Therefore, the only ρ for which we get a non-trivial contribution is the trivial one, $\rho_0 = (1, \cdots, 1)$. By Proposition 5.2, if $\rho = \rho_0$, then

$$\mathcal{M}(X_0, \rho_0, \Gamma_-, \Gamma_+) = \overline{\mathcal{M}}^{\text{disc}}(X_-, L, D, \rho_0, \Gamma_-) \times_{(\text{ev}_{\xi^-}, \text{ev}_{\xi^+})} \overline{\mathcal{M}}(X_+, D, \rho_0, \Gamma_+).$$

It remains to understand the fiber-product term on the right. Since $\overline{\mathcal{M}}(X_+, D, \rho_0, \Gamma_+)$ has virtual dimension zero and $\tau_{\mathcal{M}}$-invariant Kuranishi structure,

$$\text{ev}_{\xi^+}[\overline{\mathcal{M}}(X_+, D, \rho_0, \Gamma_+)]^{\mathfrak{s}} \subset D^k$$

is a ϕ_{X_+}-invariant zero-dimensional chain, which we denote by $\gamma_{\Gamma+}$. Then $N_{\Gamma_+}^{\text{rel}} = |\gamma_{\Gamma_+}| \in \mathbb{Q}$ is the closed relative GW invariants of the class Γ_+ counting elements of the corresponding relative moduli space. Therefore,

$$\begin{aligned}[\overline{\mathcal{M}}^{\text{disc}}(X_-, L, D, \rho_0, \Gamma_-) &\times_{(\text{ev}_{\xi^-}, \text{ev}_{\xi^+})} \overline{\mathcal{M}}(X_+, D, \rho_0, \Gamma_+)]^{\mathfrak{s}} \\ &= [\overline{\mathcal{M}}^{\text{disc}}(X_-, L, D, \rho_0, \Gamma_-) \times_{\text{ev}_{\xi^-}} \gamma_{\Gamma_+}]^{\mathfrak{s}}.\end{aligned}$$

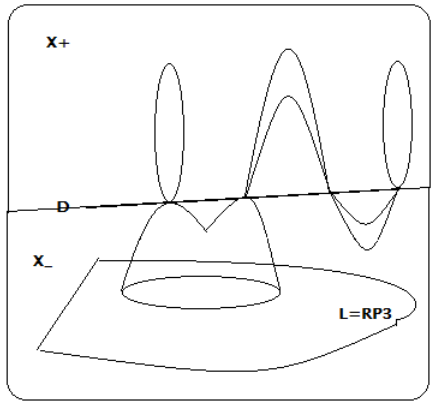


Figure 5.1: A typical J-holomorphic map in singular fiber.

Lemma 5.4. *With the notation as above,*

$$\#[\overline{\mathcal{M}}^{\text{disc}}(X_-, L, D, \rho_0, \Gamma_-) \times_{\text{ev}_{\xi-}} \gamma_{\Gamma_+}]^{\mathfrak{s}} = \alpha^{\text{rel,disc}}_{\Gamma_-} N^{\text{rel}}_{\Gamma_+},$$

where $\alpha^{\text{rel,disc}}_{\Gamma_-}$ *are the numbers defined in (4.6).*

Proof. A typical element of $\overline{\mathcal{M}}^{\text{disc}}(X_-, L, D, \rho_0, \Gamma_-) \times_{(\text{ev}_{\xi-}, \text{ev}_{\xi+})} \overline{\mathcal{M}}(X_+, D, \rho_0, \Gamma_+)$ represents a curve as in Figure 5.1. In this figure, the domain of Γ_+ consists of three rational curves, while the domain of Γ_- consists of a disc component and a rational curve. Since $\Gamma_-\#\Gamma+$ is a degeneration of the disc, the unique bordered component of Γ_-, say $(\Sigma_0, \partial\Sigma_0)$, intersects the domain of Γ_+ in disjoint irreducible components. Thus, if the topological type of Γ_- over $(\Sigma_0, \partial\Sigma_0)$ is $[\frac{d_0}{2}] \in H_2(\mathbb{CP}^3, \mathbb{RP}^3)$, then Γ_+ has at least d_0 components.

Let $(\xi_1, \cdots, \xi_{d_0})$ denote the intersection points of Σ_0 with the common divisor D, and let $(\xi_1, \cdots, \xi_k) \in D^k$ be the set of all intersection points. For $\epsilon = (\epsilon_1, \cdots, \epsilon_l) \in \mathbb{Z}_2^l$, let $\mathfrak{c}_\epsilon \colon D^l \to D^l$ be the map which is equal to the identity on i-th factor if $\epsilon_i = 0$ and to τ_3 if $\epsilon_i = 1$.

If $(\xi_1, \cdots, \xi_k) \in \gamma_{\Gamma_+}$, then $(\mathfrak{c}_\epsilon(\xi_1, \cdots, \xi_{d_0}), \mathfrak{c}_{\epsilon'}(\xi_{d_0+1}, \cdots, \xi_k))$ is also in γ_{Γ_+}, where ϵ is arbitrary and ϵ' depends on Γ_+ and ϵ. This is because Σ_1 meets Γ_+ at disjoint components Σ_i, and for every J_+-holomorphic map $u\colon \Sigma_i \to X_+$, $\tau_{\mathcal{M}}(u)$ is also a J_+-holomorphic map with same the topological type; that is, we can flip each individual component of Γ_+ using the induced involution ϕ_+ on X_+.

Given $(\xi_1, \cdots, \xi_k) \in \gamma_{\Gamma_+}$, let S be the set of the 2^{d_0} tuples obtained from $(\xi_1, \cdots, \xi_k)$. Let $q = (q_{d_0+1}, \cdots, q_k) \in D^{k-d_0}$. For each point $\mathfrak{c}_{\epsilon'}(\xi_{d_0+1}, \cdots, \xi_k)$ as above, choose a path $\gamma_{\epsilon'}(t)$, $0 \le t \le 1$, in $D^{(k-d_0)}$ connecting these two points. Let S_t be the set obtained by replacing a point of the form $(\mathfrak{c}_\epsilon(\xi_1, \cdots, \xi_{d_0}), \mathfrak{c}_{\epsilon'}(\xi_{d_0+1}, \cdot, \xi_k))$ with $(\mathfrak{c}_\epsilon(\xi_1, \cdots, \xi_{d_0}), \gamma_{\epsilon'}(t))$. Then, S_1 is of the form $\gamma_0 \times \{q\} \in D^k$ as in (4.5).

For each S_t, define

$$\mathcal{M}^{\mathrm{disc}}_{t,\Gamma_-} = \overline{\mathcal{M}}^{\mathrm{disc}}(X_-, L, D, \rho_0, \Gamma_-) \times_{\mathrm{ev}_{\xi^-}} S_t,$$

to be the zero-virtual-dimensional relative moduli space with incidence condition determined by S_t and

$$\mathcal{M}^{\mathrm{disc}}_{\Gamma_-} = \bigcup_{t \in I} \mathcal{M}^{\mathrm{disc}}_{t,\Gamma_-}$$

to be their union. Then $[\mathcal{M}_{\Gamma_-}]^{\mathfrak{s}}$ is a one-dimensional cobordism between $[\mathcal{M}^{\mathrm{disc}}_{0,\Gamma_-}]^{\mathfrak{s}}$ and $[\mathcal{M}^{\mathrm{disc}}_{1,\Gamma_-}]^{\mathfrak{s}}$, because the part of the incidence condition which corresponds to the disc part $(\Sigma_1, \partial\Sigma_1)$ of Γ_- is fixed, and so the disc part of each curve in cobordism is fixed, and so disc-bubbling does not happen in the middle. We conclude that

$$\begin{aligned} \#[\overline{\mathcal{M}}^{\mathrm{disc}}(X_-, L, D, \rho_0, \Gamma_-) \times_{\mathrm{ev}_{\xi^-}} S_0]^{\mathfrak{s}} &= \#[\overline{\mathcal{M}}^{\mathrm{disc}}(X_-, L, D, \rho_0, \Gamma_-) \times_{\mathrm{ev}_{\xi^-}} S_1]^{\mathfrak{s}} \\ &= 2^{d_0} \alpha^{\mathrm{rel,disc}}_{\Gamma_-}. \end{aligned}$$

Performing this for all points in the 0-chains γ_{Γ_+} and then adding up all the terms gives the desired result. □

Proof of Theorem 1.4. By (5.2) and Lemma 5.4,

$$N_\beta^{\text{disc}} = \sum_{\Gamma_- \# \Gamma_+ = \beta} \frac{1}{\mathfrak{Aut}(\Gamma_-, \Gamma_+)} N_{\Gamma_+}^{\text{rel}} \alpha_{\Gamma_-}^{\text{rel,disc}}. \tag{5.3}$$

Thus, the open invariants of (X, L) can be expressed as a linear function of the relative GW invariants of the symplectic manifold X_+ and the universal constants $\alpha_{\Gamma_-}^{\text{rel,disc}}$. □

5.3 Kuranishi structure on $\overline{\mathcal{M}}^{\text{disc}}(X, L, D, \rho, \beta)$

This section outlines a construction of Kuranishi structure on the relative moduli space $\overline{\mathcal{M}}^{\text{disc}}(X, L, D, \rho, \beta)$. It includes all the steps needed to put Kuranishi structure on the other moduli spaces in this thesis. The case with marked points can be treated similarly. We refer to [7] and [25] for the definition and basic properties of Kuranishi structures and to [20, 12] for the details on gluing theorems that we use here. Throughout this chapter (X, ω) denotes a symplectic manifold, $L \subset X$ and $D \subset X$ a Lagrangian submanifold and a symplectic hypersurface, $\beta \in H_2(X, L)/\sim$, $\rho = (s_1, \cdots, s_k)$, $l(\rho) = k$. We fix a compatible almost complex structure J.

5.3.1 Kuranishi neighborhood of irreducible regular maps

Let $u : (D^2, S^1) \to (X, L)$ be a regular J-holomorphic map; thus, $\text{Im}(u) \cap D$ is finite. We denote the set of such maps by $\mathcal{M}^{*,\text{reg,disc}}(X, L, D, \rho, \beta)$. There are $l(\rho)$ marked points $\vec{\xi}$ corresponding to the contact points with D. Define

$$E_u = u^*TX, \qquad F_u = u|_{S^1}^* TL, \quad \text{and} \quad E_u^{0,1} = (T^*D^2)^{0,1} \otimes_{\mathbb{C}} E_u.$$

Fix $p > 2$ and $l > \max s_i$. Let $W^{l,p}(E_u, F_u)_\rho$ be the set of vector fields of class $W^{l,p}$ vanishing to order s_i at each intersection point ξ_i with D and tangent to TL along S^1. Similarly, let $W^{l-1,p}(E_u^{0,1})_\rho$ be the set of E_u-valued $(0,1)$-forms of class

$W^{l-1,p}$ vanishing to order $s_i - 1$ at each ξ_i. The linearized Cauchy-Riemann operator then is a map

$$D_u \colon W^{l,p}(E_u, F_u)_\rho \to W^{l-1,p}(E_u^{0,1})_\rho.$$

Choose finite-dimensional subspaces $\mathcal{E}_u \subset W^{l-1,p}(E_u^{0,1})_\rho$ and $\mathcal{E}_{\tilde{u}} \subset W^{l-1,p}(E_{\tilde{u}}^{0,1})_\rho$ such that every $\eta \in \mathcal{E}_u$ is smooth and supported away from the boundary and marked points, D_u modulo $\mathcal{E}_u$ is surjective, and $\tilde{T}_\phi^1(\mathcal{E}_u) = \mathcal{E}_{\tilde{u}}$. The last condition guarantees that $\tau_{\mathcal{M}}$ induces an involution on the Kuranishi structure.

We take our Kuranishi neighborhood to be $V(u) = (\pi \circ D_u)^{-1}(0)$ (modulo the automorphism group $\mathrm{PSL}(2,\mathbb{R})$ of the disc), which is a smooth manifold of dimension

$$\mu(\beta) + n - 3 + 2l(\rho) + \dim(E_u) - 2D \cdot \beta.$$

The obstruction bundle at each $f \in V_u$ is obtained by parallel translation of $\mathcal{E}_u$ with respect to the induced metric of J. Thus, we get a vector bundle $E(u)$ and a Kuranishi neighborhood $(V(u), E(u))$. The Kuranishi map in this case is just the Cauchy-Riemann operator $f \to \bar{\partial} f$. If there are additional boundary or interior marked points, the tangent space is bigger and includes the tangent spaces of marked points. In this case the Kuranishi structure is a product of Kuranishi structure of the map and the moduli space of marked points.

5.3.2 Kuranishi neighborhoods for nodal regular maps

Let $[u, \Sigma] \in \overline{\mathcal{M}}^{\text{reg,disc}}(X, L, D, \rho, \beta) \setminus \mathcal{M}^{*,\text{reg,disc}}(X, L, D, \rho, \beta)$ be so that the domain is nodal, but the image is still regular. Write $\Sigma = \coprod \Sigma_i$, where each Σ_i is a smooth curve isomorphic to either the disc or the sphere. Then each $u_i = u \mid_{\Sigma_i}$ is an irreducible map. For simplicity we assume there are only two components Σ_1, Σ_2 in the decomposition; we can further assume that the two components are discs with a boundary point in common. The cases with more components or with sphere components can be treated similarly.

We can assume that the node is given by $1 \in \Sigma_i$ for each of the two discs. Let $q = u_i(1) \in L$. If $[u_1] = \beta_1$ and $[u_2] = \beta_2$, then

$$(u_i, \Sigma_i, 1) \in \mathcal{M}_{1,0}^{*,\mathrm{reg},\mathrm{disc}}(X, L, D, \rho_i, \beta_i)$$
$$[u, \Sigma] \in \mathcal{M}_{1,0}^{*,\mathrm{reg},\mathrm{disc}}(X, L, D, \rho_1, \beta_1) \times_{(\mathrm{ev}_1,\mathrm{ev}_1)} \mathcal{M}_{1,0}^{*,\mathrm{reg},\mathrm{disc}}(X, L, D, \rho_2, \beta_2).$$

Let $(V(u_i), E(u_i))$ be the Kuranishi neighborhoods constructed in the previous section and

$$V(u_1, u_2) = (\mathrm{ev}_1 \times \mathrm{ev}_1)^{-1}(\Delta),$$

where Δ is the diagonal in $L \times L$ and $\mathrm{ev}_1 \colon V(u_i) \to L$ are the evaluation maps. For $V(u_1, u_2)$ to be a manifold, we need $(\mathrm{ev}_1 \times \mathrm{ev}_1)$ to be a submersion. We can choose $\mathcal{E}_{u_i}$ big enough so that both evaluation maps ev_1 are submersions. For a fixed β, there are only finitely many topological types of nodal maps which can appear in the limit, and so by induction we can choose obstruction bundles at each step big enough so that the induced Kuranishi structures on the corners of the moduli space for β obey the required conditions. Thus, with correct choice of obstruction bundles $\mathcal{E}_{u_i}$, $V(u_1, u_2)$ is a smooth manifold with projections

$$(\pi_1, \pi_2) \colon V(u_1, u_2) \to V(u_1) \times V(u_2).$$

Then $\mathcal{E}_{f_1,f_2} = \pi_1^{-1}\mathcal{E}_{f_1} \oplus \pi_2^{-1}\mathcal{E}_{f_2}$ gives the fiber of corresponding obstruction bundle over $V(u_1, u_2)$ and the Kuranishi map is as before. Thus, we get a Kuranishi neighborhood $(V(u_1, u_2), E(u_1, u_2))$ of u in the boundary component

$$\mathcal{M}_{1,0}^{*,\mathrm{reg},\mathrm{disc}}(X, L, D, \rho_1, \beta_1) \times_{(\mathrm{ev}_1,\mathrm{ev}_1)} \mathcal{M}_{1,0}^{*,\mathrm{reg},\mathrm{disc}}(X, L, D, \rho_2, \beta_2).$$

In order to extend this Kuranishi neighborhood to a Kuranishi neighborhood $V(u) = V(u_1, u_2) \times [0, \epsilon)$ of u in the original moduli space, we glue the domain and deform the nodal maps in $V(u_1, u_2)$ into J-holomorphic discs modulo obstruction.

Lets z_1, z_2 be local coordinates near $1 \in D^2$, modeled on the closure of upper half-plane as neighborhoods of $0 \in \mathbb{H}$. For each positive real gluing parameter μ, consider the Riemann surface $\Sigma_\mu \cong D^2$ obtained by gluing Σ_1 and Σ_2 via $z_1 z_2 = -\mu$. This gluing respects the orientation of the boundary on each part. Since the divisor D is disjoint from L, a straightforward modification of the proof of [7, Proposition 7.2.12] yields the following.

Proposition 5.5. *There is a continuous family of embeddings*

$$\iota_\mu \colon V(u_1, u_2) \to W^{1,p}(X, \beta), \quad \mu \in (0, \epsilon),$$

with the following properties:

1. $f_\mu = \iota_\mu(f_1, f_2)$ *converges to* (f_1, f_2) *as* $\mu \to 0$*;*
2. $\bar{\partial}_J f_\mu \in \mathcal{E}_{f_1, f_2}$*, where* $\mathcal{E}_{f_1, f_2}$ *is a subspace of* $W^{l-1,p}(E^{0,1}_{f_\mu})$ *obtained via parallel translation;*
3. *every map* f' *close enough to some* $f \in V(u_1, u_2)$ *with* $\bar{\partial}_J f' \in \mathcal{E}_{f'}$ *is in the image of some* ι_μ*.*

5.3.3 Kuranishi neighborhood for non-regular maps

We now consider the case $[u, \Sigma]$ is not regular. This means some component of u is mapped into the divisor D. As we will see, the part mapped into the D will satisfy certain properties, and not every such map can be a limit of a regular stable relative maps. By Section 4.2, every non-regular map

$$[u] \in \overline{\mathcal{M}}^{\mathrm{disc}}(X, L, D, \rho, \beta) \setminus \mathcal{M}^{\mathrm{reg,disc}}(X, L, D, \rho, \beta)$$

can be modeled as a stable map into the singular space $X[n]$ as in Definition 4.7. Therefore, we can write $\Sigma = \coprod \Sigma_i$ such that $u_0 = u \mid_{\Sigma_0}$ is a regular map into the (X, D), possibly from a disconnected domain, and $u_i = u \mid_{\Sigma_i}$, $i > 0$, is a regular map into $(Y_D, D_0 \cup D_\infty)$. Thus,

$$\begin{aligned}
&[u_0] \in \mathcal{M}^{\mathrm{reg,disc}}(X, L, D, \rho_0, \Gamma_0),\\
&[u_i] \in \mathcal{M}^{\mathrm{reg}}(Y_D, D_0 \cup D_\infty, \rho_i^0 \cup \rho_i^\infty, \Gamma_i)/\mathbb{C}^*, \quad i > 0,
\end{aligned}$$

where

- Γ_i describes the topological type of the domain and β_i is the homology class of the image;
- ρ_i^0 describes the intersection pattern of the map with $D_0 \subset Y_D$ and ρ_i^∞, $i > 0$, describes the intersection pattern of u_i with $D_\infty \subset Y_D$, and $\rho_i^0 = \rho_{i+1}^\infty$;
- there is a $\mathbb{C}^*$-action on the space of maps into Y_D which comes from the $\mathbb{C}^*$-action on the $\mathbb{P}^1$ fibers.

For simplicity, we assume that $\Sigma = \Sigma_0 \cup \Sigma_1$ and u_0 and u_1 are not nodal. The general case is an easy extension of this case. From the previous subsections, we have Kuranishi neighborhoods $(V(u_i), E(u_i))$ and evaluation maps

$$\mathrm{ev}_i = \mathrm{ev}_{\vec{\zeta}_i} : V(u_i) \to D^{l(\rho_0)},$$

where $\rho_0 = \rho_1^{0\infty}$ is the intersection pattern between two components and $\vec{\zeta_i}$ are contact points with D. Let

$$V(u_0, u_1) = (\mathrm{ev}_0 \times \mathrm{ev}_1)^{-1}(\Delta)$$

be the inverse image of the diagonal map. For each $(f_0, f_1) \in V(u_0, u_1)$, let $\mathcal{E}_{f_0,f_1} = \mathcal{E}_{f_0} \oplus \mathcal{E}_{f_1}$. If $V(u_i)$ is big enough, $V(u_0, u_1)$ is a manifold. This way we get a Kuranishi neighborhood of $[u]$ in

$$\mathcal{M}^{\mathrm{reg,disc}}(X, L, D, \rho_0, \Gamma_0) \times_{(\mathrm{ev}_0, \mathrm{ev}_1)} \mathcal{M}^{\mathrm{reg}}(Y_D, D_0 \cup D_\infty, \rho_i^0 \cup \rho_i^\infty, \Gamma_i)/\mathbb{C}^*.$$

A gluing theorem similar to Proposition 5.5 is needed to extend this to a Kuranishi neighborhood of $[u]$ on $\overline{M}^{\mathrm{disc}}(X, L, D, \rho, \beta)$. Contrary to the previous case, the gluing will not be unique, and we get a gluing map from some covering space of $V(u_0, u_1)$. This is the space $\mathcal{M}(X_0, \rho, \Gamma_-, \Gamma_+)$ which appears in the statement of Proposition 5.2.

Let $(f_0, f_1) \in V(u_0, u_1)$. Since the obstruction bundle is supported away from the marked and nodal points, u_i is J-holomorphic near the intersection points $\vec{\zeta_i}$ of u_i with D. Let $\vec{q} = \mathrm{ev}_0(\vec{\zeta_0}) = \mathrm{ev}_1(\vec{\zeta_1})$ be the set of intersection points with D. By the symplectic sum procedure of Chapter 2, we can construct a family $\mathcal{X}$ over some small disc Δ whose central fiber is $X \cup_D Y_D$ and whose other fibers are isotopic to X itself. Moreover, in this case, $\mathcal{X}$ is obtained by blowing up $X \times \Delta$ along $D \times \{0\}$. We denote by J to be the complex structure on $\mathcal{X}$.

We choose a set of local coordinate charts on $\mathcal{X}$ around the points $\vec{q}$ as follows. Fix a $\mathbb{C}$-linear identification of $T_{q_i}D$ with $\mathbb{C}^{n-1}$ and extend it to normal coordinates (v_i^j) around q_i in D. Let L_- be the normal bundle of D in X and let L_+ be the normal bundle of D in Y_D. Identifying $L_- \mid_{q_i}$ with $\mathbb{C}$, taking a direct sum with the dual, and parallel translating along radial lines in the v-coordinates, we obtain coordinates $(x, y)\colon (L_- \oplus L_+) \to \mathbb{C} \oplus \mathbb{C}$. This gives a coordinate chart (v, x, y) near each q_i such that the projection $\mathcal{X} \to \Delta$ is given by $(v, x, y) \to xy \in L_- \otimes L_+ = \mathbb{C}$. In these coordinates, the almost-complex structure J on $\mathcal{X}$ agrees with the standard almost-complex structure on $\mathbb{C}^{n-1} \oplus \mathbb{C} \oplus \mathbb{C}$ at the origin, and J has the form $J_D \oplus J_{\mathbb{C}} \oplus J_{\mathbb{C}}$ along D. By [11, Lemmas 3.2,3.4], in these coordinates and around each q_i, $f \in V(u_0, u_1)$ can be written in the form

$$f(z_i, w_i) = f_0(z_i) \# f_1(w_i) = (h^v(z, w), a_i z_i^{s_i}(1 + h^x), b_i w_i^{s_i}(1 + h^y)),$$

with $h^v(0,0) = h^x(0,0) = h^y(0,0) = 0$. In order to glue the two parts of a map f to get a map $f_\mu \in W^{1,p}(X, L)$ with $\bar{\partial}_J f_\mu \in \mathcal{E}_{f_\mu}$, we first have to glue the domains. Let $\mu = (\mu_1, \cdots \mu_k)$ be a tuple of (sufficiently small) complex numbers and define Σ_μ to be the Riemann surface obtained by gluing the domains around the interior marked points $\vec{\zeta_i}$ via the equation $z_j w_j = \mu_j$. So we are replacing the node with a small cylinder described by the gluing parameters μ_i at each node. If we can glue two parts of f and get a map f_μ as above for small μ, then the part of f_μ which is mapped to the neck can be written in the form $f_\mu(z_i, w_i) = (v_{\mu_i}, x_{\mu_i}, y_{\mu_i})$ with $x_{\mu_i} y_{\mu_i} = \epsilon$ for some fixed $\epsilon \in \Delta$. For small ϵ, the maps f_μ are closely approximated by $(q_i, a_i z_i^{s_i}, b_i w_i^{s_i})$ near the intersection point q_i, see [12, Section 5]. So for f_μ to be in X_μ, we need

$$a_i b_i \mu_i^{s_i} = \epsilon. \tag{5.4}$$

This shows that there are altogether $|\rho| = \prod s_i$ possibilities for choosing μ (for a fixed ϵ), and each choice leads to a different map. The coefficients a_i and b_i are the s_i-jets of the components of f_j normal to D at ζ_j^i modulo higher order terms, and so

$$a_i \in (T^*_{\zeta_0^i}\Sigma_0)^{s_i} \otimes L_{-,q_i}, \quad b_i \in (T^*_{\zeta_1^i}\Sigma_1)^{s_i} \otimes L_{+,q_i}.$$

Let $\mathcal{L}_i^j$, $i = 0, 1$, be the relative cotangent bundle to Σ_i at ζ_i^j. These are complex line bundles over the Deligne-Mumford moduli space and the leading coefficients are sections

$$a_i \in \Gamma((\mathcal{L}_0^i)^{s_i} \oplus \mathrm{ev}_{0i}^* L_-) \quad \text{and} \quad b_i \in \Gamma((\mathcal{L}_1^i)^{s_i} \oplus \mathrm{ev}_{1i}^* L_+).$$

We conclude that μ is a multisection of the bundle

$$\bigotimes [(\mathcal{L}_0^i)^* \otimes (\mathcal{L}_1^i)^*] \to V(u_0, u_1).$$

We define $\tilde{V}(u_0, u_1)$ to be the total space of this multisection. This is an ètale covering of $V(u_0, u_1)$ and we can pull back the obstruction bundle to get a Kuranishi chart $(\tilde{V}(u_0, u_1), E(u_0, u_1))$.

In order to finish the construction of a Kuranishi structure on $\overline{\mathcal{M}}^{\mathrm{disc}}(X, L, D, \rho, \beta)$, we need a gluing theorem to extend the Kuranishi neighborhood $(\tilde{V}(u), E(u))$ to a Kuranishi neighborhood of $[u]$ on $\overline{\mathcal{M}}^{\mathrm{disc}}(X, L, D, J, \beta)$. This gluing theorem is provided by a slight modification (to accommodate obstruction bundles) of the gluing theorem in [12, Sections 5–8].

Proposition 5.6. *There is a continuous family of orientation-preserving embeddings*

$$\iota_\mu \colon \tilde{V}(u_0, u_1) \to W^{1,p}(X, \beta), \qquad a_i b_i \mu^{s_i} = \epsilon$$

with the following properties:

1. $f_\mu = \iota_\mu(f_0, f_1)$ *converges to* (f_1, f_2) *as* $\mu \to 0$*;*
2. $\bar{\partial}_J f_\mu \in \mathcal{E}_{f_\mu}$, *where* $\mathcal{E}_{f_\mu}$ *is the subspace of* $E^{0,1}_{f_\mu}$ *obtained via parallel translation;*
3. *any map* f' *close enough to some* $f \in \tilde{V}(u)$ *with* $\bar{\partial}_J f' \in \mathcal{E}_{f'}$ *is in the range of some* ι_μ.

Chapter 6

Computations

Let $[d] \in H_2(\mathbb{P}^3) \cong \mathbb{Z}$ be the homology class of a degree d curve. The elements of the moduli space $\overline{\mathcal{M}}_d(\mathbb{P}^3, [d])^{\eta_3,\eta}$ whose elements are degree d real curves with d pairs of complex conjugate marked points, $\xi_1 = (z_1, \overline{z_1}), \cdots, \xi_n = (z_d, \overline{z_d})$. By the result of Section 3.3, this moduli space is oriented. Let

$$\mathrm{ev}_i \colon \overline{\mathcal{M}}_d(\mathbb{P}^3, [d])^{\eta_3,\eta} \to \mathbb{P}^3, \quad \mathrm{ev}_i((u, \Sigma, (\xi_j))) = u(z_i) \in \mathbb{P}^3,$$

be the evaluation maps corresponding to the marked points. Let $H^3 = \mathrm{PD}(\mathrm{pt})$ be the homology class of a point, where H is the class of hyperplane. We are interested in the real invariants

$$N_d^{\mathrm{real}} = \int_{[\overline{\mathcal{M}}_d(\mathbb{P}^3,[d])^{\eta_3,\eta}]} \mathrm{ev}_1^*(H^3) \wedge \cdots \wedge \mathrm{ev}_d^*(H^3). \tag{6.1}$$

Geometrically, this is the number of genus zero real curves of degree d passing through d generic points of $\mathbb{P}^3$. For instance, $N_1 = 1$ because for any point of $\mathbb{P}^3$, there is a unique real line passing though that point (and its conjugate). In this section, we compute these invariants and compare them to the invariants of $\mathbb{P}^3$ obtained by counting J-holomorphic discs with respect to the other involution on $\mathbb{P}^3$ which has fixed points (denoted by τ_3).

6.1 Preliminaries

Let $\mathbf{T} = \{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$ be the unit circle. The torus $\mathbf{T}^4$ acts on $\mathbb{P}^3$ by

$$(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \cdot [z_1, z_2, z_3, z_4] = [\zeta_1 \cdot z_1, \zeta_2 \cdot z_2, \zeta_3 \cdot z_3, \zeta_4 \cdot z_4].$$

Under the injection $i\colon \mathbf{T}^2 \to \mathbf{T}^4$, given by $(\zeta_1, \zeta_2) \to (\zeta_1, \zeta_1^{-1}, \zeta_2, \zeta_2^{-1})$, we get an action of $\mathbf{T}^2$ on $\mathbb{P}^3$ which commutes with the action of involution η_3. The action of $\mathbf{T}^4$ on $\mathbb{P}^3$ has four fixed points,

$$p_1 = [1, 0, 0, 0], \quad \cdots, \quad p_4 = [0, 0, 0, 1].$$

The action of $\mathbf{T}^2$ has the same set of fixed points and the involution η_3 permutes these points,

$$p_1 \longleftrightarrow p_2, \quad p_3 \longleftrightarrow p_4.$$

By composition on the left, $\mathbf{T}^2$ also acts on the moduli space $\overline{\mathcal{M}}_d(\mathbb{P}^3, [d])^{\eta_3, \eta}$.

Lemma 6.1. *The irreducible $\boldsymbol{T}^2$-invariant curves in $\mathbb{P}^3$ are the lines L_{ij}, connecting p_i and p_j. Moreover, the irreducible real $\boldsymbol{T}^2$-invariant curves are L_{12} and L_{34}.*

Let λ_i to be the equivariant first Chern class of $\mathcal{O}(1)_{\mathbb{P}^3}$ restricted to p_i in $H^*_{\mathbf{T}^4}(\mathrm{pt})$. Then,

$$H^*_{\mathbf{T}^4}(\mathrm{pt}) = \mathbb{Q}[\lambda_1, \lambda_2, \lambda_3, \lambda_4].$$

The weights of $T\mathbb{P}^3$ at the point p_i are $\{\lambda_i - \lambda_j\}_{j\neq i}$; see [10, Chapter 27]. Let α, β be the generators of $H^*_{\mathbf{T}^2}(\mathrm{pt})$ defined by

$$i^*\colon H^*_{\mathbf{T}^4}(\mathrm{pt}) \to H^*_{\mathbf{T}^2}(\mathrm{pt}), \quad i^*(\lambda_1) = \alpha,\ i^*(\lambda_2) = -\alpha,\ \ i^*(\lambda_3) = \beta,\ i^*(\lambda_4) = -\beta.$$

For notational convenience, we often omit the pull-back map i^* and write

$$\lambda_1 = \alpha,\ \lambda_2 = -\alpha\ \ \lambda_3 = \beta,\ \lambda_4 = -\beta.$$

Let $u\colon (\Sigma, (\zeta_i)) \to \mathbb{P}^3$ be a holomorphic map in $\overline{\mathcal{M}}_d(\mathbb{P}^3, [d])^{\eta_3,\eta}$ defined on nodal domain Σ with marked points (ξ_i) whose image as a marked curve is fixed under the action of $\mathbf{T}^2$. Since there are no $\mathbf{T}^2$-fixed points in $\mathbb{P}^3$ that are also fixed by η_3,

$$\Sigma = \Sigma_0 \cup \coprod \Sigma_i \coprod \Sigma_{\bar{i}}$$

has a unique central component $\Sigma_0 \cong \mathbb{P}^1$, which is invariant under the involution, and others come in conjugate pairs; see Section 3.1. Every nodal and marked point is mapped to one of the fixed points p_i. If $u\,|_{\Sigma_i}$ has degree d_i, then $d = d_0 + 2\sum d_i$. We call such a curve of type d_0. By Remark 3.6, d_0 is odd, i.e. $[\overline{\mathcal{M}}_{2k}(\mathbb{P}^3, [2k])^{\eta_3,\eta}]^{\mathbf{T}^2} = \emptyset$.

Corollary 6.2. *For $d \in 2\mathbb{Z}$, $N_d^{\mathrm{real}} = 0$.*

Remark 6.3. For d even, a localization calculation shows that $N_d^{\tau_3}$ is zero. This implies that the conclusion of Theorem 1.10 for $L \cong \mathbb{RP}^3$ and β even holds, i.e. $N_\beta^\phi(X) = N_{\tilde{\beta}}^{\mathrm{real}}(X_+)$.

Therefore, from now on, we assume d_0 is odd. For every $\mathbf{T}^2$-fixed curve of type d_0, after removing the central component, we get two irreducible closed curve

$$(u', \Sigma' = \coprod \Sigma_i), \qquad (u'', \Sigma'' = \coprod \Sigma_{\bar{i}}),$$

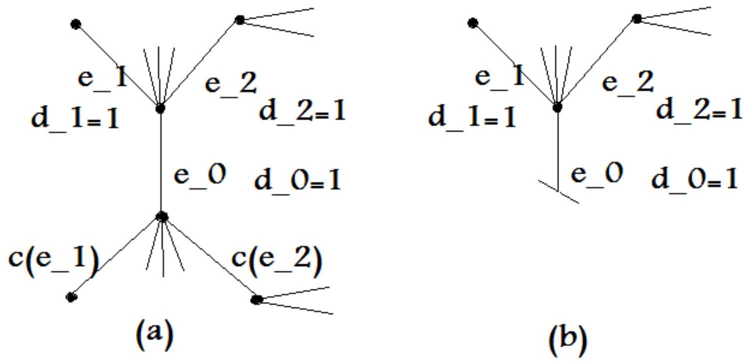


Figure 6.1: A typical decorated graph (a) and its half (b).

each of degree $l = \frac{d-d_0}{2}$. Each of these has $d+1$ marked points, $(z_i)_{i=0}^{d}$, so that z_0 corresponds to the nodal point shared with Σ_0 and rest of the marked points are decorated by $\pm$ signs depending on whether it is the first point of ξ or not. If the new marked point z_0 is mapped into p_i, we say that the map is of type (d_0, p_i).

As described above, every curve in $[\overline{\mathcal{M}}_d(\mathbb{P}^3, [d])^{\eta_3,\eta}]^{\mathbf{T}^2}$ can be modeled on a symmetric labeled tree, Γ, where the symmetry comes from complex conjugation.

6.2 Localization calculation of odd degree real GW invariants

Such a tree has a central edge e_0 corresponding to the central component, a degree for each component, and open edges corresponding to the marked points. Figure 6.1(a) shows one such Γ of total degree 5 and 5 pairs of conjugate marked points. Removing e_0 from Γ, we get a disconnected graph $\Gamma' \cup \eta(\Gamma')$. Choose one of the components (say Γ') and add the corresponding half edge in place of the central edge, Figure 6.1(b). We denote the total half graph by Γ_{half}; it has a unique half edge corresponding to the central part. All calculations below are based on this half graph; one can check that the result is independent of which half we choose.

For any flag $F = (v, e)$, let $v(F) = \mathrm{Im}(v) \in \{p_i\}$ be the corresponding fixed point. For every $(f, \Sigma, (\xi_i))$ in the fixed point locus, there is an exact sequence

$$0 \to \mathrm{Aut}(\Sigma, (\xi_i))_{\mathbb{R}} \to \mathrm{Def}(f)_{\mathbb{R}} \to \mathrm{Def}(f, \Sigma, (\xi_i))_{\mathbb{R}} \to \mathrm{Def}(\Sigma, (\xi_i))_{\mathbb{R}} \to 0.$$

Thus,

$$e(N_\Gamma) = e(\mathrm{Def}(f, \Sigma, (\xi_i))^{\mathrm{mov}}_{\mathbb{R}}) = \frac{e(\mathrm{Def}(f)^{\mathrm{mov}}_{\mathbb{R}})e(\mathrm{Def}(\Sigma, (\xi_i))^{\mathrm{mov}})_{\mathbb{R}}}{e(\mathrm{Aut}(\Sigma, (\xi_i))^{\mathrm{mov}}_{\mathbb{R}})}, \tag{6.2}$$

where "mov" means the moving part (the part with non-zero weights) and $e(-)$ means the top Chern class. Following [10, Chapter 27], we drive a formula for $e(N_\Gamma)$ for any odd-dimensional projective space, $\mathbb{P}^{2m-1}$.

- **The bundle** $\mathrm{Aut}(\Sigma, (\xi_i))_{\mathbb{R}}$. For each non-contracted component of Σ, there is a torus-fixed piece coming from the infinitesimal automorphisms of the component fixing the two special points. This fixed part cancels with a similar weight-zero term in $\mathrm{Def}(f)$. There are, however, more automorphisms. They correspond to vertices v of valence 1 (note that they come in conjugate pairs). In this case, the point mapping to some fixed point p_i is not a special point. Hence, we have an additional automorphism corresponding to moving these points. For each such point $v \in \Sigma'$, this is a complex one-dimensional space, isomorphic (as a $\mathbf{T}$-representation) to the tangent space to v, i.e.

$$e(\mathrm{Aut}(\Sigma, (\xi_i))^{\mathrm{mov}}_{\mathbb{R}}) = \prod_{\mathrm{val}(v)=1} w_F,$$

  where F is the unique flag containing $v \in \Gamma_{\mathrm{half}}$, connecting v to v' by edge e of degree d_e, and $w_F = \frac{\lambda(v)-\lambda(v')}{d_e}$.

- **The bundle** $\mathrm{Def}(\Sigma, (\xi_i))_{\mathbb{R}}$. A deformation of the contracted components (as a marked curve) is a weight-zero deformation of the map that yields the tangent space $\mathrm{Def}(\Sigma)^{\mathrm{fix}}_{\mathbb{R}}$ as a summand of $\mathrm{Def}(f, \Sigma, (\xi_i))^{\mathrm{fix}}_{\mathbb{R}}$. The other deformations come from smoothing (conjugate pairs of) nodes of Σ. The one-dimensional space associated to each node is identified with the tensor product of the tangent spaces of the two components at the node. For every node corresponding to a vertex v, there are two possibilities. If $\mathrm{val}(v) = 2$ and v connects two non-contracted components (we also have valence two vertices which correspond to a single marked point on a non-contracted component), the contribution is

$$\prod_{\mathrm{val}(v)=2} (w_{F_1} + w_{F_2}),$$

where F_1, F_2 are the flags attached to $v \in \Gamma_{\mathrm{half}}$. If $\mathrm{val}(v) \geq 3$, there is a stable component mapping to a fixed point and the contribution is

$$\prod_{\substack{F \in \Gamma_{\mathrm{half}} \\ \mathrm{val}(v(F))>2}} (w_F - \psi_F),$$

where ψ_F is the ψ-class corresponding to the marked point determinded by F on the contracted curve.

- **The bundle** $\mathrm{Def}(f)_{\mathbb{R}} = H^0(\Sigma, f^*T\mathbb{P}^{2m-1})_{\mathbb{R}}$. There is an exact sequence

$$\begin{aligned} 0 &\to H^0(\Sigma, f^*T\mathbb{P}^{2m-1})_{\mathbb{R}} \\ &\to H^0(\Sigma_{e_0}, f^*T\mathbb{P}^{2m-1})_{\mathbb{R}} \oplus \bigoplus_v H^0(\Sigma_v, f^*T\mathbb{P}^{2m-1}) \oplus \bigoplus_{e \neq e_0} H^0(\Sigma_e, f^*T\mathbb{P}^{2m-1}) \\ &\to \bigoplus_F T_{v(F)}\mathbb{P}^{2m-1} \to 0. \end{aligned}$$

For $e \neq e_0$, the contribution of $H^0(\Sigma_e, f^*T\mathbb{P}^{2m-1})$ to $e(\mathrm{Def}(f)^{\mathrm{mov}})$ is classical and is equal to

$$(-1)^{d_e} \frac{d_e!^2}{d_e^{2d_e}} (\lambda_i - \lambda_j)^{2d_e} \prod_{a=0}^{d_e} \prod_{k \neq i,j} (\frac{a}{d_e}\lambda_i + \frac{d_e - a}{d_e}\lambda_j - \lambda_k),$$

where i and j are the two vertex labels of the edge e and d_e is the degree of the edge e; see [10, Section 27.4].

The half edge e_0 corresponds to a map of odd degree to one of the lines $L_{2i-1,2i}$. So it is given by

$$f_{2i-1}\colon [z_0, z_1] \to [0, \cdots, 0, z_0^l, z_1^l, 0, \cdots, 0] \text{ or}$$
$$f_{2i}\colon [z_0, z_1] \to [0, \cdots, 0, z_1^l, z_0^l, 0, \cdots, 0],$$

with the convention that $[z_0, z_1] = [1, 0]$ is in the half graph. As an example, we consider f_1. In the open chart $\{z_1 \neq 0\} = \mathbb{C}^{2m-1} \subset \mathbb{P}^{2m-1}$, f_1 is given by

$$f_1(z) = (z^l, 0, \cdots, 0).$$

The weights of the torus action on $T_{p_1}\mathbb{P}^{2m-1}$ are $-2\alpha_1, \alpha_2 - \alpha_1, -\alpha_2 - \alpha_1, \cdots$. Every section of $H^0(\Sigma_{e_0}, f_1^*T\mathbb{P}^{2m-1})_{\mathbb{R}}$ is then of the form

$$(\sum_{a=0}^{2l} b_{2,a}z^a, \sum_{a=0}^{l} b_{34,a}z^a, \sum_{a=0}^{l}(-1)^a\overline{b_{34,a}}z^{l-a}, \cdots), \quad b_{2,a} = (-1)^a\overline{b_{2,2l-a}}.$$

Therefore, $b_{2,0}, \cdots, b_{2,l-1}, b_{34,0}, b_{34,l}, \cdots \in \mathbb{C}$ and $b_{2,l} \in i^l\mathbb{R}$ give coordinates on $H^0(\Sigma_{e_0}, f_1^*T\mathbb{P}^{2m-1})_{\mathbb{R}}$. The part given by $b_{2,0} + b_{2,l}z^l + \overline{b_{2,0}}z^{2l}$ comes from tangent vectors to Σ_{e_0}. The middle term has weight zero and cancels with the weight zero factor in $\mathrm{Aut}(\Sigma_{e_0})$. So the non-zero weights correspond to $b_{2,0}, \cdots, b_{2,l-1}, b_{34,0}, b_{34,l}, \cdots \in \mathbb{C}$ with weights

$$2\alpha_1(1 - \frac{a}{l}),\ 0 \leq a < l, \quad \alpha_1(1 - \frac{2a}{l}) - \alpha_2,\ \cdots,\ \alpha_1(1 - \frac{2a}{l}) - \alpha_m,\ 0 \leq a \leq l.$$

Therefore, up to a sign factor, the contribution is

$$I_{f_1} = (-1)^{\frac{l+1}{2}(m-1)} \left(\frac{2\alpha_1}{l}\right)^l l! \prod_{j=2}^{m} \prod_{\substack{0 \le a \le l \\ a \text{ odd}}} \left(\left(\frac{a\alpha_1}{l}\right)^2 - \alpha_j^2\right).$$

- The only term left is the contribution of $T_{v(F)}\mathbb{P}^{2m-1}$, which is $\prod_{i \neq v(f)}(\lambda_v - \lambda_i)$.

Putting all this together we get

$$\begin{aligned}\frac{1}{e(N_\Gamma)} = \frac{1}{e(N_{\Gamma_{\text{half}}})} = & \prod_{\substack{F \in \text{Flags} \\ \text{val}(v(F))>2}} \frac{1}{w_F - \psi_F} \prod_{\substack{F \in \text{Flags} \\ u \neq v(F)}} (\lambda_{v(F)} - \lambda_u) \\ & \prod_{v \in \text{vertices}} \prod_{u \neq v} \frac{1}{\lambda_v - \lambda_u} \prod_{\text{val}(v)=2} \frac{1}{w_{F_1} + w_{F_2}} \prod_{\text{val}(v)=1} w_F \\ & \prod_{e \neq e_0} \left(\frac{(-1)^{d_e} d_e^{2d_e}}{d_e!^2 (\lambda_i - \lambda_j)^{2d_e}} \prod_{a+b=d_e} \prod_{k \neq i,j} \frac{1}{\frac{a}{d_e}\lambda_i + \frac{b}{d_e}\lambda_j - \lambda_k} \right) \\ & (-1)^\epsilon (-1)^{\frac{l+1}{2}(m-1)} \left(\frac{l}{2\alpha_i}\right)^l \frac{1}{l!} \prod_{\substack{j=1 \\ j \neq i}}^{m} \prod_{\substack{a=1 \\ a \text{ odd}}}^{l} \frac{1}{(\frac{a\alpha_i}{l})^2 - \alpha_j^2},\end{aligned}$$

where $\epsilon = 1$ if the half edge is built over f_{2i-1} and $\epsilon = 0$ if the half edge is built over f_{2i}. This extra ± 1 factor is the consequence of the orientation.

Example 6.4 ($d = 1$)**.** We denote the contribution of f_1 by $C(\alpha, \beta)$. Summing over all four possible cases, we get

$$N_1^{\text{real}} = 2(C(\alpha, \beta) + C(\beta, \alpha)), \qquad C(\alpha, \beta) = \frac{\alpha^2}{2(\alpha^2 - \beta^2)}.$$

From this we get,

$$N_1^{\text{real}} = \frac{\alpha^2}{\alpha^2 - \beta^2} + \frac{\beta^2}{\beta^2 - \alpha^2} = 1,$$

as it should be.

	$d=1$	$d=3$	$d=5$
$\lvert N_d^{\text{real}}\rvert$	1	1	5

Table 6.1: N_d^{real} in low degrees.

Example 6.5 ($d=3$)**.** In this case threre are five possible types of Γ_{half}. Doing the calculations we get

$$
\begin{aligned}
N_3^{\text{real}} = 4\times [\, -\left(\frac{3}{2}\right)^5 & \left(\frac{\alpha^4}{(\alpha^2-\beta^2)(\alpha^2-9\beta^2)}\frac{\beta^4}{(\beta^2-\alpha^2)(\beta^2-9\alpha^2)}\right)\\
& + \frac{1}{32}\frac{\alpha^4+\beta^4}{(\alpha^2-\beta^2)^2} + \frac{-\beta^8(3\alpha^2+5\beta^2)}{(\alpha^2-\beta^2)^4(9\alpha^2-\beta^2)}\frac{-\alpha^8(3\beta^2+5\alpha^2)}{(\alpha^2-\beta^2)^4(9\beta^2-\alpha^2)}\\
& + \frac{3}{16}\frac{\alpha^4+\beta^4}{(\alpha^2-\beta^2)^2} + 3\frac{\alpha^2\beta^6+\beta^2\alpha^6}{(\beta^2-\alpha^2)^4}\\
& + \frac{3}{8}\frac{\alpha^4+\beta^4}{(\alpha^2-\beta^2)^2} + 3\frac{-\beta^2\alpha^4(3\alpha^2+\beta^2)-\alpha^2\beta^4(3\beta^2+\alpha^2)}{4(\alpha^2-\beta^2)^4}\\
& + \frac{\alpha^4+\beta^4}{4(\alpha^2-\beta^2)^2} + \frac{\alpha^6(3\alpha^2-\beta^2)+\beta^6(3\beta^2-\alpha^2)}{2(\alpha^2-\beta^2)^4}\Bigg] = -1.
\end{aligned}
$$

From a similar but longer calculation, we get $N_5 = 5$.

6.3 Comparison with the open GW invariants

The other involution, τ_3, on $\mathbb{P}^3$ has fixed locus $\text{Fix}(\tau_3) = \mathbb{RP}^3$. Let $[d/2] \in H_2(\mathbb{P}^3, \mathbb{RP}^3)$ be the relative homology class of d times of a half line. Let $\overline{\mathcal{M}}_{0,d}^{\text{disc}}(\mathbb{P}^3, \mathbb{RP}^3, [d/2])_{\text{dec}}$ be the moduli space of degree d discs with one un-decorated and $d-1$ decorated (decorated with $\pm$) interior marked points. If d is odd, $\partial^1\overline{\mathcal{M}}_{0,d}^{\text{disc}}(\mathbb{P}^3, \mathbb{RP}^3, [d/2])_{\text{dec}}$ consists of only nodal discs. By gluing the boundary components of $\overline{\mathcal{M}}_{0,d}^{\text{disc}}(\mathbb{P}^3, \mathbb{RP}^3, [d/2])_{\text{dec}}$ via $\tau_{\mathfrak{glue}}$, we get a moduli space

$$\widetilde{\mathcal{M}}_{0,d}^{\text{disc}}(\mathbb{P}^3, \mathbb{RP}^3, [d/2])_{\text{dec}} = \overline{\mathcal{M}}_{0,d}^{\text{disc}}(\mathbb{P}^3, \mathbb{RP}^3, [d/2])_{\text{dec}}/\sim_{\text{glue}} \tag{6.3}$$

without boundary. We can then define open GW-invariants $N^{\text{disc}}_{d/2}$ by counting J-holomorphic discs passing through d generic points as before. Applying localization, $\widetilde{\mathcal{M}}^{\text{disc}}_{0,d}(\mathbb{P}^3, \mathbb{RP}^3, [d/2])_{\text{dec}}$ and $\overline{\mathcal{M}}_d(\mathbb{P}^3, [d])^{\eta_3,\eta}$ have the same fixed point loci and thus

$$N^{\text{disc}}_{d/2} = N^{\text{real}}_d.$$

Conjecture 6.6. *The closed moduli spaces* $\widetilde{\mathcal{M}_{0,d}}^{\text{disc}}(\mathbb{P}^3, \mathbb{RP}^3, [d/2])_{\text{dec}}$ *and* $\overline{\mathcal{M}_d}(\mathbb{P}^3, [d])^{\eta_3,\eta}$ *are isomorphic.*

For example, for $d = 1$ both moduli spaces are isomorphic to the quadratic hypersuface in $\mathbb{P}^3$.

Bibliography

[1] M. Aganagic, A. Klemm, and C. Vafa, *Disk instantons, mirror symmetry and the duality web*, Z. Naturforsch. A 57 (2002), no. 1–2, 1–28.

[2] M. Audin, *Lagrangian skeletons, periodic geodesic flows and symplectic cuttings*, Manuscripta Math. 124 (2007), no. 4, 533–550.

[3] R. Castano-Bernard, D. Matessi, and J. Solomon, *Symmetries of Lagrangian fibrations*, Adv. Math. 225 (2010), no. 3, 1341–1386.

[4] Y. Eliashberg, A. Givental, and H. Hofer, *Introduction to Symplectic Field Theory*, Geom. Funct. Anal. 2000, Special Volume, Part II, 560-673.

[5] K. Fukaya, *Counting pseudo-holomorphic discs in Calabi-Yau 3 fold*, Tohoku Math. J. (2) 63 (2011), no. 4, 697-727.

[6] K. Fukaya, and K. Ono, *Arnold conjecture and Gromov-Witten invariant*, Topology 38 (1999), no. 5, 933-1048.

[7] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian intersection Floer theory: anomaly and obstruction, Volumes I,II*, AMS/IP Studies in Advanced Mathematics, 2009.

[8] K. Fukaya, Y-G, Oh. H, Ohta and K. Ono, *Anti-symplectic involution and Floer cohomology*, arXiv:0912.2646.

[9] P. Georgieva, *Orientability of moduli spaces and open Gromov-Witten invariants*, PhD Thesis, 2011.

[10] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, E. Zaslow, *Mirror Symmetry*, Clay Mathematics Monographs.

[11] E. Ionel and T. Parker, *Relative Gromov-Witten Invariants*, Ann. of Math. (2) 157 (2003), no. 1, 45-96.

[12] E. Ionel and T. Parker, *The symplectic sum formula for Gromov-Witten invariants*. Ann. of Math. (2) 159 (2004), no. 3, 935-1025.

[13] J. M. F. Labastida, M. Marino, and C. Vafa, *Knots, links and branes at large N*, J. High Energy Phys. 2000, no. 11, Paper 7, 42 pp.

[14] E. Lerman, *Symplectic cuts*, Mathematical Research Letters 2 (1995), 247–258.

[15] M. Levine and R. Panharipande, *Algebraic cobordism revisited*, Invent. Math. 176 (2009), no. 1, 63–130.

[16] J. Li, *A degeneration formula of GW-invariants.* J. Differential Geom. 60 (2002), no. 2, 1992–93.

[17] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. Amer. Math. Soc. 11 (1998), no. 1, 119–174.

[18] M. Liu, *Moduli of J-holomorphic curves with Lagrangian boundary conditions and open Gromov-Witten invariants for a S^1-equivariant pair*, math.SG/0210257.

[19] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxford Mathematical Monographs, 1998.

[20] D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*, American Mathematical Society Colloquium Publications, 52, American Mathematical Society, Providence, RI, 2004.

[21] J. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Reports on Mathematical Physics 5 (1974), no. 1, 121–130.

[22] H. Ooguri and C. Vafa, *Knot invariants and topological strings*, Nuclear Phys. B 577 (2000), no. 3, 419–438.

[23] R. Pandharipande, J. Solomon, and J. Walcher, *Disk enumeration on the quintic 3-fold* J. Amer. Math. Soc. 21 (2008), no. 4, 1169-1209.

[24] Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology*, J.Differential Geom. 42 (1995), no. 2, 259–367.

[25] J. Solomon, *Intersection theory on the moduli space of holomorphic curves with Lagrangian boundary conditions*, arXiv:math/0606429.

[26] J-Y. Welschinger, *Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry*, Invent. Math., Volume 162, Issue 1, Oct 2005, 195–234.

[27] – , *Spinor states of real rational curves in real algebraic convex 3-manifolds and enumerative invariants*, Duke Math. J. 127 (2005), no. 1, 89–121.

[28] – , *Enumerative invariants of strongly semi-positive real symplectic manifolds*, arXiv: math.AG/0509121.

[29] – , *Open strings, Lagrangian conductors and Floer functor*, arXiv:math.SG/0812.0276v1.

[30] – , *Open Gromov-Witten invariants in dimension four*, arXiv:1110.2705.

[31] – , *Open Gromov-Witten invariants in dimension six* arXiv:1201.3518.

[32] E. Witten, *Chern-Simons gauge theory as a string theory*, The Floer memorial volume, 637-678, Progr. Math., 133, Birkhauser, Basel, 1995.

[33] A. Zinger, *A comparison theorem for Gromov-Witten invariants in the symplectic category.* Adv. Math. 228 (2011), no. 1, 535-574.