On Moduli Spaces of J-holomorphic curves in Symplectic Manifolds

Mohammad Farajzadeh-Tehrani

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Outline

■ Part 1

■ What are symplectic manifolds, what are $J$-holomorphic curves, and why do we care about them?

■ What is known about the (moduli) space of $J$-holomorphic maps

■ In particular, how do we construct a compact moduli space of $J$-holomorphic maps?

■ Part 2

■ I will introduce certain moduli spaces of $J$-holomorphic curves with extra tangency conditions

■ I will discuss the motivation for studying such moduli spaces and the known results

■ I will finish by stating a NEW compactification result
Part 1
What is a Symplectic manifold?

- A $2n$-dim manifold $X$ with a closed ($d\omega = 0$) and non-degenerate ($\omega^n \neq 0$) 2-form $\omega$.

- Example: $X = \mathbb{R}^{2n} = \mathbb{C}^n$ with $\omega_{\text{std}} = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$.

- Every $(X, \omega)$ is locally isomorphic to $(\mathbb{R}^{2n}, \omega_{\text{std}})$.

- In real dimension 2 (C-dim 1), all Riemann surfaces are symplectic manifolds.
Almost-complex structures compatible with $\omega$

- **Almost complex structure**: $J: TX \to TX$ s.t. $J^2 = -\text{id}$

- **Compatible with $\omega$**: $\omega(\cdot, J\cdot)$ is a metric

- **Crucial observation**: Space of compatible $J$ is non-empty (infinite dimensional) and contractible
  In particular it is connected

- A triple $(X, \omega, J)$ is called an **almost Kähler** manifold

- **Kähler manifold**: If $J$ comes from a complex structure on $X$

- (Newlander-Nirenberg 1957) **Nijenhuis** $(2, 1)$-tensor measures how far $J$ is from defining a complex structure

\[ N_J(u, v) \equiv [u, v] + J[u, Jv] + J[Ju, v] - [Ju, Jv] \in T_xX \quad \forall u, v \in T_xX \]

- In dimension 2 all $J$’s are holomorphic ($N_J \equiv 0$)
What are $J$-holomorphic maps?

- $(X, \omega, J)$ as before
- $(\Sigma, j)$: a Riemann Surface with complex structure $j$
- (Gromov 85) $J$-holomorphic map

$$u: (\Sigma, j) \longrightarrow (X, J) \quad \text{s.t.} \quad \bar{\partial} u = du + Jduj = 0$$

i.e. $d_x u: T_x \Sigma \longrightarrow T_{u(x)} X$ is $\mathbb{C}$-linear \(\forall \ x \in \Sigma\)

- This is a non-linear Cauchy-Riemann (CR) equation
- Im$(u) \subset X$ is called a $J$-holomorphic curve
  It could be singular at some points

- One can also define $J$-holomorphic maps from Riemann surfaces with boundary subject to some boundary conditions (Floer theories)
Moduli spaces of $J$-holomorphic maps

- If $h : \Sigma \longrightarrow \Sigma$ is a holomorphic reparametrization $\Rightarrow$
  - $u' = u \circ h$ is also $J$-holomorphic
  - $u$ and $u'$ are equivalent (define the same curve $\text{Im}(u) = \text{Im}(u')$)
- The homology class $A \in H_2(X, \mathbb{Z})$ represented by $u$ and the Genus $g$ of $\Sigma$ characterize the topological type of $u$

- $(\text{Set}) \, M_g(X, A) = \{ (u, (\Sigma, j)) : \bar{\partial}u = 0 \} / \sim$
- $(\text{Set}) \, M_{g,k}(X, A) = \{ (u, (\Sigma, j, z_1, \ldots, z_k)) : \bar{\partial}u = 0 \} / \sim$

genus 2 Riemann surface with 3 marked points

![genus 2 Riemann surface with 3 marked points](image)
Why are these moduli spaces useful?

- Powerful tool to study global geometry/topology of symplectic manifolds
- To find periodic orbits of Hamiltonian ODE’s: Floer homology
- Defining enumerative invariants: Gromov-Witten theory
- String theory, Mirror Symmetry
- Topology of 3-manifolds: Heegaard-Floer homology
- Connections to Seiberg-Witten theory and other Gauge theories
Example of Gromov-Witten theory

- Gromov-Witten theory **formalizes** and **generalizes** the enumerative geometry in algebraic geometry which is about **finding/counting** holomorphic curves of specific type

- **Example 1:** There is a unique holomorphic sphere of homology class \([1] \in H_2(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z}\) (called a **line**) passing through 2 points in \(\mathbb{CP}^n\)

- **Example 2:** There are **2875 lines** in a (generic) degree 5 Calabi-Yau hypersurface in \(\mathbb{CP}^4\)

- **Evaluation maps:** \(\text{ev} = \prod_{i=1}^{k} \text{ev}_i: \mathcal{M}_{g,k}(X, A) \longrightarrow X^k\)
  
  \[(u, \Sigma, j, z_1, \ldots, z_k) \xrightarrow{\text{ev}_i} u(z_i) \in X\]
Example of Gromov-Witten theory (continued)

- We want $\text{GW}^{X,A}_{g,k} = \text{ev}_*(\mathcal{M}_{g,k}(X,A)) \subset X^k$ to be a “nice” cycle

- Then, for cycles $\beta_1, \ldots, \beta_k$ representing homology classes $B_1, \ldots, B_k \in H_*(X)$, we define

  $$\text{GW}^X_{g,A}(B_1, \ldots, B_k) = \# \left( \text{GW}^{X,A}_{g,k} \cap (\beta_1 \times \cdots \times \beta_k) \right) \in \mathbb{Z}$$

- For this construction to work we need

  1. a **Topology/Smooth structure** on $\mathcal{M}_{g,k}(X,A)$
  2. a “nice” **Compactification** $\overline{\mathcal{M}}_{g,k}(X,A)$ of the correct “expected dimension”
  3. some control of **Smoothness** of $\overline{\mathcal{M}}_{g,k}(X,A)$
Topology and smoothness of $\mathcal{M}_{g,k}(X, A)$

- For a fixed $(\Sigma, j)$, $\ell \in \mathbb{Z}_+$, and $p > 1$ with $\ell p > 2$,

  \[
  \mathcal{E} \xrightarrow{\partial} \mathcal{B} = W^{\ell,p}(\Sigma, X) = \{(u, \Sigma, j) : u \in L^{\ell,p}\}
  \]

  \[
  \mathcal{E}_u = W^{\ell-1,p}(\Sigma, \Omega^{0,1}_{\Sigma,j} \otimes u^*TX)
  \]

- By elliptic regularity: $\bar{\partial}^{-1}(0) = \{(u, \Sigma, j) : \bar{\partial} u = 0\}$

- $D_u \bar{\partial} : W^{\ell,p}(\Sigma, u^*TX) \longrightarrow W^{\ell-1,p}(\Sigma, \Omega^{0,1}_{\Sigma,j} \otimes u^*TX)$ is a linear CR operator

- $(u^*TX, u^*J)$ is a holomorphic vector bundle and

  \[
  D_u \bar{\partial} = \bar{\partial}_{std} + \text{first order operator}
  \]
Smoothness of $\mathcal{M}_{g,k}(X, A)$ (continued)

- In particular $D_u \bar{\partial}$ has finite dimensional kernel and cokernel (it is Fredholm) and by Riemann-Roch

$$\dim_{\mathbb{R}} \ker(D_u \bar{\partial}) - \dim_{\mathbb{R}} \text{coker}(D_u \bar{\partial}) = 2 \left( \langle c_1(TX), A \rangle + n(1 - g) \right)$$

- If $D_u \bar{\partial}$ is surjective $\Leftrightarrow$ $\bar{\partial}$-section is transverse at $u$

Then, the Implicit Function Theorem implies that the space of $J$-holomorphic maps on $(\Sigma, j)$ around $u$ is a manifold with tangent space $\ker(D_u \bar{\partial})$ at $u$

- Expected real dimension of $\mathcal{M}_{g,k}(X, A)$ is

$$2 \left( \langle c_1(TX), A \rangle + (n - 3)(1 - g) + k \right)$$

- The idea around the transversality issue is to consider global (Ruan-Tian) or local (Li-Tian, Fukaya-Ono, etc) deformations $\bar{\partial}u = \nu$ of CR equation
How do we compactify $\mathcal{M}_{g,k}(X, A)$?

- Symplectic area (energy) of $J$-holomorphic maps in $\mathcal{M}_{g,k}(X, A)$ is fixed and coincides with the $L^2$-norm
  \[
  \langle \omega, A \rangle = \int_{\Sigma} u^* \omega = \int_{\Sigma} \| du \|^2
  \]

- $J$-holomorphic maps are minimal surfaces

- $L^p$-bound with $p > 2$ would have implied compactness but energy bound is not enough

- For a sequence of $J$-holomorphic maps over a fixed domain $(\Sigma, j)$, energy may bubble off at finitely many points
Compactification (continued)

- (Gromov 85) The limiting curves can be realized as the image of $J$-holomorphic maps from nodal domains

$$\overline{M}_{g,k}(X, A) \equiv \{\text{stable} \ J\text{-holomorphic maps of total homology class } A \text{ from genus } g \ k\text{-marked nodal domains} \}/\sim$$

- **Theorem.** If $(X, \omega)$ is a closed symplectic manifold and $J$ is compatible with $\omega$, then $\overline{M}_{g,k}(X, A)$ has a sequential convergence topology which is compact and metrizable.
Part 2

Moduli space of $J$-holomorphic maps for pairs $(X, D)$ of a symplectic manifold and a “divisor”

**End Goal:** Construct a compactification of the correct expected dimension
What is a divisor?

- Divisor in holomorphic manifolds: a holomorphic hypersurface, i.e. of \( \mathbb{C} \)-codimension 1 (possibly singular)

- Curves and Divisors are **dual**:

  (1) \( C \) holomorphic curve in \( X \), (2) \( D \) divisor in \( X \), (3) \( C \not\subset D \)

  \[ \Rightarrow D \cap C = \text{a finite set of points with positive multiplicities} \]

- Smooth symplectic divisors:
  - \( \mathbb{R} \)-codimension 2 symplectic submanifolds

- \((X, \omega)\) symplectic manifold, \( D \) smooth symplectic divisor, then
  - the space of \( J \) compatible with both \( \omega \) and \( D \) is still non-empty and contractible

- How do we define singular divisors (varieties) in symplectic topology?
**Simple Normal Crossings (SNC) divisors**

- **SNC divisor** in a holomorphic manifold: a **transverse** union
  
  \[ D = \bigcup_{i=1}^{m} D_i \]
  
  of smooth divisors

  \[ X = \mathbb{C}^3, \quad D_i = (x_i = 0) \cong \mathbb{C}^2, \text{ etc.} \]

- **Definition (2014, –, McLean, Zinger).** An SNC symplectic divisor is a transverse union of smooth ones which are “positively intersecting” along each stratum

  \[ D_I = \bigcap_{i \in I} D_i \quad \forall I \subset \{1, \ldots, m\} \]
SNC divisors and $J$-holomorphic Curves

- **Theorem (2014, –, McLean, Zinger).** For an SNC symplectic divisor $D \subset (X, \omega)$, there is a “good” space of compatible $J$

- Given $A \in H_2(X, \mathbb{Z}), \ k \in \mathbb{N}, \ D = \bigcup_{i=1}^{m} D_i$ as above with $A \cdot D_i \geq 0$ for all $i = 1, \ldots, m$, fix $k$ vectors

\[ s_1, \ldots, s_k \in \mathbb{N}^m, \ s_i = (s_{ij})_{j=1}^{m} \text{ s.t.} \]

\[ A \cdot D_j = \sum_{i=1}^{k} s_{ij} \quad \forall \ j = 1, \ldots, m \]

- With $s = (s_1, \ldots, s_k)$, define

\[ \mathcal{M}_{g,k}(X, A) \supset \mathcal{M}_{g,s}(X, D, A) \equiv \]

\[ \left\{ (u, \Sigma, j, z_1, \ldots, z_k) : \text{Im}(u) \nsubseteq D \text{ and } \text{ord}_{z_i}(u, D_j) = s_{ij} \right\} \]

- $s_{ij} = 0 \Rightarrow u(z_i) \nsubseteq D_j$
Example

\[ \mathcal{M}_{0,s=((3,2)(0,1)(0,0))}(\mathbb{C}\mathbb{P}^2, D, [3]) \subset \mathcal{M}_{0,3}(\mathbb{C}\mathbb{P}^2, [3]) \]

\[ D_1 = (x_1 = 0) \]
\[ D_{12} = \text{pt} = [0, 0, 1] \]
\[ s_1 = (3, 2) \]
\[ u(z_1) \]
\[ s_3 = (0, 0) \]
\[ u(z_3) \bullet \]
\[ z_2 = \infty \]
\[ z_1 = 0 \]
\[ Z = \mathbb{C}\mathbb{P}^2[x_1, x_2, x_3] \]
\[ u(z) = [z^3, z^2, az^3 + bz^2 + cz + d] \]
\[ S^2 = \mathbb{C} \cup \infty \]
\[ D_2 = (x_2 = 0) \]
\[ s_2 = (0, 1) \]
\[ u(z_2) \bullet \]
Big Question:

How to construct a compactification $\overline{M}_{g,s}(X, D, A)$ of the correct expected dimension?
Why are the moduli spaces $\mathcal{M}_{g,s}(X, D, A)$ interesting?

- **Geometry of singularities.** Hironaka’s Theorem (1964): Singular varieties can be blown up to a smooth variety with a snc exceptional divisor.

- **Exact complements:** If $\text{PD}(\omega)$ is a multiple of $D$, the complement would be an “exact” symplectic manifold.

- **Atiyah-Floer conjecture** is about a relation between the instanton Floer homology of suitable 3-dimensional manifolds with the symplectic Floer homology of moduli spaces of flat connections over surfaces. Proposed proof of Fukaya-Daemi (2017) uses Floer homology relative to snc divisors.

- **Mirror symmetry**

- **Smooth divisors could be complicated**
Previous works (smooth $D$, early 2000)

- Jun Li (algebraic), Ionel-Parker and Li-Ruan (symplectic)
  
  - **idea:** In order to construct a (so called *relative*) compactification, they also degenerate the target.
  
  - **issue 1:** Changing the target makes the analysis hard (still incomplete after 15 years).
  
  - **issue 2:** It does not generalize to snc case.
Previous works (SNC $D$ and more, mid 2000-current)

- Gross-Siebert, Abramovich-Chen, ... (algebraic case)
  
  - **idea:** They consider pairs of holomorphic maps and maps between certain *sheaves of monoids* on domains and a fixed sheaf of monoids on the target

  - **issue 1:** complicated for computations

  - **issue 2:** specific to the algebraic category

- Brett Parker (analytical, certain *almost Kähler* cases)

  - **idea:** Similarly, pairs of holomorphic maps and maps of certain analytical sheaves

  - **issue 1:** very very complicated

  - **issue 2:** it essentially works in the Kähler category
The main difficulty for constructing a compactification

- A sequence of $J$-holomorphic maps can partially sink into the divisor in the limit
- The intersection data $s$ gets lost in the limit
- Observation:

In the limit we get $C_2 = \text{Im}(u_2)$ and $C_1 = \text{Im}(u_1)$ + a meromorphic section $\zeta \in \Gamma_{\text{mero}}(u_1^*N_D X)$

- $\text{ord}_{z_1} (\zeta) = s_1$, $\text{ord}_{z_2} (\zeta) = s_2$, $\text{ord}_p (\zeta) = -\text{ord}_p (u_2, D)$
A new definition: log tuple

- Only \( \mathbb{C}^*-\)equivalence class \([\zeta]\) of \(\zeta\) is well-defined

**Definition.** Given \((X, D = \bigcup_{i=1}^{m} D_i, J)\), a log tuple \(f \equiv (u, \Sigma, j, [\zeta_i]_{i \in I})\) supported at \(p_1, \ldots, p_\ell \in \Sigma\) consists of

- a smooth Riemann Surface \((\Sigma, j)\)
- a \(J\)-holomorphic map \(u: \Sigma \to D_I = \cap_{i \in I} D_i\) (with \(D_\emptyset \equiv X\))
- \(\mathbb{C}^*\)-class of meromorphic sections \(\zeta_i \in \Gamma_{\text{mero}}(u^*\mathcal{N}_{D_i} X)\)

such that \(\text{ord}_x(f) = 0 \in \mathbb{Z}^m\) for \(x \neq p_1, \ldots, p_\ell\)
A new definition: log $J$-holomorphic maps

Definition (–, 2017). Given $(X, D = \bigcup_{i=1}^{m} D_i, J)$, a $k$-marked genus $g$ degree $A$ log map $f$ of contact type $s_1, \ldots, s_k \in \mathbb{N}^m$ consists of

- a $k$-marked genus $g$ nodal Riemann Surface $\Sigma$
- a log tuple $f_v = (u_v, \Sigma_v, j_v, [\zeta_{v,i}]_{i \in I_v})$ for each component $\Sigma_v$ of $\Sigma$ supported at the nodes and marked points on $\Sigma_v$

such that the following conditions hold:
1. the underlying map $u = (u_v)_{v \in V}$ represents the homology class $A$

2. contact order at the $i$-th marked point $z_i$ is $s_i$

3. contact orders at the nodal points are negative of each other

4. there exist vectors $\{s_v \in \mathbb{Z}^m\}_{v \in V}$ such that

$$s_{v_2} - s_{v_1} = \lambda_e s_{\bar{e}} \quad \text{for some } \lambda_e > 0$$

for any oriented edge $\bar{e}$ from $v_1$ to $v_2$
5. **AND:** there exist a group $G_G$ associated to $G$, and a group element $g_f \in G_G$ associated to $f$; we want this group element to be $1 \in G_G$

- **Theorem (–, 2017).** For any $(X, \omega, D = \bigcup_{i=1}^{m} D_i)$, suitable choice of $J$, and $s_1, \ldots, s_k \in \mathbb{N}^m$,

  - the moduli space $\overline{M}_{g,s}(X, D, A)$ of all equivalence classes of $k$-marked genus $g$ degree $A$ log maps of contact type $s = (s_1, \ldots, s_k) \in \mathbb{N}^m$ is compact, metrizable, and of the expected dimension

  - the natural forgetful map $\overline{M}_{g,s}(X, D, A) \longrightarrow \overline{M}_{g,k}(X, A)$ is an embedding if $g = 0$, and it is a locally-embedding if $g > 0$
What is left to be done?

- Extending to a bigger class of $J$
- Deformation theory
- Constructing Virtual Fundamental Cycle (addressing the transversality problem)
- Comparing to the log moduli spaces constructed in the algebraic case
- Calculating the resulting Gromov-Witten type invariants
- ...

Thank you for your attention