On Moduli Spaces of J-holomorphic curves in Symplectic Manifolds

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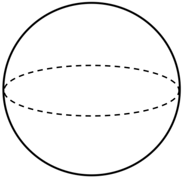
Outline

- **Part 1**
  - What are **symplectic manifolds**, what are J-**holomorphic curves**, and why do we care about them?
  - What is known about the (moduli) space of J-holomorphic maps
  - In particular, how do we construct a **compact** moduli space of J-holomorphic maps?
- **Part 2**
  - I will introduce certain moduli spaces of J-holomorphic curves with **extra tangency** conditions
  - I will discuss the motivation for studying such moduli spaces and the known results
  - I will finish by stating a **NEW compactification result**

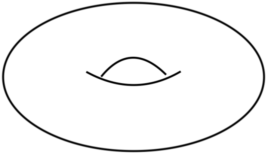
Part 1

What is a Symplectic manifold?

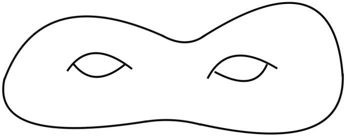
- $2n$-dim manifold X with
  a closed ($\mathrm{d}\omega = 0$) and non-degenerate ($\omega^n \neq 0$) 2-form ω
- Example: $X = \mathbb{R}^{2n} = \mathbb{C}^n$ with $\omega_{\mathsf{std}} = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \cdots + \mathrm{d}x_n \wedge \mathrm{d}y_n$
- Every (X, ω) is locally isomorphic to $(\mathbb{R}^{2n}, \omega_{\mathsf{std}})$
- In real dimension 2 ($\mathbb{C}$-dim 1), all Riemann surfaces are symplectic manifolds



Sphere: g=0



Torus: g=1



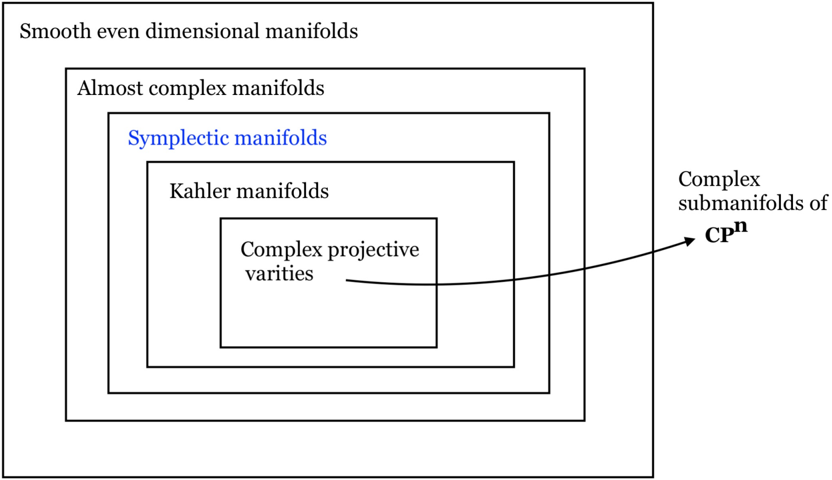
g=2

Almost-complex structures compatible with ω

- Almost complex structure: $J\colon TX \longrightarrow TX$ s.t. $J^2 = -\mathrm{id}$
- Compatible with ω: $\omega(\cdot, J\cdot)$ is a metric
- **Crucial observation**: Space of compatible J is non-empty (infinite dimensional) and contractible
  In particular it is connected
- A triple (X, ω, J) is called an **almost Kähler** manifold
- **Kähler manifold**: If J comes from a complex structure on X
- (Newlander-Nirenberg 1957) **Nijenhueis** $(2,1)$-tensor measures how far J is from defining a complex structure

  $$N_J(u,v) \equiv [u,v] + J[u,Jv] + J[Ju,v] - [Ju,Jv] \in T_xX \qquad \forall u,v \in T_xX$$

- In dimension 2 all J's are holomorphic ($N_J \equiv 0$)


Smooth even dimensional manifolds
Almost complex manifolds
Symplectic manifolds
Kahler manifolds
Complex projective varities
Complex submanifolds of $\mathbf{CP^n}$

What are J-holomorphic maps?

- (X, ω, J) as before
- $(\Sigma, \mathfrak{j})$: a Riemann Surface with complex structure $\mathfrak{j}$
- (Gromov 85) J-**holomorphic map**

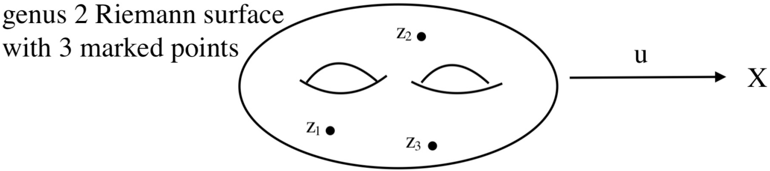
$$u\colon (\Sigma, \mathfrak{j}) \longrightarrow (X, J) \qquad \text{s.t.} \qquad \bar{\partial} u = \mathrm{d}u + J \mathrm{d}u \mathfrak{j} = 0$$

  i.e. $\mathrm{d}_x u\colon T_x\Sigma \longrightarrow T_{u(x)}X$ is $\mathbb{C}$-linear $\forall\ x \in \Sigma$

- This is a non-linear Cauchy-Riemann (CR) equation
- $\mathrm{Im}(u) \subset X$ is called a J-**holomorphic curve**
  It could be singular at some points
- One can also define J-holomorphic maps from Riemann surfaces **with boundary** subject to some boundary conditions (Floer theories)

Moduli spaces of J-holomorphic maps

- If $h\colon \Sigma \longrightarrow \Sigma$ is a holomorphic reparametrization $\Rightarrow$
  - $u' = u \circ h$ is also J-holomorphic
  - u and u' are **equivalent** (define the same curve $\mathrm{Im}(u) = \mathrm{Im}(u')$)
- The homology class $A \in H_2(X, \mathbb{Z})$ represented by u and the Genus g of Σ characterize the topological type of u
- (Set) $\mathcal{M}_g(X, A) = \left\{ \big(u, (\Sigma, \mathrm{j})\big) : \bar{\partial} u = 0 \right\} / \sim$
- (Set) $\mathcal{M}_{g,k}(X, A) = \left\{ \big(u, (\Sigma, \mathrm{j}, z_1, \ldots, z_k)\big) : \bar{\partial} u = 0 \right\} / \sim$



Why are these moduli spaces useful?

- Powerful tool to study global geometry/topology of symplectic manifolds
- To find periodic orbits of Hamiltonian ODE's: Floer homology
- Defining enumerative invariants: Gromov-Witten theory
- String theory, Mirror Symmetry
- Topology of 3-manifolds: Heegaard-Floer homology
- Connections to Seiberg-Witten theory and other Gauge theories

Example of Gromov-Witten theory

- Gromov-Witten theory **formalizes** and **generalizes** the enumerative geometry in algebraic geometry which is about **finding/counting** holomorphic curves of specific type
- **Example 1**: There is **a unique** holomorphic sphere of homology class $[1] \in H_2(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z}$ (called a **line**) passing through **2** points in $\mathbb{CP}^n$
- **Example 2**: There are **2875 lines** in a (generic) degree **5** Calabi-Yau hypersurface in $\mathbb{CP}^4$
- **Evaluation maps**: $\mathsf{ev} = \prod_{i=1}^{k} \mathsf{ev}_i : \mathcal{M}_{g,k}(X, A) \longrightarrow X^k$

$$(u, \Sigma, \mathbf{j}, z_1, \ldots, z_k) \xrightarrow{\mathsf{ev}_i} u(z_i) \in X$$

Example of Gromov-Witten theory (continued)

- We want $\mathsf{GW}^{X,A}_{g,k} = \mathsf{ev}_*(\mathcal{M}_{g,k}(X,A)) \subset X^k$ to be a "nice" cycle
- Then, for cycles $\beta_1, \ldots, \beta_k$ representing homology classes $B_1, \ldots, B_k \in H_*(X)$, we define

$$\mathsf{GW}^X_{g,A}(B_1, \ldots, B_k) = \#\left(\mathsf{GW}^{X,A}_{g,k} \cap (\beta_1 \times \cdots \times \beta_k)\right) \in \mathbb{Z}$$

- For this construction to work we need
  1. a **Topology/Smooth structure** on $\mathcal{M}_{g,k}(X,A)$
  2. a "nice" **Compactification** $\overline{\mathcal{M}}_{g,k}(X,A)$ of the correct "expected dimension"
  3. some control of **Smoothness** of $\overline{\mathcal{M}}_{g,k}(X,A)$

Topology and smoothness of $\mathcal{M}_{g,k}(X,A)$

- For a fixed $(\Sigma, \mathfrak{j})$, $\ell \in \mathbb{Z}_+$, and $p > 1$ with $\ell p > 2$,

$$
\begin{array}{ll}
\mathcal{E} & \mathcal{E}_u = W^{\ell-1,p}(\Sigma, \Omega^{0,1}_{\Sigma,\mathfrak{j}} \otimes u^*TX) \\
\bar\partial \uparrow \downarrow \pi & \\
\mathcal{B} & = W^{\ell,p}(\Sigma, X) = \{(u, \Sigma, \mathfrak{j}) \colon u \in L^{\ell,p}\}
\end{array}
\qquad
\begin{array}{c}
\bar\partial u \\
\bar\partial \uparrow \\
u
\end{array}
$$

- By **elliptic regularity**: $\bar\partial^{-1}(0) = \left\{(u, \Sigma, \mathfrak{j}) \colon \bar\partial u = 0\right\}$

- $D_u\bar\partial \colon W^{\ell,p}(\Sigma, u^*TX) \longrightarrow W^{\ell-1,p}(\Sigma, \Omega^{0,1}_{\Sigma,\mathfrak{j}} \otimes u^*TX)$
  is a linear CR operator

- (u^*TX, u^*J) is a holomorphic vector bundle and

$$D_u\bar\partial = \bar\partial_{\mathsf{std}} + \text{first order operator}$$

Smoothness of $\mathcal{M}_{g,k}(X, A)$ (continued)

- In particular $D_u\bar\partial$ has finite dimensional **kernel** and **cokernel** (it is **Fredholm**) and by **Riemann-Roch**

$$\dim_{\mathbb{R}}\ker(D_u\bar\partial) - \dim_{\mathbb{R}}\mathrm{coker}(D_u\bar\partial) = 2\Big(\langle c_1(TX), A\rangle + n(1-g)\Big)$$

- If $D_u\bar\partial$ is surjective $\Leftrightarrow$ $\bar\partial$-section is **transverse** at u
Then, the **Implicit Function Theorem** implies that the space of J-holomorphic maps on $(\Sigma, \mathfrak{j})$ around u is a manifold with tangent space $\ker(D_u\bar\partial)$ at u

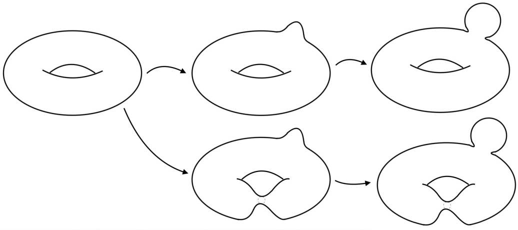
- Expected real dimension of $\mathcal{M}_{g,k}(X, A)$ is

$$2\big(\langle c_1(TX), A\rangle + (n-3)(1-g) + k\big)$$

- The idea around the transversality issue is to consider global (Ruan-Tian) or local (Li-Tian, Fukaya-Ono, etc) deformations $\bar\partial u = \nu$ of CR equation

How do we compactify $\mathcal{M}_{g,k}(X, A)$?

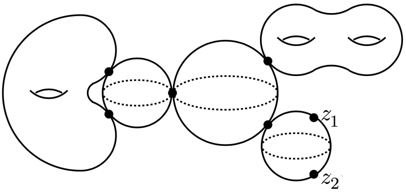
- Symplectic area (energy) of J-holomorphic maps in $\mathcal{M}_{g,k}(X, A)$ is fixed and coincides with the L^2-norm
$$\langle \omega, A \rangle = \int_\Sigma u^*\omega = \int_\Sigma ||du||^2$$
- J-holomorphic maps are minimal surfaces
- L^p-bound with $p > 2$ would have implied compactness but energy bound is not enough
- For a sequence of J-holomorphic maps over a fixed domain (Σ, j), energy may bubble off at finitely many points



Compactification (continued)

- (Gromov 85) The limiting curves can be realized as the image of J-holomorphic maps from nodal domains

A genus 4 nodal Riemann surface with 2 marked points



$$\overline{\mathcal{M}}_{g,k}(X,A) \equiv \{\textbf{stable } J\text{-holomorphic maps of total homology class } A \text{ from genus g k-marked nodal domains } \}/\sim$$

- **Theorem.** If (X, ω) is a closed symplectic manifold and J is compatible with ω, then $\overline{\mathcal{M}}_{g,k}(X,A)$ has a sequential convergence topology which is compact and metrizable

Part 2

Moduli space of J-holomorphic maps for pairs (X, D) of a symplectic manifold and a "divisor"

End Goal: Construct a compactification of the correct expected dimension

What is a divisor?

- Divisor in holomorphic manifolds: a holomorphic hypersurface, i.e. of $\mathbb{C}$-**codimension** 1 (possibly singular)

- Curves and Divisors are **dual**:

  (1) C holomorphic curve in X, (2) D divisor in X, (3) $C \not\subset D$
  $\Rightarrow D \cap C =$ a finite set of points with positive multiplicities

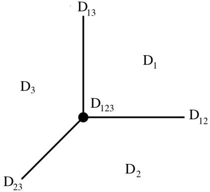
- Smooth symplectic divisors:
  $\mathbb{R}$-codimension 2 symplectic submanifolds

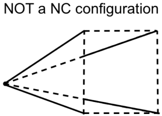
- (X, ω) symplectic manifold, D smooth symplectic divisor, then the space of J compatible with both ω and D is still non-empty and contractible

- How do we define singular divisors (varieties) in symplectic topology?

Simple Normal Crossings (SNC) divisors

- **SNC divisor** in a holomorphic manifold: a **transverse** union $D = \bigcup_{i=1}^{m} D_i$ of smooth divisors





$X = \mathbb{C}^3$, $D_i = (x_i = 0) \cong \mathbb{C}^2$, etc.

- **Definition (2014, –, McLean, Zinger).** An SNC symplectic divisor is a transverse union of smooth ones which are "positively intersecting" along each stratum

$$D_I = \bigcap_{i \in I} D_i \qquad \forall I \subset \{1, \dots, m\}$$

SNC divisors and J-holomorphic Curves

- **Theorem (2014, –, McLean, Zinger).** For an SNC symplectic divisor $D \subset (X, \omega)$, there is a "good" space of compatible J

- Given $A \in H_2(X, \mathbb{Z})$, $k \in \mathbb{N}$, $D = \bigcup_{i=1}^m D_i$ as above with $A \cdot D_i \geq 0$ for all $i = 1, \ldots, m$, fix k vectors

$$s_1, \ldots, s_k \in \mathbb{N}^m, \quad s_i = (s_{ij})_{j=1}^m \quad \text{s.t.}$$

$$A \cdot D_j = \sum_{i=1}^k s_{ij} \quad \forall\, j = 1, \ldots, m$$

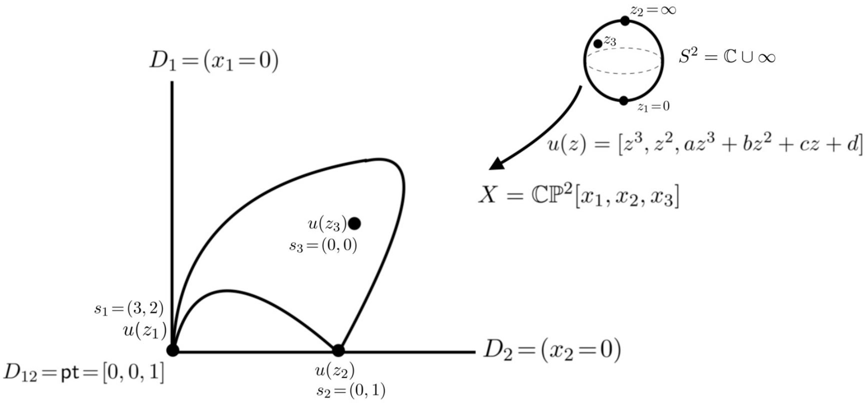
- With $\mathbf{s} = (s_1, \ldots, s_k)$, define

$$\mathcal{M}_{g,k}(X, A) \supset \mathcal{M}_{g,\mathbf{s}}(X, D, A) \equiv \big\{(u, \Sigma, \mathfrak{j}, z_1, \ldots, z_k) \colon \mathrm{Im}(u) \not\subset D \quad \text{and} \quad \mathrm{ord}_{z_i}(u, D_j) = s_{ij}\big\}$$

- $s_{ij} = 0 \Rightarrow u(z_i) \not\subset D_j$

Example

$$\mathcal{M}_{0,\mathsf{s}=((3,2)(0,1)(0,0))}(\mathbb{CP}^2, D, [3]) \subset \mathcal{M}_{0,3}(\mathbb{CP}^2, [3])$$



Big Question:

How to construct a compactification $\overline{\mathcal{M}}_{g,\mathbf{s}}(X, D, A)$ of the correct expected dimension?

Why are the moduli spaces $\mathcal{M}_{g,\mathbf{s}}(X, D, A)$ interesting?

- Geometry of **singularities**. Hironaka's Theorem (1964): Singular varieties can be blown up to a smooth variety with a snc exceptional divior
- **Exact complements**: If PD(ω) is a multiple of D, the complement would be an "exact" symplectic manifold
- **Atiyah-Floer conjecture** is about a relation between the instanton Floer homology of suitable 3-dimensional manifolds with the symplectic Floer homology of moduli spaces of flat connections over surfaces. Proposed proof of Fukaya-Daemi (2017) uses Floer homology relative to snc divisors.
- Mirror symmetry
- Smooth divisors could be complicated

Previous works (smooth D, early 2000)

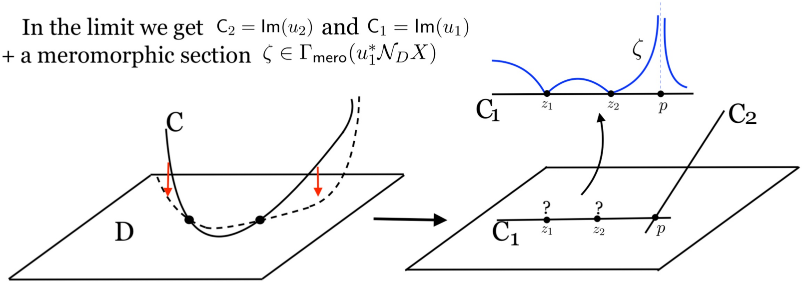
- Jun Li (algebraic), Ionel-Parker and Li-Ruan (symplectic)
  - **idea:** In order to construct a (so called **relative**) compactification, they also degenerate the target
  - **issue 1:** Changing the target makes the analysis hard (still incomplete after 15 years)
  - **issue 2:** It does not generalize to snc case

Previous works (SNC D and more, mid 2000-current)

- Gross-Siebert, Abramovich-Chen, ... (algebraic case)
  - **idea:** They consider pairs of holomorphic maps and maps between certain **sheaves of monoids** on domains and a fixed sheaf of monoids on the target
  - **issue 1:** complicated for computations
  - **issue 2:** specific to the algebraic category
- Brett Parker (analytical, certain **almost Kähler** cases)
  - **idea:** Similarly, pairs of holomorphic maps and maps of certain analytical sheaves
  - **issue 1:** very very complicated
  - **issue 2:** it essentially works in the Kähler category

The main difficulty for constructing a compactification

- A sequence of J-holomorphic maps can partially sink into the divisor in the limit
- The intersection data **s** gets lost in the limit
- Observation:

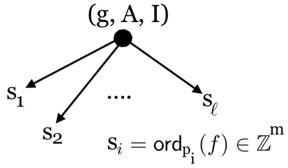


- $\mathrm{ord}_{z_1}(\zeta) = s_1$, $\mathrm{ord}_{z_2}(\zeta) = s_2$, $\mathrm{ord}_p(\zeta) = -\mathrm{ord}_p(u_2, D)$

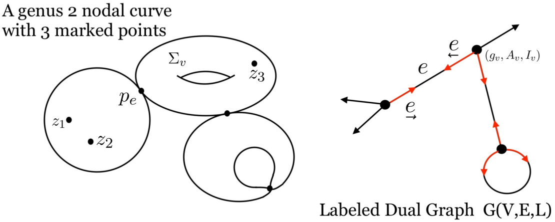
A new definition: log tuple

- Only $\mathbb{C}^*$-equivalence class $[\zeta]$ of ζ is well-defined
- **Definition.** Given $\big(X, D = \bigcup_{i=1}^m D_i, J\big)$, a **log tuple** $f \equiv (u, \Sigma, \mathfrak{j}, [\zeta_i]_{i\in I})$ supported at $p_1, \ldots, p_\ell \in \Sigma$ consists of
  - a smooth Riemann Surface $(\Sigma, \mathfrak{j})$
  - a J-holomorphic map $u \colon \Sigma \longrightarrow D_I = \cap_{i\in I} D_i$ (with $D_\emptyset \equiv X$)
  - $\mathbb{C}^*$-class of meromorphic sections $\zeta_i \in \Gamma_{\text{mero}}(u^*\mathcal{N}_{D_i}X)$

  such that $\text{ord}_x(f) = 0 \in \mathbb{Z}^m$ for $x \neq p_1, \ldots, p_\ell$



A new definition: log J-holomorphic maps



- **Definition (–, 2017).** Given $\left(X, D = \bigcup_{i=1}^{m} D_i, J\right)$, a k-marked genus g degree A **log map** f of contact type $s_1, \ldots, s_k \in \mathbb{N}^m$ consists of
  - a k-marked genus g nodal Riemann Surface Σ
  - a log tuple $f_v = (u_v, \Sigma_v, \mathfrak{j}_v, [\zeta_{v,i}]_{i \in I_v})$ for each component Σ_v of Σ supported at the nodes and marked points on Σ_v

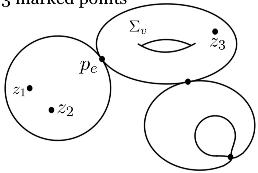
such that the following conditions hold:

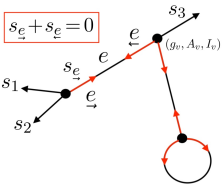
1. the underlying map $u = (u_v)_{v \in \mathsf{V}}$ represents the homology class A
2. contact order at the i-th marked point z_i is s_i
3. contact orders at the nodal points are negative of each other
4. there exist vectors $\{s_v \in \mathbb{Z}^m\}_{v \in \mathsf{V}}$ such that

$$s_{v_2} - s_{v_1} = \lambda_e s_{\underset{\rightarrow}{e}} \quad \text{for some} \quad \lambda_e > 0$$

   for any oriented edge $\underset{\rightarrow}{e}$ from v_1 to v_2

A genus 2 nodal curve
with 3 marked points





Labeled Dual Graph G(V,E,L)

5. **AND:** there exist a group $\mathcal{G}_G$ associated to G, and a group element $\mathfrak{g}_f \in \mathcal{G}_G$ associated to f; we want this group element to be $1 \in \mathcal{G}_G$

- **Theorem (–, 2017).** For any $\left(X, \omega, D = \bigcup_{i=1}^{m} D_i\right)$, suitable choice of J, and $s_1, \ldots, s_k \in \mathbb{N}^m$,
  - the moduli space $\overline{\mathcal{M}}_{g,\mathbf{s}}(X, D, A)$ of all *equivalence classes* of k-marked genus g degree A **log maps** of contact type $\mathbf{s} = (s_1, \ldots, s_k) \in \mathbb{N}^m$ is compact, metrizable, and of the expected dimension
  - the natural forgetful map $\overline{\mathcal{M}}_{g,\mathbf{s}}(X, D, A) \longrightarrow \overline{\mathcal{M}}_{g,k}(X, A)$ is an embedding if $g = 0$, and it is a locally-embedding if $g > 0$

What is left to be done?

- Extending to a bigger class of J
- Deformation theory
- Constructing Virtual Fundamental Cycle (addressing the transversality problem)
- Comparing to the log moduli spaces constructed in the algebraic case
- Calculating the resulting Gromov-Witten type invariants
- ...

Thank you for your attention