Topics in Differential Topology

Mohammad F. Tehrani

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Preface

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2.1 Manifolds

Roughly speaking, a manifold is a (topological) space that locally resembles Euclidean space, and globally, it is obtained by attaching countably many such local pieces (known as charts). Globally most manifolds are not homeomorphic to Euclidean space or an open subset of that. For example, the sphere is not homeomorphic to the plane. In the following sections, we will learn about some tools for distinguishing different manifolds.

Manifolds have applications to different fields and thus they are often equipped with additional structures such as metric, holomorphic structure, symplectic structure, and more. In the following sections, we mainly focus on manifolds admitting just a differentiable or holomorphic structure. While there are topological manifolds that do not admit any smooth structure, category of smooth manifolds contains all well-known examples. A differentiable structure allows calculus to be done on manifolds.

We start this section by recalling a few important definitions and theorems from general topology; see [7].

Definition 2.1. Let M be a topological space; we say M is

(1) *Hausdorff*, if every two distinct points in M can be separated by disjoint open sets;

(2) *regular*, if (one-point-sets are closed[1] and) every point p and a closed subset C not containing p can be separated by disjoint open sets;

(3) *normal*, if (one-point-sets are closed and) every two disjoint closed subsets in M can be separated by disjoint open sets;

(4) *paracompact*: if every open covering $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ of M admits a locally-finite[2] open covering refinement;

(5) *metrizable*: if the topology of M comes from a metric;

(6) *second-countable*, if M has a countable basis. A basis is a collection of open sets $\mathcal{B} = \{U_\alpha\}_{\alpha\in\mathcal{I}}$ covering M such that every open set in M is a union of open sets in $\mathcal{B}$.

The first theorem below shows that, essentially, being second-countable is stronger than metrizability. It is easy to see that a metrizable space is normal, and thus Hausdorff and regular. The second theorem below shows that it is also paracompact.

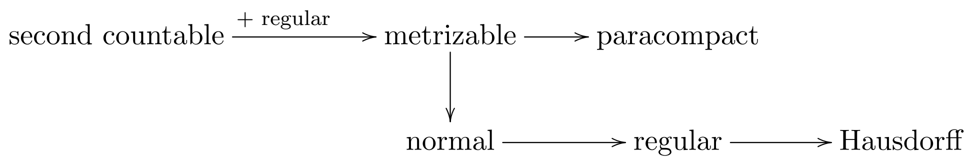
Theorem 2.2 (Urysohn Metrization Theorem ([7], Thm 34.1))**.** *Every regular and second-countable topological space is metrizable.*

Theorem 2.3 (Smirnov Metrization Theorem ([7], Thm 42.1))**.** *A topological space M is metrizable if and only if it is Hausdorff, paracompact, and locally-metrizable.*

[1] Or one may assume M is Hausdorff

[2] A collection $\{U_\alpha\}$ of subsets of M is called locally finite if for every $x\in M$ there exists a neigborhood $U \ni x$ such that U intersects only finitely many U_α.

The following diagram roughly summarizes the hierarchy between these notions.



We are now ready to state the definition of a C^0-manifold (topological manifold).

Definition 2.4. A topological or C^0-manifold M is a Hausdorff and second-countable topological space such that for each $p \in M$ there exists an open neighborhood $U \ni p$ and a homeomorphism $\varphi\colon U \longrightarrow V$ to an open subset V of $\mathbb{R}^m$.

A local homeomorphism $\varphi\colon U \longrightarrow V$ as in Definition 2.4 is called a **chart** for M around p. If

$$\varphi = (x^1, \ldots, x^m)\colon U \longrightarrow \mathbb{R}^m,$$

the functions $\{x^i\}_{i=1}^m$ are the **local coordinates** around p corresponding to φ.

It is natural to ask whether the integer m in Definition 2.4 can vary from chart to chart? Fortunately, the following theorem of Brouwer implies that the integer m is a topological invariant of M; we will call m the **dimension** of M, or equally, we say M is an m-manifold.

Theorem 2.5 (Brouwer's Invariance Domain Theorem [1])**.** *If V is an open subset of $\mathbb{R}^n$ and $f\colon V \longrightarrow \mathbb{R}^n$ is an injective continuous map, then $f(V)$ is open in $\mathbb{R}^n$ and f is a homeomorphism between V and $f(V)$.*

HW 2.6. Use this theorem to prove that the integer m does not depend on the particular choice of a chart on M.

The three conditions of Definition 2.4 are independent of the each other. In other words, there are examples that satisfy exactly any two of these conditions. The following example, known as the "double origin line" is a topological space that is second-countable and admits local charts, but it is not Hausdorff. Let

$$M := \mathbb{R} \times \{1,2\}/\{(x,1) \sim (x,2) \qquad \forall x \in \mathbb{R} - \{0\}\Big\}$$

with the quotient topology. In other words, M is the topological space obtained by identifying two copies of $\mathbb{R}$ along $\mathbb{R} - \{0\}$. It has two "zero points", denoted by 0_1 and 0_2, which are the images of $(0,1)$ and $(0,2)$ in the quotient space, respectively. Non-Hausdorff spaces like the above example are not desirable. In particular, they are not metrizable. They also do not admit continuous functions that have different values at un-separable points such as 0_1 and 0_2. Thus, they are not suitable for doing Calculus!

Next, we explain the motivations for the second-countability condition. First, with little effort, it follows from the Urysohn Metrization Theorem that M is metrizable. Thus, it has all the nice properties listed in Definition 2.1. More importantly, many constructions on manifolds involve the following two steps. First, we construct a collection local functions/vector fields/etc. on individual local charts where we have Calculus on an open subset of $\mathbb{R}^m$ at our disposal. Then, we need to add up the resulting local data to construct a global structure on M. For the second step to be feasible, we need to have at most a locally-finite or a convergent countable summation.

The existence of a countable cover, or equally, the para-compactness of M facilitates the second step. In particular, every manifold admits [7, Thm 41.7] a partition of unity in the following sense.

Definition 2.7. Suppose M is a topological space. A **partition of unity** for M is a collection of continuous functions

$$\{\theta_\alpha\colon U_\alpha \longrightarrow [0,1]\}_{\alpha\in\mathcal{I}}$$

such that

- $\text{support}(\theta_\alpha) = \overline{\{x\in M\colon \theta_\alpha(x)\neq 0\}} \subset U_\alpha$;
- the collection of closed sets $\{\text{support}(\theta_\alpha)\}_{\alpha\in\mathcal{I}}$ is locally-finite[3];
- $\sum_{\alpha\in\mathcal{I}}\theta_\alpha \equiv 1$.

The last summation is well-defined at any $x\in M$ by the second condition. The last condition implies that $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ is an open covering of M.

By the existence of local charts, every manifold is locally path-connected. Therefore, the notions of connectedness and path-connectedness are the same for manifolds; see [7, Thm 25.5].

HW 2.8. Describe a Hausdorff topological space that admits local charts but it is not second-countable.

Definition 2.9. A collection

$$\mathcal{A} = \{\varphi_\alpha\colon U_\alpha \longrightarrow V_\alpha \subset \mathbb{R}^m\}_{\alpha\in\mathcal{I}}$$

of charts on M such that $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ is a covering of M is called an atlas.

It is often difficult to find the the minimum number of charts needed to cover a manifold. Before we progress further, we discuss a few examples to illustrate the idea of covering a topological space with charts.

Example 2.10. The easiest example of a manifold is any open subset $M\subset\mathbb{R}^m$. In this case, we can take inclusion map $\iota\colon M \longrightarrow \mathbb{R}^m$ as a single chart covering the entire M.

Example 2.11. Let

$$S^m := \{(x_1,\ldots,x_{m+1}) \in \mathbb{R}^{m+1}\colon x_1^2+\ldots+x_{m+1}^2 = 1\} \subset \mathbb{R}^{m+1}; \tag{2.1}$$

with the standard subspace topology (which is metrizable). Putting $m=1$, we a get a circle in $\mathbb{R}^2$ and putting $m=2$ we a get a sphere in $\mathbb{R}^3$. For $m>2$, (2.1) is the m-dimensional unit sphere in $\mathbb{R}^{m+1}$ centered at the origin. Let

$$U_\pm = S^m - \{(0,\ldots,0,\pm1)\}$$

and consider the stereographic projections

$$\varphi_\pm\colon U_\pm \longrightarrow V_\pm = \mathbb{R}^m, \qquad \varphi_\pm(x_1,\ldots,x_{m+1}) = \frac{(x_1,\ldots,x_m)}{1\mp x_{m+1}},$$

of S^m (minus a point) to $\mathbb{R}^m$ from the north and south poles, $(0,\ldots,0,1)$ and $(0,\ldots,0,-1)$, respectively.

[3]For every $x\in M$, there exists a neighborhood $U\ni x$ such that $U\cap\text{support}(\theta_\alpha)\neq\emptyset$ for only finitely many α.

HW 2.12. Show that $\varphi_\pm$ are surjective homeomorphisms. Therefore, the collection

$$\{\varphi_\pm : U_\pm \longrightarrow \mathbb{R}^m\}$$

is a 2-chart atlas of M.

HW 2.13. Show that S^m is not homeomorphic to $\mathbb{R}^m$. Therefore, we can not cover S^m with a single chart!

HW 2.14. Let

$$M := \{(x,y,z) \in \mathbb{R}^3 \colon x^3 + y^3 + z^3 = 1\} \subset \mathbb{R}^3; \tag{2.2}$$

with the subspace topology. Describe a set of charts covering M proving that M is a 2-manifold. Is M connected? Use a software to draw this surface.

HW 2.15. Let

$$M := \{(x,y) \in \mathbb{R}^2 \colon xy = 0\} \subset \mathbb{R}^2; \tag{2.3}$$

with the subspace topology. Prove that M is not a manifold.

The common feature of Example 2.11, HW 2.2, and HW 2.3 is that they are all the zero set of a single function in Euclidean space. Later, we find a criteria for when the zero set of a function is a (smooth) manifold.

HW 2.16. Show that the product of two manifolds is a manifold. For example, $T^n := (S^1)^n$ is called the n-torus. For $n = 2$, we get the donut-shaped 2-torus.

HW 2.17. Show that every manifold M can be written as a countable union $M = \bigcup_{i=1} U_n^n$ such that each $K_n = \overline{U_n}$ is compact and $K_n \subset U_{n+1}$ for all $n \geq 1$. Such a union is known as an *exhaustion* of M.

Next, we go over the definition of a differentiable manifold. We need an extra structure on a topological manifold M that allows us to differentiate functions and thus extend the notion of derivative from Calculus to manifolds. Suppose M is a topological m-manifold and $f\colon M \longrightarrow \mathbb{R}$ is a continuous function. Fix a chart $\varphi\colon U \longrightarrow V \subset \mathbb{R}^m$ on M. The composition

$$f \circ \varphi^{-1} \colon V \longrightarrow \mathbb{R}$$

gives us a function on an open subset of $\mathbb{R}^m$ for which the notion of derivative/partial derivative can be defined. In particular, we say f is smooth or C^∞ (with respect to φ) if $f \circ \varphi^{-1}$ is a smooth function; or if $m = 2k$ and we identify $\mathbb{R}^{2m}$ with $\mathbb{C}^k$, we say f is holomorphic if $f \circ \varphi^{-1}$ is a holomorphic function. This notion of smoothness, however, depends on the choice of a chart in the following way. Suppose $\widetilde{\varphi}\colon \widetilde{U} \longrightarrow \widetilde{V}$ is another chart such that $U \cap \widetilde{U} \neq \emptyset$. Restricted to $\varphi(U \cap \widetilde{U}) \subset \mathbb{R}^m$, we have

$$f \circ \varphi^{-1} = (f \circ \widetilde{\varphi}^{-1}) \circ (\widetilde{\varphi} \circ \varphi^{-1})$$

where the so called **transition map**

$$\widetilde{\varphi} \circ \varphi^{-1} \colon \varphi(U \cap \widetilde{U}) \longrightarrow \widetilde{\varphi}(U \cap \widetilde{U})$$

is a homeomorphism between open subsets of $V \subset \mathbb{R}^m$ and $\widetilde{V} \subset \mathbb{R}^m$. If the transition map $\widetilde{\varphi} \circ \varphi^{-1}$ is smooth, the smoothness of $f \circ \widetilde{\varphi}^{-1}$ implies the smoothness of $f \circ \varphi^{-1}$ and vice versa. In other words, if the transition maps are smooth, the smoothness of $f \circ \varphi^{-1}$ is independent of the particular choice of a chart. In this situation, we say f is smooth.

Definition 2.18. Suppose $\mathcal{A}$ is an atlas for a topological manifold M. We say $\mathcal{A}$ defines a C^k, smooth, analytic, or holomorphic structure on M if the transition maps of the charts in $\mathcal{A}$ are C^k, smooth, analytic, or holomorphic, respectively.

HW 2.19. Show that the atlas in Example 2.11 defines a smooth structure on S^m. If $m=2$, show that it defines a holomorphic structure on S^2.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are two smooth atlases on M. If the transition maps between every chart of $\mathcal{A}$ and every chart of $\mathcal{B}$ are smooth, the union $\mathcal{A}\cup\mathcal{B}$ is a larger atlas defining the same smooth structure on M. In this situation we write $\mathcal{A}\sim\mathcal{B}$. Otherwise, $\mathcal{A}$ and $\mathcal{B}$ define different smooth structures on the same topological manifold M. Similarly, we can ask the same question for C^k, analytic, and holomorphic structures on M. In any case, the relation $\sim$ defines an equivalence relation.

Definition 2.20. A smooth structure on a topological manifold M is the equivalence class $[\mathcal{A}]$ of an atlas $\mathcal{A}$ that defines a smooth structure on M.

Equivalently, the equivalence class $[\mathcal{A}]$ can be thought of as the *maximal* atlas $\mathcal{A}_{\max}$ containing $\mathcal{A}$; it includes all charts on M whose transition maps with charts of $\mathcal{A}$ are smooth. The three important questions are:

(a) Does every topological manifold admit a C^1 structure?

(b) Given a C^r structure $\mathcal{A}_{\max}$ on M and $r<k\leq\infty$, is there a sub-atlas $\mathcal{B}_{\max}\subset\mathcal{A}_{\max}$ that defines a C^k structure on M?

(c) Can a given topological manifold admit more than two (or infinitely many) smooth structures (up to conjugation by homeomorphisms of M)?

Regrading question (a), the first example of a topological manifold that does not admit a C^1 structure was discovered by .... 's example is an 8-dimensional manifold. Subsequently, various 4-dimensional examples were found ..... . In dimensions 1, 2, and 3, the answer to Question (a) is Yes. In dimension one, the only connected manifolds are $\mathbb{R}$ and S^1; see [**?**, Ch 2]. In dimension two, ... Hatcher Icon. In dimension three, first, a 1976 result of Hamilton shows that every C^0 manifold admits a "piece-wise linear structure". Then, on shows that every piece-wise linear 3-manifold admits a unique smooth structure; see [8, Section 3.10]. In dimensions five and higher, there is a classification of smooth, piece-wise linear, and topological structures by Kirby and Sieberman [**?**] in terms of various groups in algebraic topology. There are still many open questions in dimension four. A full classification of smooth 4-manifolds is unknown.

Regrading question (b), by .... the answer to question (2) is positive; every maximal C^r atlas on M include a maximal smooth sub-atlas. Furthermore, by .... every C^∞ structure contains an (real) analytic structure. For this reason, in what follows, we will restrict our attention to smooth and holomorphic manifolds.

For question (c), let us first elaborate on the meaning of the sentence in the parenthesis. ... Most notably, $\mathbb{R}^4$ has a non-standard smooth structure.

So far, we have defined a manifold to be a given topological space M with some additional properties. However, sometimes, the space M is not explicitly given. Instead, we build it by gluing countably many open sets in $\mathbb{R}^m$ using transition maps which are homeomorphisms or diffeomorphisms. In this scenario, by construction, the resulting (quotient) space admits local charts and is second-countable. We then have to prove that it is Hausdorff to show that it is a manifold.

More precisely, let $\{V_\alpha\}_{\alpha\in\mathcal{I}}$ be a countable collection of open subsets in $\mathbb{R}^m$ (or $\mathbb{C}^m$). For each pair of indices $\alpha,\beta\in\mathcal{I}$, suppose

$$V_{\alpha,\beta}\subset V_\alpha \qquad \text{and} \qquad V_{\beta,\alpha}\subset V_\beta$$

are open subsets admitting transition maps

$$V_{\alpha,\beta} \underset{\varphi_{\beta,\alpha}=\varphi^{-1}_{\alpha,\beta}}{\overset{\varphi_{\alpha,\beta}}{\rightleftarrows}} V_{\beta,\alpha}\ ,$$

which depending on the context, are homeomorphisms, smooth diffeomorphisms (diffeomorphisms), or holomorphic homeomorphisms (biholomorphisms). Furthermore, suppose that

- $V_{\alpha,\alpha}=V_\alpha$ for all $\alpha\in\mathcal{I}$, and $\varphi_{\alpha,\alpha}=\mathrm{id}_{V_\alpha}$;
- for all $\alpha,\beta,\gamma\in\mathcal{I}$, we have

$$V_{\alpha,\beta\gamma}=V_{\alpha,\gamma\beta}:=V_{\alpha,\beta}\cap V_{\alpha,\gamma}=V_{\alpha,\beta}\cap\varphi^{-1}_{\alpha,\beta}(V_{\beta,\gamma})$$

- for all $\alpha,\beta,\gamma\in\mathcal{I}$, restricted to $V_{\alpha,\beta\gamma}$, the transition maps satisfy the cocycle condition

$$\varphi_{\alpha,\gamma}=\varphi_{\beta,\gamma}\circ\varphi_{\alpha,\beta}.$$

Under these conditions, the equation

$$x\sim y \Leftrightarrow x\in V_\alpha,\ y\in V_\beta,\ y=\varphi_{\alpha,\beta}(x),$$

defines an equivalence relation on the disjoint-union-topological space $\widetilde{M}=\coprod_{\alpha\in\mathcal{I}}V_\alpha$.

HW 2.21. Check that all the conditions of being an equivalence relation are satisfied.

Let

$$M:=\widetilde{M}/\sim \tag{2.4}$$

denote the quotient topological space; M is a (C^0, smooth, or holomorphic, depending on the condition on the transition maps) manifold if and only if it is Hausdorff. It is second countable because each V_α is second-countable and $\mathcal{I}$ is countable. Let $\pi\colon\widetilde{M}\longrightarrow M$ denote the projection map. A natural atlas for M consists of the open sets $U_\alpha=\pi(V_\alpha)$ with the chart maps $\varphi_\alpha=\mathrm{id}_{V_\alpha}\circ\pi^{-1}$.

The example of the double origin line after HW 2.6 is a bad example of this construction where

$$\mathcal{I}=\{1,2\},\quad V_1=V_2=\mathbb{R},\quad V_{1,2}\cong V_{2,1}=\mathbb{R}^*,\quad \varphi_{1,2}=\mathrm{id}_{\mathbb{R}^*},$$

but the resulting quotient space is not Hausdorff. On the other hand, S^1 is obtained by gluing the same collection of open sets

$$\mathcal{I}=\{1,2\},\quad V_1=V_2=\mathbb{R},\quad V_{1,2}\cong V_{2,1}=\mathbb{R}^*,$$

via a different transition map $\varphi_{1,2}(x)=1/x$.

Remark 2.22. Constructing a manifold as a quotient space of an easier (e.g. linear) space is a common technique in the literature. Often, this is done by considering group quotients. We will discuss such quotient spaces in Section 2.4.

Remark 2.23. The gluing construction above also allows for a categorial description of a manifold which can be generalized to a setting suitable for constructing moduli spaces; see [2, 3, 6, 5]. A category is described by a set of objects and morphism spaces between the objects. Here, we consider the category whose set of objects are $\{V_\alpha\}_{\alpha\in\mathcal{I}}$, and the morphism space between V_α and V_β is $\mathrm{Hom}(V_\alpha, V_\beta) = V_{\alpha,\beta}$.

Example 2.24. We finish this section with the description of real and complex projective spaces. Define $\mathbb{RP}^m$ to be the set of all lines ℓ in $\mathbb{R}^{m+1}$. Similarly define $\mathbb{CP}^m$ to be the set of all complex lines ℓ in $\mathbb{R}^{m+1}$. Sometimes, especially the choice of $\mathbb{R}$ or $\mathbb{C}$ is not important, we will simply denote them by $\mathbb{P}^m$. We show that the real (resp. complex) projective space $\mathbb{RP}^m$ (resp. $\mathbb{CP}^m$) has the structure of a real (resp. complex) manifold of dimension m. It is easy to describe $\mathbb{P}^m$ as a quotient manifold in the sense of Remark 2.22, but we postpone that to Section 2.4. Note that we have not described the topology on each of these spaces. We will describe an atlas which, via (2.4), gives us both the topology and the manifold structure on each of these spaces. Let

$$V_\alpha = \mathbb{R}^{\mathcal{I}-\{\alpha\}} \cong \mathbb{R}^m \quad \forall \alpha\in\mathcal{I}=\{1,\ldots,m+1\}, \quad V_{\alpha,\beta} = \{(x_\gamma)_{\gamma\in\mathcal{I}-\{\alpha\}} \in \mathbb{R}^{\mathcal{I}-\{\alpha\}} \colon x_\beta \neq 0\}$$

$$\varphi_{\alpha,\beta}\big((x_\gamma)_{\gamma\in\mathcal{I}-\{\alpha\}}\big) = (y_\gamma)_{\gamma\in\mathcal{I}-\{\beta\}} \ \text{ s.t. } \ y_\gamma = \begin{cases} x_\gamma/x_\beta & \text{if } \gamma\neq\alpha \\ 1/x_\beta & \text{if } \gamma=\alpha\,. \end{cases}$$

The claim is the manifold $\mathbb{RP}^m$ obtained as in (2.4) from the collection of charts and transition maps above is (set-wise) equal to $\mathbb{RP}^m$. In fact, each line $\ell\in\mathbb{RP}^m$ is of the form $\mathbb{R}\cdot(x_\gamma)_{\gamma\in\mathcal{I}}\subset\mathbb{R}^{\mathcal{I}}$ such that $(x_\gamma)_{\gamma\in\mathcal{I}}\in\mathbb{R}^{\mathcal{I}}-0$ is a non-zero vector/point. Two points $(x_\gamma)_{\gamma\in\mathcal{I}}$ and $(x'_\gamma)_{\gamma\in\mathcal{I}}$ define the same line if and only if one of them is a non-zero multiple of the other. Let $U_\alpha\subset\mathbb{RP}^m$ be the (open) subset of lines where $x_\alpha\neq 0$ (this is well-defined by the last sentence). The bijective map

$$\varphi_\alpha\colon U_\alpha \longrightarrow V_\alpha, \qquad (x_\gamma)_{\gamma\in\mathcal{I}} \longrightarrow (x_\gamma/x_\alpha)_{\gamma\in\mathcal{I}-\{\alpha\}}$$

identifies U_α with V_α such that $\varphi_\beta\varphi_\alpha^{-1} = \varphi_{\alpha,\beta}$. Replacing $\mathbb{R}$ with $\mathbb{C}$, the description of $\mathbb{CP}^m$ is exactly the same.

HW 2.25. Show that $\mathbb{CP}^1 \cong S^2$; i.e. under the natural identification $\mathbb{C}\cong\mathbb{R}^2$, show that the charts and the transition maps in (2.24) and (2.11) are the same.

The half-space

$$\mathbb{H}_m = \{(x_1, x_2, \ldots, x_m)\colon x_1 \geq 0\}$$

is not a manifold in the sense of Definition 2.4 along its boundary points $\partial\mathbb{H}_m=\{0\}\times\mathbb{R}^{m-1}\subset\mathbb{R}^m$. We can modify Definitions 2.4 and 2.20 by including charts that have in image $\mathbb{H}$ to define a topological or smooth manifold M with boundary ∂M in the following way.

Definition 2.26. A topological m-manifold M is a Hausdorff and second-countable topological space such that for each $p\in M$ there exists an open neighborhood $U\ni p$ and a homeomorphism $\varphi\colon U\longrightarrow V$ to an open subset $V\subset\mathbb{H}^m$.

For all $p\in M$, the property $\varphi(p)\in\partial\mathcal{H}_m$ is independent of the choice of φ. We call the set of such points the boundary of M and denote it by ∂M; ∂M is a topological $(m-1)$-manifold. Smooth manifolds with boundary can be defined similarly: we need an atlas where the transition maps are smooth; ∂M will be a smooth $(m-1)$-manifold.

2.2 Vector bundles

If M is a manifold and W is a real or complex finite dimensional vector space, associated to a function $f\colon M \longrightarrow W$ we get its graph

$$\mathrm{Gr}(f) = \{(x, f(x))\colon x \in M\} \subset X \times W$$

which generalizes the visual concept of a graph in Calculus. The natural projection map

$$\pi_M\colon M \times W \longrightarrow M \tag{2.5}$$

restricts to an identification (homeomorphism or more depending on the context) of $\mathrm{Gr}(f)$ with the base manifold M. Also, $\mathrm{Gr}(f)$ intersects each fiber of (2.5) in exactly one point. The product $M \times W$ is a trivial example of a vector bundle. The map

$$s\colon M \longrightarrow M \times W, \qquad s(x) = (x, f(x)),$$

is a section of this vector bundle in the sense that $\pi_M \circ s = \mathrm{id}_M$. The function f and the section s carry the same amount of information. More generally, an arbitrary vector bundle E is a family of vector spaces over a manifold M that is locally "isomorphic" to a product as in (2.5), but globally, it may not be a product. The notion of section is defined for every vector bundle and generalizes the notion of function.

Definition 2.27. Suppose M is a topological manifold and W is a finite dimensional real or complex vector space. A topological W-**vector bundle** over M is a topological manifold E together with a continuous (surjective) projection map $\pi\colon E \longrightarrow M$ admitting a collection of **local trivializations** in the following sense: There is an open covering $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ of M such that

(a) for each $\alpha \in I$, there exists a homeomorphism $h_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times W$ so that $\pi \circ h_\alpha^{-1}$ coincides with the natural projection map $\pi_{U_\alpha}\colon U_\alpha \times W \longrightarrow U_\alpha$;

(b) for each $\alpha, \beta \in I$, there is a continuous function

$$h_{\alpha,\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(W)$$

such that the composition $h_\beta \circ h_\alpha^{-1}\colon (U_\alpha \cap U_\beta) \times W \longrightarrow (U_\alpha \cap U_\beta) \times W$ has the form

$$h_\beta \circ h_\alpha^{-1}(x, w) = (x, h_{\alpha,\beta}(x)w).$$

A **section** of E is a map $s\colon M \longrightarrow E$ such that $\pi \circ s = \mathrm{id}_M$. When $W \cong \mathbb{R}^k$, we say E is a real vector bundle of rank k. When $W \cong \mathbb{C}^k$, we say E is a complex vector bundle of rank k.

By Condition (a), using the trivialization h_α, each fiber $E_x = \pi^{-1}(x)$ of $E|_{U_\alpha}$ inherits a linear structure from W. By Condition (b), this linear structure is independent of the choice of $\alpha \in \mathcal{I}$ such that $x \in U_\alpha$. Therefore, we can rephrase Definition 2.27 in the following way:

A topological W-vector bundle over M is a topological manifold E together with a continuous (surjective) projection map $\pi\colon E \longrightarrow M$ and a vector bundle structure on each fiber E_x, admitting a collection of local trivializations in the following sense. For each $x \in M$, there is an open neighborhood $U \ni x$ and a local trivialization $h\colon \pi^{-1}(U) \longrightarrow U \times W$ such that (i) $\pi \circ h^{-1}$ coincides with the natural projection map $\pi_U\colon U \times W \longrightarrow U$; (ii) for each $y \in U$, the restriction

$$h_y := h|_{\{y\}\times W}\colon \{y\} \times W \longrightarrow E_y$$

is a linear isomorphism.

In order to define a smooth (resp. holomorphic) vector bundle, we need to consider a smooth (resp. holomorphic) structure on M and require the **change of trivialization maps** $h_{\alpha,\beta}$ above from one chart to another to be smooth (resp. holomorphic) instead of just continuous in the following way.

Definition 2.28. Let $\pi\colon E\longrightarrow M$ be a real rank k topological W-vector bundle. Suppose that the atlas $\mathcal{A}=\{\varphi_\alpha\colon U_\alpha\longrightarrow V_\alpha\}_{\alpha\in\mathcal{I}}$ on M defines a smooth structure. We say $\mathcal{A}$ lifts to a smooth structure on E if there exists a collection of trivialization maps

$$\{h_\alpha\colon \pi^{-1}(U_\alpha)\longrightarrow U_\alpha\times W\}_{\alpha\in\mathcal{I}} \tag{2.6}$$

as in Definition 2.27 such that

$$h_{\alpha,\beta}\circ\varphi_\alpha^{-1}\colon V_{\alpha,\beta}\longrightarrow GL(W)\cong GL(\mathbb{R}^k)\subset\mathbb{R}^{k\times k}$$

is a smooth function on $V_{\alpha,\beta}$ for all $\alpha,\beta\in\mathcal{I}$. Similarly, suppose $\mathcal{A}$ defines a holomorphic structure on M and $W\cong\mathbb{C}^k$. We say $\mathcal{A}$ lifts to a holomoprhic structure on E if there exists a collection of trivialization maps

$$\{h_\alpha\colon \pi^{-1}(U_\alpha)\longrightarrow U_\alpha\times W\}_{\alpha\in\mathcal{I}}$$

such that each

$$h_{\alpha,\beta}\circ\varphi_\alpha^{-1}\colon V_{\alpha,\beta}\longrightarrow GL(W)\cong GL(\mathbb{C}^k)\subset\mathbb{C}^{k\times k}$$

is holomorphic.

It is easy to see that if Definition 2.28 holds for an atlas $\mathcal{A}$, it holds for any atlas $\mathcal{A}'$ that is finer[4] than $\mathcal{A}$. Therefore, the smooth structure defined by (2.6) only depends on the equivalence class $[\mathcal{A}]$ of $\mathcal{A}$ (i.e. the smooth structure defined by $\mathcal{A}$).

In (2.4), we provided a rather different way of thinking about manifolds as a space obtained by gluing local Euclidean pieces via transition maps. When M is described as in (2.4), it is better to think of local trivializations of E as a product over V_α instead of U_α. If $\mathcal{A}$ is an atlas as in Definition 2.28, that can be done easily by considering the compositions

$$\Phi_\alpha\colon \pi^{-1}(U_\alpha)\longrightarrow V_\alpha\times W,\qquad \Phi_\alpha=(\varphi_\alpha\times\mathrm{id}_W)\circ h_\alpha,$$

of

$$h_\alpha\colon \pi^{-1}(U_\alpha)\longrightarrow U_\alpha\times W\quad\text{and}\quad \varphi_\alpha\times\mathrm{id}_W\colon U_\alpha\times W\longrightarrow V_\alpha\times W\qquad\forall\alpha\in\mathcal{I}.$$

The new transition maps

$$\Phi_\beta\circ\Phi_\alpha^{-1}\colon V_{\alpha,\beta}\times W\longrightarrow V_{\beta,\alpha}\times W\qquad\forall\alpha,\beta\in\mathcal{I}$$

have the form

$$\Phi_\beta\circ\Phi_\alpha^{-1}(x,w)=(\varphi_{\alpha,\beta}(x),\Phi_{\alpha,\beta}(x)w),$$

where

$$\Phi_{\alpha,\beta}\colon V_{\alpha,\beta}\longrightarrow \mathrm{GL}(W),\qquad \Phi_{\alpha,\beta}(x)=h_{\alpha,\beta}\circ\varphi_\alpha^{-1}.$$

[4] We say $\mathcal{A}'$ is finer than $\mathcal{A}$ if every chart in $\mathcal{A}'$ is the restriction to a sub open set of a chart in $\mathcal{A}$.

If $\mathcal{A}$ is a smooth (resp. holomoprhic) atlas on M which satisfies Definition 2.28, by assumption, the change of trivialization maps $\Phi_{\alpha,\beta}$ are smooth (resp. holomorphic).

Conversely, suppose

$$\{\{V_\alpha\}_{\alpha\in\mathcal{I}}, \{\varphi_{\alpha,\beta}\colon V_{\alpha,\beta} \longrightarrow V_{\beta,\alpha}\}_{\alpha,\beta\in\mathcal{I}}\}$$

is a collection as in the argument leading to (2.4), and

$$\{\Phi_{\alpha,\beta}\colon V_{\alpha,\beta} \longrightarrow \mathrm{GL}(W)\}_{\alpha,\beta} \tag{2.7}$$

is a collection satisfying the cocycle condition

$$\Phi_{\alpha,\gamma}(x)w = \Phi_{\beta,\gamma}(\varphi_{\alpha,\beta}(x))\,\Phi_{\alpha,\beta}(x)\,w \tag{2.8}$$

on $V_{\alpha,\beta\gamma}$. Then the equation

$$(x,w)\sim(y,u) \Leftrightarrow (x,w)\in V_\alpha\times W,\ (y,u)\in V_\beta\times W,\ y=\varphi_{\alpha,\beta}(x), u=\Phi_{\alpha,\beta}(x)w$$

extends the equivalence relation in (2.4) to an equivalence relation on the disjoint union

$$\widetilde{E} = \coprod_{\alpha\in\mathcal{I}} (V_\alpha\times W).$$

The quotient space

$$E \coloneqq \widetilde{E}/\sim \tag{2.9}$$

is automatically Hausdorff and defines a vector bundle over M. The projection map $\pi\colon E\longrightarrow M$ is the map induced by

$$\widetilde{\pi} \coloneqq \coprod_{\alpha\in\mathcal{I}} \pi_{V_\alpha}\colon \widetilde{E}\longrightarrow \widetilde{M}.$$

Recall that a section of a vector bundle $\pi\colon E\longrightarrow M$ is a map $s\colon M\longrightarrow E$ such that $\pi\circ s=\mathrm{id}_M$. When E is obtained by gluing local trivial pieces $V_\alpha\times W$ as in (2.9), a section s is equivalent to a collection of local functions

$$s_\alpha\colon V_\alpha\longrightarrow W$$

such that

$$s_\beta(\phi_{\alpha,\beta}(x)) = \Phi_{\alpha,\beta}(x)s_\alpha(x).$$

One of the most important examples of a vector bundle is the tangent bundle of a smooth manifold, whose sections generalize the notion of vector fields in Euclidean space. Suppose M is a smooth manifold as in (2.4) constructed from gluing local charts $\{V_\alpha\}_{\alpha\in\mathcal{I}}$ via the transition maps $\{\varphi_{\alpha,\beta}\colon V_{\alpha,\beta}\longrightarrow V_{\beta,\alpha}\}_{\alpha,\beta\in\mathcal{I}}$. Then the tangent bundle $E=TM$ constructed as in (2.9) corresponds to the change of trivialization maps

$$\Phi_{\alpha,\beta} = \mathrm{d}\varphi_{\alpha,\beta}\in GL(\mathbb{R}^m),$$

where $\mathrm{d}\varphi_{\alpha,\beta}$ is the matrix of partial derivatives. If $(x_1,\ldots,x_m)$ are local coordinates on V_α, $(y_1,\ldots,y_m)$ are local coordinates on V_β, a vector field (= section of tangent bundle) ξ on M is given be a collection of local vector fields

$$\zeta_\alpha = \sum_{i=1}^m a_i\partial_{x_i}$$

on the open sets $V_\alpha \subset \mathbb{R}^m$ that are identified on the overlaps via the relation

$$\zeta_\beta = \sum_{i=1}^{m} b_i \partial_{y_i} = \mathrm{d}\Phi_{\alpha,\beta}(\zeta_\alpha) = \sum_{i=1}\Big(\sum_{j=1}^{m} a_j \frac{\partial y_i}{\partial_{x_j}}\Big)\partial_{y_i}. \tag{2.10}$$

The cocycle condition (2.8) holds by Chain Rule.

Holomorphic tangent bundle of a complex manifold is defined similarly.

HW 2.29. Consider the real projective space $\mathbb{RP}^m$ and the atlas in Example 2.24 (Repeat the following with $\mathbb{CP}^m$ instead of $\mathbb{RP}^m$). For $\alpha = m+1$, let ξ_{m+1} denote the local vector field

$$\xi_{m+1}(x_1,\ldots,x_m) = \sum_{i=1}^{m} x_i \partial x_i$$

on V_{m+1}. Show that ξ_{m+1} extends to a unique smooth vector field ξ on $\mathbb{RP}^m$ (ξ is smooth if each of its local sections are smooth functions).

Example 2.30. Consider the real or complex projective space $\mathbb{P}^m$ and the atlas in Example 2.24. The so called tautological line bundle $\gamma \longrightarrow \mathbb{PP}^m$ (resp $\gamma \longrightarrow \mathbb{CP}^m$) is the real (resp. complex) line bundle whose fiber over the point $\ell \in \mathbb{RP}^m$ (resp. $\ell \in \mathbb{CP}^m$) is the real (resp. complex) line $\ell \subset \mathbb{R}^{m+1}$ (resp. $\ell \subset \mathbb{R}^{m+1}$) itself. We will show that γ is non-trivial (not isomorphic to the trivial bundle); γ plays an important role in the classification of complex line bundles.

HW 2.31. Write down the transition maps $\Phi_{\alpha,\beta}$ in (2.7) of γ with respect to the atlas in Example 2.24.

Operations such as direct sum, quotient, tensor product, taking dual, etc. naturally extend to vector bundles. Here are a few example.

(1) If E and F are vector spaces on M, then $E \oplus F$ is a vector space whose fiber over $x \in M$ is the direct sum of fibers $E_x \oplus F_x$.

(2) If E is a real (resp. complex) vector space, the dual vector space E^* is a real (resp. complex) vector space of the same rank whose fiber over $x \in M$ is $\mathrm{Hom}_{\mathbb{R}}(E,\mathbb{R})$ (resp. $\mathrm{Hom}_{\mathbb{C}}(E,\mathbb{C})$). If M is a smooth manifold, the dual of the tangent bundle TM is the cotangent bundle T^*M. Sections of T^*M are called differential 1-forms.

(3) If M is a real smooth manifold, the top exterior power $\Lambda^{\max}TM$ is a real line bundle on M called the orientation line bundle. We say M is orientable if $\Lambda^{\max}TM$ is trivial; i.e. $\Lambda^{\max}TM \cong M \times \mathbb{R}$. An orientation on M is a choice of this isomorphism. Similarly, if $E \longrightarrow M$ is any C^0-vector bundle over a topological manifold M, we say E is orientable if $\Lambda^{\mathrm{top}}E$ is trivial.

HW 2.32. Describe the transition maps $\Phi_{\alpha,\beta}$ of $\Lambda^{\max}TM$ in terms of the partial derivatives $\frac{\partial y_i}{\partial_{x_j}}$ in (2.10).

2.3 Maps between manifolds and vector bundles

Definition 2.33. Suppose M and M' are smooth (resp. holomorphic) manifolds, and let $f\colon M\longrightarrow M'$ be a continuous map. We say f is smooth (resp. holomorphic) if for every pair of charts $\varphi\colon U\longrightarrow V$ on M and $\varphi'\colon U'\longrightarrow V'$ on M' (in the corresponding maximal atlases), the composition function

$$\varphi'\circ f\circ\varphi^{-1}\colon \varphi(f^{-1}(U'))\longrightarrow V'$$

is smooth (resp. holomorphic).

Definition 2.33 is well-defined, because changing the charts on the source and target manifolds results in compositions by transition maps on the right or left, respectively, which are smooth (resp. holomorphic).

Definition 2.33 generalizes the notion of smooth function $f\colon M\longrightarrow\mathbb{R}$ that we discussed after HW 2.17. Just like for smooth functions in Calculus, we can differentiate smooth maps between manifolds. First we need to define the notion of a vector bundle homomorphism.

Definition 2.34. Suppose $f\colon M\longrightarrow M'$ is a continuous map between two manifolds, E is a vector bundle over M, and E' is a vector bundle over M'. We say $F\colon E\longrightarrow E'$ a homomorphism lifting/over f if F is continuous, it maps the fiber over $x\in M$ to the fiber $f(x)\in M'$, and its restriction to each fiber is linear.

In terms of local charts, suppose

$$E=\Big(\coprod_{\alpha\in\mathcal{I}}V_\alpha\times W\Big)\Big/\ V_{\alpha,\beta}\times W\ni(x,w)\sim(\varphi_{\alpha,\beta}(x),\Phi_{\alpha,\beta}(x)w)\in V_{\beta,\alpha}\times W$$

and

$$E'=\Big(\coprod_{\alpha'\in\mathcal{I}'}V'_\alpha\times W'\Big)\Big/V'_{\alpha',\beta'}\times W'\ni(x',w')\sim(\varphi_{\alpha',\beta'}(x'),\Phi'_{\alpha',\beta'}(x')w')\in V'_{\beta',\alpha'}\times W'$$

as in (2.9). For each $\alpha\in\mathcal{I}$ and $\alpha'\in\mathcal{I}'$, let $V_{\alpha,\alpha'}$ denote the pre-image of $V_{\alpha'}$ in V_α under f and

$$f_{\alpha,\alpha'}\colon V_{\alpha,\alpha'}\longrightarrow V_{\alpha'}$$

denote the restriction of f to $V_{\alpha,\alpha'}$. Then, the restriction $F_{\alpha,\alpha'}$ of F to $V_{\alpha,\alpha'}\times W$ has the form

$$F_{\alpha,\alpha'}(x,w)=(f_{\alpha,\alpha'}(x),\Theta_{\alpha,\alpha'}(x)w)$$

for some matrix-valued function

$$\Theta_{\alpha,\alpha'}\colon V_{\alpha,\alpha'}\longrightarrow\mathrm{Hom}(W,W').$$

Conversely, a collection of such functions $\Theta_{\alpha,\alpha'}$ glue along the overlaps to define a homomorphism

$$F\colon E\longrightarrow E'$$

if and only if they satisfy the following compatibility conditions on the relevant overlaps

$$\begin{aligned}\Theta_{\beta,\alpha'}(\varphi_{\alpha,\beta}(x))\,\Phi_{\alpha,\beta}(x)&=\Theta_{\alpha,\alpha'}(x);\\ \Phi'_{\alpha',\beta'}(f_{\alpha,\alpha'}(x))\,\Theta_{\alpha,\alpha'}(x)&=\Theta_{\alpha,\beta'}(x).\end{aligned}\tag{2.11}$$

In other words, the homomorphisms $\Theta_{\alpha,\alpha'}$ must commute with the change of trivialization maps on M and M'. Since $\Phi_{\alpha,\beta} \in \mathrm{GL}(W)$ and $\Phi'_{\alpha',\beta'} \in \mathrm{GL}(W')$, for each $x \in M$, the rank and nullity of the linear map

$$F_x = F|_{E_x} \colon E_x \longrightarrow E'_{f(x)}$$

is equal to the rank and nullity of $\Theta_{\alpha,\alpha'}(x)$.

A homomorphism F is continuous/smooth/holomorphic if and only if the matrix-valued functions $\Theta_{\alpha,\alpha'}$ are continuous/smooth/holomorphic in the corresponding atlas.

Lemma 2.35. *Suppose $f \colon M \longrightarrow M'$ is a smooth map between manifolds. Associated to f there exists a natural derivative map $\mathrm{d}f$ which is a smooth vector bundle homomorphism $\mathrm{d}f \colon TM \longrightarrow TM'$ lifting f.*

Proof. If f is locally given by

$$f_{\alpha,\alpha'} \colon V_{\alpha,\alpha'} \longrightarrow V_{\alpha'}, \qquad (x_1, \ldots, x_m) \longrightarrow (y_1, \ldots, y_n),$$

it follows from Chain Rule that the collection of $n \times m$ partial derivative matrices

$$\Theta_{\alpha,\alpha'}(x) = \mathrm{d}f_{\alpha,\alpha'}(x) = \left[\frac{\partial y_i}{\partial x_j}(x)\right]_{\substack{1\le j\le m\\ 1\le i\le n}} \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n)$$

satisfies (2.11), and thus defines a vector bundle homomorphism $\mathrm{d}f \colon TM \longrightarrow TN$. □

Just like in linear algebra, by putting certain restrictions on the rank and nullity of $\mathrm{d}f$ we obtain interesting spacial cases that are useful in the study of manifolds.

Definition 2.36. Let $f \colon M \longrightarrow M'$ be smooth. We say

(1) f is an immersion if $\mathrm{Ker}(\mathrm{d}f) = 0$ at every point on M;

(2) f is a smooth embedding if it is a one-to-one immersion;

(3) f is a submersion if $\mathrm{Image}(\mathrm{d}f) = TM'$ (in particular, f is surjective).

HW 2.37. Show that the map

$$f \colon S^2 = \{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\} \longrightarrow \mathbb{R}^2, \qquad (x,y,z) \longrightarrow (x,y)$$

is smooth. Find the set of points at which $\mathrm{d}f$ is not full rank.

HW 2.38. Show that the map

$$f \colon S^1 \longrightarrow \mathbb{R}^3, \qquad \theta \longrightarrow (x,y,z) = (\sin(\theta) + 2\sin(2\theta), \cos(\theta) - 2\cos(2\theta), -3\sin(3\theta))$$

is an embedding of S^1 into $\mathbb{R}^3$. Here θ is the multi-valued ($\theta \cong \theta \pm 2k\pi$) angle function in the polar coordinates (r, θ) of $\mathbb{R}^2$ which restricts to a multivalued[5] coordinate function on S^1. Use a software to draw the image of f. The image is known as **trefoil knot** in knot theory.

Let's elaborate on each item of Definition 2.36. To do that, we first need to recall the Rank Theorem.

[5] On every open set of the form S^1 − point $\subset S^1$, we can fix a branch of the multi-valued angle function to get a chart on S^1.

Theorem 2.39 (Rank Theorem). *Suppose $f\colon M \longrightarrow M'$ is a smooth[6] map. Assume that the rank of $\mathrm{d}_p f$ is a constant number r for all $p \in M$. Then, for every $p \in M$, there exist a local chart $\varphi\colon U \longrightarrow V \subset \mathbb{R}^m$ around p on M and a local chart $\varphi'\colon U' \longrightarrow V' \subset \mathbb{R}^{m'}$ around $f(p)$ on M' such that $\varphi(p) = 0 \in \mathbb{R}^m$, $\varphi'(f(p)) = 0 \in \mathbb{R}^{m'}$, and $\varphi' \circ f \circ \varphi\colon U \longrightarrow V$ is a linear map. In particular, we can choose φ and φ' such that*

$$\varphi' \circ f \circ \varphi(x_1, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0),$$

where the number of zeros on the right-hand side is $m' - r$ if $m' > r$, and there are no zeros if $r = m'$.

The statement is clearly local; i.e., if the restriction of f to a neighborhood $\widetilde{U}$ of p has constant rank r, we can find a smaller neighborhood $U \subset \widetilde{U}$ that satisfies the result. Therefore, this theorem is an immediate corollary of the Rank Theorem in analysis. We are now ready to dig into different cases of Definition 2.36.

First, we define the notion of submanifold. There are two equivalent ways to define a submanifold. We say $N \subset M$ is a submanifold of a manifold M if N is a manifold and the inclusion map $\iota\colon N \longrightarrow M$ is an embedding[7]. With this definition, if $\dim(N) = n$ and $\dim(M) = m$, for every $p \in N$, by Rank Theorem, there exist a chart $\varphi\colon U \longrightarrow V$ around p in N and a chart $\varphi'\colon U' \longrightarrow V'$ around p in M such that

$$\varphi' \circ \varphi^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0).$$

In other words, for every $p \in N$ there is a neighborhood $U \subset M$ around p with local coordinates $(x_1, \ldots, x_m)$ such that

$$N \cap U = (x_{m+1} = 0) \cap \cdots \cap (x_n = 0). \tag{2.12}$$

The latter gives us another way of defining submanifolds. Either way, it is easy to show the image of a smooth/holomorphic embedding $f\colon M \longrightarrow M'$ in the sense of Definition 2.36.(2) is a smooth/holomorphic **sub-manifold** of the target manifold M'; see HW 2.36 for an embedding of S^1 into $\mathbb{R}^3$.

By Rank Theorem, every immersion $f\colon M \longrightarrow M'$ in the sense of Definition 2.36.(1) is locally an embedding; i.e. for every $p \in M$, there is a neighborhood $U \ni p$ in M such that $f|_U\colon U \longrightarrow M'$ is an embedding. Globally, however, the image of f may have self-intersections. For any point $p \in N = f(M)$ its pre-image $f^{-1}(p) \in M$ can be more than a singleton, but it will be a discrete set. Therefore, locally around such p, N is a union of intersecting manifolds (or branches). Extra conditions on f will control how different branches intersect at p.

HW 2.40. Describe

$$N := \{(x, y) \in \mathbb{R}^2 \colon xy = 0\} \subset \mathbb{R}^2,$$

as the image of an immersion; c.f. HW 2.15.

If $f\colon M \longrightarrow M'$ is a submersion in the sense of Definition 2.36.(3), by Rank Theorem, for every $q \in M'$ and $p \in f^{-1}(q)$, there are charts $\varphi\colon U \longrightarrow V \subset \mathbb{R}^m$ around p and $\varphi'\colon U' \longrightarrow V' \subset \mathbb{R}^{m'}$ around q such that

$$\varphi' \circ f \circ \varphi^{-1}(x_1, \ldots, x_m) = (x_1, \ldots, x_{m'}).$$

[6] C^1 is enough.

[7] Depending on the context, we require ι to be a continuous, smooth, or holomorphic embedding.

Therefore, the local coordinates $(x_{m'+1},\ldots,x_m)$ define a chart on $U\cap f^{-1}(q)$. In other words,

$$\pi\colon \varphi\colon U\cap f^{-1}(q)\longrightarrow \mathbb{R}^{m-m'},$$

where

$$\pi\colon \mathbb{R}^m\longrightarrow \mathbb{R}^{m-m'}\colon (x_1,\ldots,x_m)\longrightarrow (x_{m'+1},\ldots,x_m)$$

is the projection map to the last $m-m'$ coordinates, defines a chart on $\pi^{-1}(q)$. It is easy to show that such charts around different point of $f^{-1}(q)$ are compatible with each other and thus $f^{-1}(q)$ is a submanifold of M; $f^{-1}(q)$ is called a level set of f. Again, the argument is local, in the sense that the conclusion we just made about $f^{-1}(q)$ only requires $\mathrm{d}f$ to be full rank at every point on preimage of q.

Definition 2.41. Suppose $f\colon M\longrightarrow M'$ is a smooth map, we say $q\in M'$ is a **regular value** of f if $\mathrm{d}_p f$ is surjective at every $p\in f^{-1}(q)$.

Since rank $\mathrm{d}_p f$ is a lower semi-continuous function in p, if q is a regular value, then there is an open neighborhood W of $f^{-1}(q)$ such that $\mathrm{d}_p f$ is surjective at every $p\in W$. Therefore, the argument before Definition 2.41 proves the following Lemma.

Lemma 2.42. *Suppose $f\colon M\longrightarrow M'$ is a smooth map and $q\in M'$ is a regular value. Then the level set $f^{-1}(q)\subset M$ is a submanifold of codimension* $\dim_{\mathbb{R}} M'$.

If $f^{-1}(q)$ is not compact, then W does not necessarily include an open set of the form $f^{-1}(V)$ where V is a neighborhood of q. However, if $f^{-1}(q)$ is compact, then W contains an open set of the form $f^{-1}(V)$. Moreover, in this case, one can show that, for sufficiently small V, there is a diffeomorphism (or so called product structure)

$$\varphi\colon V\times f^{-1}(q)\longrightarrow f^{-1}(V)$$

such that $f\circ(q',p)=q'$ for all $q'\in V$, $p\in f^{-1}(q)$. In particular, $f^{-1}(q)$ is diffeomorphic to every other level set $f^{-1}(q')$ over V. The proof of this statement needs a metric on M; we will come back to it later.

We have thus far learned two methods for getting a submanifold; either by embedding a manifold into another, or by looking at a regular level set of a function. In the first approach, we often try to embed a (complicated) manifold into a larger and simpler manifold (such as the Euclidean space), or we study the set of different embeddings of one manifold into another. In the second approach, we often build new manifolds by looking a level sets of smooth functions on known manifolds (again, such as the Euclidean space), or we use the level sets to foliate a manifold and divide it into pieces. For example, it is natural to wonder if every manifold embeds into Euclidean space. Here is a result (Whitney embedding theorem).

Theorem 2.43. *Every smooth m-manifold M admits a smooth embedding into $\mathbb{R}^{2m}$.*

A stronger version of Theorem 2.43 due to John Nash shows that every Riemannian manifold (a smooth manifold equipped with a Riemannian metric) admits an isometric embedding into some Euclidean space.

HW 2.44. Explain why no compact holomorphic manifold admits a holomorphic embedding into some $\mathbb{C}^n$ (Those that admit such an embedding are necessarily open and are called Affine varieties.)

HW 2.45. Explain why

$$N := \{(x,y,z) \in \mathbb{R}^3 \colon x^2+y^2+z^2=0\} \subset \mathbb{R}^3,$$

is not a manifold. Can you describe it as the image of an immersion?

The concepts in Definition 2.36 extend to an arbitrary smooth vector bundle homomorphism (not just the derivative) in the following way. Suppose

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{f} & M' \end{array}$$

is a vector bundle homomorphism as in Definition 2.34. Just like linear maps between vector spaces, we define

$$\mathrm{Ker}(F) = \big\{ v \in E \colon F(v) = 0 \in E' \big\}, \qquad \mathrm{Coker}(F)_{f(x)} = \frac{E'_{f(x)}}{F(E_x)} \quad \forall\, x \in M.$$

In general, rank $\mathrm{Ker}(F)_x$ (and thus rank $\mathrm{Coker}(F)_{f(x)}$) can be different at different $x \in M$. Therefore, $\mathrm{Ker}(F)$ and $\mathrm{Coker}(F)$ are not always vector bundles.

HW 2.46. With notation as above, suppose rank $\mathrm{Ker}(F)_x \equiv r$ for every $x \in M$. Then $\mathrm{Ker}(F)$ and $\mathrm{Coker}(F)$ are vector bundles over M and M', respectively. (See [4, p. 266].)

Example 2.47. In Example 2.30, the natural inclusion $\ell \in \mathbb{R}^{m+1}$ (resp. $\ell \subset \mathbb{C}^{m+1}$), for each line ℓ, gives an embedding of γ into the trivial line bundle $\mathbb{RP}^m \times \mathbb{R}^{m+1}$ (resp. $\mathbb{CP}^m \times \mathbb{C}^{m+1}$). Here, by an embedding we mean a homomorphism over the identity map of $\mathbb{P}^m$ which is injective on each fiber.

HW 2.48. With notation as in Example 2.30, show that there is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^m} \stackrel{\iota}{\longrightarrow} (\gamma^*)^{\oplus m+1} \longrightarrow T\mathbb{P}^m \longrightarrow 0,$$

where $\mathcal{O}_{\mathbb{P}^m}$ is the trivial (real or complex) line bundle over $\mathbb{P}^m$. Here "exact sequence" means that ι is an embedding and the $T\mathbb{P}^m = \mathrm{Coker}(\iota)$. In the real case, use this exact sequence to prove that the orientation line bundle of $\mathbb{RP}^m$ is isomorphic to γ if m is even, and to $\mathcal{O}_{\mathbb{P}^m}$ if m is odd.

The concepts of regular value and regular level sets can be generalized in the following way. Suppose $f_1 \colon M_1 \longrightarrow M'$ and $f_2 \colon M_2 \longrightarrow M'$ are smooth maps from manifolds M_1 and M_2 into M', respectively. We say f_1 and f_2 are transversal maps, and write $f_1 \pitchfork f_2$ if for every $q \in f_1(M_1) \cap f_2(M_2)$, and all $p_1 \in f_1^{-1}(q)$, $p_2 \in f_2^{-1}(q)$, we have

$$\mathrm{d}_{p_1} f_1(T_{p_1} M_1) + \mathrm{d}_{p_2} f_1(T_{p_2} M_2) = T_q M.$$

Definition 2.41 is a special case of with $(f_1, M_1) = (f, M)$ and $(f_2, M_2) = (\iota, q)$, where $\iota \colon q \longrightarrow M'$ is the embedding map of singleton q into M'. As another special case, we say the smooth submanifolds $M_1, M_2 \subset M'$ are intersecting transversely if the embedding maps $\iota_1 \colon M_1 \longrightarrow M'$ and $\iota_2 \colon M_2 \longrightarrow M'$ are transverse; i.e.

$$T_q M_1 + T_q M_2 = T_q M \qquad \forall\, q \in M_1 \cap M_2. \tag{2.13}$$

The the following lemma holds by Rank Theorem.

Lemma 2.49. *Suppose $M_1, M_2 \subset M'$ are transversely intersecting submanifolds. Then $M_1 \cap M_2$ is a submanifold of codimension*

$$\mathrm{codim}_{M'}(M_1 \cap M_2) = \mathrm{codim}_{M'} M_1 + \mathrm{codim}_{M'} M_2$$

Suppose $\pi\colon E \longrightarrow M$ is a smooth vector bundle and $s\colon M \longrightarrow E$ is a smooth section; s is an embedding of M into E. We say s is a transverse section if s and the zero section $s_0(x) \equiv 0$ are transverse embeddings. For every $x \in s^{-1}(0)$ (note that 0 is not a single value here; it is the zero section), there is a canonical decomposition

$$T_x E \cong T_x M \oplus E_x.$$

Remark 2.50. Away from the zero section, a canonical decomposition as above does not exist. We will discuss this issue in Section 2.7.

For each $x \in s^{-1}(0)$ let $\mathrm{d}_x^{\perp} s\colon T_x M \longrightarrow E_x$ denote the E_x-component of $\mathrm{d}_x s\colon T_x M \longrightarrow T_{x,0} E$ in the decomposition above. By (2.13), s is a transverse section if and only if $\mathrm{d}_x^{\perp} s$ is surjective. Then, $s^{-1}(0) \subset M$ is a submanifold of codimension rank E.
For example, suppose ζ is a transverse vector field on M; i.e. a transverse section of the tangent bundle TM. Then $\zeta^{-1}(0)$ is a submanifold of dimension 0; i.e. it is a collection of points. If M is compact, $\zeta^{-1}(0)$ is a finite collection. For each $x \in \zeta^{-1}(0)$,

$$\mathrm{d}_x^{\perp} \zeta\colon T_x M \longrightarrow T_x M$$

is an isomorphism. We define $\varepsilon(x) \in \{\pm 1\}$ to be the sign of $\det(\mathrm{d}_x^{\perp})$.

Theorem 2.51. *Suppose M is closed (compact with boundary) and ζ is a transverse vector field. The integer*

$$\mathcal{X}(\zeta) = \sum_{x \in \zeta^{-1}(0)} \varepsilon(x) \tag{2.14}$$

is independent of the choice of ζ. Thus, it is an invariant of M; we denote it by $\mathcal{X}(M)$.

The invariant $\mathcal{X}(M)$ above is called the Euler characteristic of M.

HW 2.52. Use the transverse section in HW 2.32 to compute $\mathcal{X}(\mathbb{RP}^n)$ and $\mathcal{X}(\mathbb{CP}^n)$.

Lemma 2.49 and 2.42 are special cases of the following statement.

Lemma 2.53. *Suppose $f_1\colon M_1 \longrightarrow M'$ and $f_2\colon M_2 \longrightarrow M'$ are smooth maps from manifolds M_1 and M_2 into M', respectively. If $f_1 \pitchfork f_2$, then the fiber product space*

$$M_1\, {}_{f_1}\!\times_{f_2} M_2 := \{(x_1, x_2) \in M_1 \times M_2 \colon f_1(x_1) = f_2(x_2)\} \subset M_1 \times M_2$$

is a submanifold of dimension

$$\dim\, M_1\, {}_{f_1}\!\times_{f_2} M_2 = \dim\, M_1 + \dim\, M_2 - \dim\, M'.$$

2.4 Quotient manifolds

In the previous section, we learned how to obtain construct new manifolds out of known manifolds and maps between them; e.g. as a level set or fiber product. Another common method for constructing new manifolds out of known ones is by considering quotients. First, we start with quotients by discrete groups and then extend it to quotients by Lie groups.

Suppose M is a smooth[8] manifold and G is a discrete group (probably finite). By a smooth (right-) action of G on M we mean a function

$$\varphi\colon M \times G \longrightarrow M, \qquad (x,g) \longrightarrow x\cdot g := \varphi(x,g) \in M$$

such that $\varphi(-,g)\colon M \longrightarrow M$ is smooth for all $g \in G$ and $\varphi(-,g_1g_2) = (\varphi(-,g_1),g_2)$ for all $g_1, g_2 \in G$. In particular, by the second property, each $\varphi(-,g)$ is a diffeomorphism. Let

$$M/\varphi \equiv M/G := M/\big(x \sim x\cdot g\colon \forall\ x \in M,\ g \in G\big)$$

denote the quotient space with the quotient topology.

Theorem 2.54. *With notation as above, suppose G is a discrete group that acts freely and properly on M in the following sense:*

- *(free) for every point $x \in M$ the stabilizer subgroup $G_x = \{g \in G\colon x\cdot g = x\}$ is the trivial subgroup;*
- *(proper) for every compact subset $K \subset M$, the subset $G_K = \{g \in G : (K\cdot g)\cap K \neq \emptyset\}$ is finite.*

Then the smooth manifold structure on M induces a unique smooth manifold structure on M/G.

Example 2.55. The action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ by translations,

$$\mathbb{R}^2 \times \mathbb{Z}^2 \longrightarrow \mathbb{R}^2, \qquad (x,y)\times(m,n) \longrightarrow (x+m, y+n)$$

is smooth, free, and proper. The quotient manifold $\mathbb{R}^2/\mathbb{Z}^2$ is the 2-dimensional torus $\mathbb{T}^2$. In the holomorphic category, there are many holomorphically non-equivalent ways to define an action of $\mathbb{Z}^2$ on $\mathbb{C} \cong \mathbb{R}^2$. For each $\tau \in \mathcal{H} = \{z \in \mathbb{C}\colon \mathrm{Im}(z) > 0\}$, let

$$\varphi_\tau\colon \mathbb{C}\times\mathbb{Z}^2 \longrightarrow \mathbb{C}, \qquad z\times(m,n) \longrightarrow z + m + n\tau.$$

Then, $\mathbb{T}^2_\tau = \mathbb{C}/\varphi_\tau$ is a 2-torus with a holomorphic structure (called an elliptic curve). Since $\mathbb{R}^2$ is the universal covering space of $\mathbb{T}^2$, every elliptic curve is of the form $\mathbb{T}^2_\tau$ for some $\tau \in \mathcal{H}$. The following HW characterizes different holomorphic structures on $\mathbb{T}^2$.

HW 2.56. With notation as in Example 2.55, show that $\mathbb{T}^2_\tau$ is biholomorphic to $\mathbb{T}^2_{\tau'}$ if and only if

$$\tau' = A\cdot\tau = \frac{a\tau + b}{c\tau + d} \tag{2.15}$$

for some

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z}) = \{A \in M_{2\times 2}(\mathbb{Z}) : \det(A) = 1\}.$$

[8] The same story holds for topological, holomorphic, or other types of manifolds.

Next, we define Lie groups and extend Theorem 2.54 to non-discrete actions. A Lie group G is a manifold with a group structure such that the product map

$$G \times G \longrightarrow G$$

is smooth. Examples of Lie groups include S^1, $\mathrm{GL}(N,\mathbb{R})$, $\mathrm{GL}(N,\mathbb{C})$, and many of its subgroups.

Theorem 2.57 ([4], Theorem 9.19)**.** *With notation as above, suppose G is a Lie group that acts freely and properly on M in the following sense:*

- *(free) for every point $x \in M$ the stabilizer subgroup $G_x = \{g \in G\colon x \cdot g = x\}$ is the trivial subgroup;*
- *(proper) for every compact subset $K \subset M$, the subset $G_K = \{g \in G : (K \cdot g) \cap K \neq \emptyset\}$ is compact.*

Then the smooth manifold structure on M induces a unique smooth manifold structure on M/G such that the quotient map $\pi\colon M \longrightarrow M/G$ is a smooth submersion.

If G is a compact Lie group (such as a finite group), every smooth action of G on a manifold M is proper; see [4, Cor 7.2]. Therefore, one only needs to check that the action is free.

HW 2.58. Describe $\mathbb{P}^m$ (in both real and complex versions) as a quotient manifold.

HW 2.59. Let W be a real/complex vector space of rank n and $1 \leq k \leq n$. The real/complex Grassmannian $\mathrm{Gr}_k(W)$ space is the set of k-dimensional subspaces of W. For example,

$$\mathbb{P}^{n-1} = \mathrm{Gr}_1(W).$$

Describe $\mathrm{Gr}_k(W)$ as a quotient manifold. Generalizing Example 2.47, every $\mathrm{Gr}_k(W)$ admits a tautological rank r vector bundle $\gamma_{k,W} \longrightarrow \mathrm{Gr}_k(W)$ whose fiber over the k-dimensional subspace $K \subset W$ is K. Find the analogue of the exact sequence in HW 2.48 for $\mathrm{Gr}_k(W)$ and $\gamma_{k,W}$.

HW 2.60. In HW 2.56, (2.15) defines a left-action of the discrete group $\mathrm{SL}(2,\mathbb{Z})$ on $\mathcal{H}$. The action descends to an action of $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\pm I_{2\times 2}$. Explain why the quotient space is not a manifold. This quotient space is the "moduli space" of elliptic curves; the space parametrizing isomorphism classes of elliptic curves.

2.5 Differential forms and de Rham cohomology

By definition, for every smooth manifold M, its cotangent bundle T^*M is the vector space of linear maps on tangent vectors. A 1-form η is a section of T^*M. If $x = (x_1, \ldots, x_m)$ are local coordinates on a chart V_α, $\eta_\alpha = \eta|_{V_\alpha}$ can be written in the form

$$\eta_\alpha = \sum_{i=1}^{m} a_i(x)\mathrm{d}x_i,$$

where $\{\mathrm{d}x_i\}_{i=1}^m$ is the basis dual to $\{\partial_{x_i}\}_{i=1}^m$ under the natural bi-linear pairing map

$$TX \otimes T^*X \longrightarrow X \times \mathbb{R}, \quad (\zeta, \eta) \longrightarrow \eta(\zeta).$$

If $y = (y_1, \ldots, y_m)$ are local coordinates on another chart V_β and

$$\eta_\alpha = \sum_{i=1}^{m} b_i(y)\mathrm{d}y_i,$$

by Chain Rule and the definition of the dual bundle, the coefficients a_i and b_i are related by the change of trivialization map

$$\eta_\alpha = \varphi^*_{\alpha,\beta}\eta_\beta := \eta_\beta \circ \mathrm{d}\varphi_{\alpha,\beta} = \sum_{i=1}^{m} b_i(\varphi_{\alpha,\beta}(x))\mathrm{d}y_i(x) = \sum_{j=1}^{m}\Big(\sum_{i=1}^{m} b_i(\varphi_{\alpha,\beta}(x))\frac{\partial y_i}{\partial_{x_j}}\Big)\mathrm{d}x_j \qquad (2.16)$$

on the overlap; i.e.

$$a_j = \sum_{i=1}^{m} b_i(\varphi_{\alpha,\beta}(x))\frac{\partial y_i}{\partial_{x_j}} \qquad \forall\ j = 1,\ldots,m.$$

We say η_α is equal to the pullback of η_β by $\varphi_{\alpha,\beta}$.

Differential k-forms of are sections of the k-th exterior power $\Lambda^k T^*M$ of T^*M. We denote the space of (real-valued) smooth differential k-forms by $\Omega^k(M,\mathbb{R})$. Locally, every k-form η can be written as

$$\eta_\alpha = \sum_{i_1<\cdots<i_k} a_{i_1,\ldots,i_k}(x)\ \mathrm{d}x_{i_1}\wedge\ldots\wedge \mathrm{d}x_{i_k},$$

with transition maps

$$\eta_\alpha = \varphi^*_{\alpha,\beta}\eta_\beta = \sum_{i_1<\cdots<i_k} b_{i_1,\ldots,i_k}(\varphi_{\alpha,\beta}(x))\ \mathrm{d}y_{i_1}(x)\wedge\ldots\wedge \mathrm{d}y_{i_k}(x) =$$

$$\sum_{j_1<\cdots<j_k}\Big(\sum_{i_1<\cdots<i_k}\sum_{\sigma\in\mathbb{S}_k}\varepsilon(\sigma)b_{i_1,\ldots,i_k}(\varphi_{\alpha,\beta}(x))\ \frac{\partial y_{i_{\sigma(1)}}}{\partial_{x_{j_1}}}\cdots\frac{\partial y_{i_{\sigma(k)}}}{\partial_{x_{j_k}}}\Big)\ \mathrm{d}x_{j_1}\wedge\ldots\wedge \mathrm{d}x_{j_k},$$

where $\varepsilon(\sigma)\in\{\pm 1\}$ is the sign of permutation.

Lemma 2.61. *Prove that the local derivative maps*

$$\mathrm{d}|_{V_\alpha}\colon \Omega^k(V_\alpha,\mathbb{R})\longrightarrow \Omega^{k+1}(V_\alpha,\mathbb{R}),$$

$$\mathrm{d}|_{V_\alpha}\Big(\sum_{i_1<\cdots<i_k} a_{i_1,\ldots,i_k}(x)\ \mathrm{d}x_{i_1}\wedge\ldots\wedge\mathrm{d}x_{i_k}\Big) = \sum_{i_0<\cdots<i_k}\sum_{\ell=0}^{k}\ (-1)^\ell\frac{\partial a_{i_0,\ldots\widehat{i_\ell}\ldots i_k}(x)}{\partial_{x_{i_\ell}}}\ \mathrm{d}x_{i_0}\wedge\ldots\wedge\mathrm{d}x_{i_k}$$

are compatible along the overlaps and define a so-called exterior derivative map

$$\mathrm{d}\colon \Omega^k(M,\mathbb{R})\longrightarrow\Omega^{k+1}(M,\mathbb{R}) \qquad \forall\ k\geq 0.$$

Also, show that $\mathrm{d}\circ\mathrm{d}=0$*.*

By the last property above, the quotient spaces in the following definition are well-defined.

Definition 2.62. For $k\geq 0$, the k-th de Rham cohomology group of a smooth manifold M is the qoutient space

$$H^k_{\mathrm{dR}}(M,\mathbb{R}) := \frac{\ker\big(\mathrm{d}\colon\Omega^k(M,\mathbb{R})\longrightarrow\Omega^{k+1}(M,\mathbb{R})\big)}{\mathrm{Image}\big(\mathrm{d}\colon\Omega^{k-1}(M,\mathbb{R})\longrightarrow\Omega^k(M,\mathbb{R})\big)}.$$

HW 2.63. Explain why $H^k_{\mathrm{dR}}(M,\mathbb{R})=0$ for $k>\dim(M)$. Show that if M is connected then $H^0_{\mathrm{dR}}(M,\mathbb{R})\cong\mathbb{R}$. Calculate H^1_{dR} of S^1. Calculate H^1_{dR} and H^2_{dR} of S^2.

We will see later that for a compact manifold M the groups $H^k_{\mathrm{dR}}(M,\mathbb{R})$ are finite dimensional (vector spaces).

2.6 Integration and Stokes' theorem

In this section, first, we show how differential m-forms can be integrated on oriented m-dimensional manifolds. Then, we state the stokes' theorem and some of its consequences.

Suppose M is a smooth m-manifold (possibly with boundary) and η is an m-form on M. For the moment, suppose η is compactly supported and $\mathrm{supp}(\eta) \subset V$ for some chart $V \subset \mathbb{R}^m$. By this assumption,

$$\eta(x) = f(x)\, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_m$$

where $f\colon V \longrightarrow \mathbb{R}$ is a smooth function with compact support. We can define the integral of η on V and thus on M to be

$$\int_M \eta = \int_V \eta = \int_{\mathbb{R}^m} f(x)\ \mathrm{d}x_1 \ldots \mathrm{d}x_m \tag{2.17}$$

where the right-hand side is the standard Euclidean integration from Calculus. The question is whether (2.17) is independent of local chart/coordinates $(x_1, \ldots, x_n)$. The righthand side of (2.17) does not depend on the order of m integrations, but the effect of the permutation change of coordinate map

$$\varphi_\sigma\colon \mathbb{R}^m \longrightarrow \mathbb{R}^m, \qquad (x_1, \ldots, x_m) \longrightarrow (y_1, \ldots, y_m) = (x_{\sigma(1)}, \ldots, x_{\sigma(m)}), \qquad \sigma \in \mathbb{S}_m,$$

on η is multiplication by $(-1)^{\mathrm{sign}(\sigma)}$. Therefore, (2.17) depends on the order of the coordinates $x_1, \ldots, x_m$. An orientation on M in the sense of Item (3) in Page 12 fixes this sign problem in the following way.

By definition, an orientation on M is a choice of trivialization $\Lambda^m T^*M \cong M \times \mathbb{R}$. Under such a trivialization, the nowhere-vanishing constant section $s(x) \equiv (x, 1)$ for $M \times \mathbb{R}$ corresponds to a nowhere-vanishing differential m-form ω on M. Such a nowhere-vanishing ω is called a volume-form on M. We say local coordinates $x = (x_1, \ldots, x_m)$ are positively oriented if in the local presentation

$$\omega = g(x)\mathrm{d}x_1 \wedge \mathrm{d}x_m$$

of ω with respect to x, we have $g(x) > 0$. On oriented manifolds, we can choose a positive atlas consisting only of positively oriented charts. If $\varphi_{\alpha,\beta}\colon V_{\alpha,\beta} \longrightarrow V_{\beta,\alpha}$ is a transition map in such a positive atlas, then $\det(\mathrm{d}\varphi_{\alpha,\beta}(x)) > 0$ for all $x \in V_{\alpha,\beta}$. Therefore, the sign issue will not arise on a positive atlas. If $\det(\mathrm{d}\varphi_{\alpha,\beta}(x)) > 0$, the equality

$$\int_{V_{\alpha,\beta}} \eta_\alpha = \int_{V_{\beta,\alpha}} \eta_\beta$$

follows from Change of Variable Theorem for integral of a function on $\mathbb{R}^m$.

Suppose η is a differential m-form with compact support on an oriented manifold M. Fix a locally-finite positive atlas $\mathcal{A} = \{V_\alpha\}_\alpha$ admitting a partition of unity $\mathcal{P} = \{\theta_\alpha\colon V_\alpha \longrightarrow [0,1]\}$. It is straightforward to check that

$$\int_M \eta := \sum_\alpha \int_{V_\alpha} \theta_\alpha \eta$$

is independent of the choice of $\mathcal{A}$ and $\mathcal{P}$.

Suppose M is a smooth m-manifold with boundary ∂M (possibly empty). If M is orientable, then ∂M is also orientable. Given an orientation on M, there are two conventions for orienting ∂M. The inclusion $\partial M \subset M$ gives an exact sequence

$$0 \longrightarrow T\partial M \longrightarrow TM|_{\partial M} \longrightarrow \mathcal{N}_M\partial M \longrightarrow 0$$

where $\mathcal{N}_M\partial M$ is the normal line bundle of ∂M in M. It is easy to show that $\mathcal{N}_M\partial M$ is isomorphic to the trivial bundle $\partial M \times \mathbb{R}$. We choose an isomorphism such that constant section 1 corresponds to an outward-pointing normal vector field n_{out} along ∂M. The exact sequence above and the choice of an outward-pointing normal vector field n_{out} gives an isomorphism

$$T^*M|_{\partial M} \cong \mathcal{N}_M\partial M \oplus T^*\partial M. \tag{2.18}$$

We choose the orientation on $T^*\partial M$ such that (2.18) is an oriented isomorphism. We call it the induced orientation on ∂M.

Theorem 2.64 (Stokes' theorem)**.** *Suppose M is an oriented smooth m-manifold with boundary ∂M and η is a compactly supported $(m-1)$-form on M. Then*

$$\int_M \mathrm{d}\eta = \int_{\partial M} \eta,$$

where the righthand side is with respect to the induced orientation on ∂M

Suppose M is a connected oriented closed (compact without boundary) m-manifold. For every $(m-1)$-form η, by Stokes' theorem, we have

$$\int_M \mathrm{d}\eta = 0.$$

Therefore, $\int_M$ descends to a linear map

$$\int_M : H^m_{\text{dR}}(M,\mathbb{R}) = \frac{\Omega^m(M,\mathbb{R})}{\text{Image}\big(\mathrm{d}\colon \Omega^{m-1}(M,\mathbb{R}) \longrightarrow \Omega^m(M,\mathbb{R})\big)} \longrightarrow M. \tag{2.19}$$

For a volume form ω on M (which we know exists), we have $\int_M \omega > 0$ by definition. Therefore, the linear map above is surjective.

Proposition 2.65. *Suppose M is a connected oriented closed m-manifold. Then (2.19) is an isomorphism; i.e. $H^m_{\text{dR}}(M,\mathbb{R}) \cong \mathbb{R}$ is generated by the cohomology class of a volume form.*

We will prove this later in the discussion of Poincare duality.

Suppose M is a smooth manifold and ζ is a vector field on M. Contraction with ζ defines a degree lowering (by one) map on all positive degree differential forms

$$(\iota_\zeta\eta)(v_2,\dots,v_k) = \eta(\zeta(x),v_2,\dots,v_k) \qquad \forall\, x \in M,\ v_2,\dots,v_k \in \mathbb{T}_xM.$$

We have

$$\cdots \quad \Omega^k(M,\mathbb{R}) \xrightarrow{\iota_\zeta} \Omega^{k-1}(M,\mathbb{R}) \xrightarrow{\iota_\zeta} \cdots \Omega^0(M,\mathbb{R})$$

with $\iota_\zeta \circ \iota_\zeta = 0$. The Lie derivative of a differential form with respect to ζ is the degree preserving map

$$L_\zeta\eta = \mathrm{d}\iota_\zeta\eta + \iota_\zeta\mathrm{d}\eta$$

with the following geometric meaning. Let

$$\varphi\colon M\times\mathbb{R}\longrightarrow M, \qquad (x,t)\longrightarrow \varphi_t(x)$$

denote the ODE flow of ζ by the family of diffeomorphisms $\{\varphi_t\}_{t\in\mathbb{R}}$. Then

$$\varphi_s^*L_\zeta\eta=\frac{\mathrm{d}\varphi_t^*\eta}{\mathrm{d}t}|_{t=s}:=\lim_{t\longrightarrow s}\frac{\varphi_t^*\eta-\varphi_s^*\eta}{t-s}. \tag{2.20}$$

In particular, putting $s=0$, we get

$$L_\zeta\eta=\lim_{t\longrightarrow 0}\frac{\varphi_t^*\eta-\eta}{t}.$$

Now, suppose M is oriented and compact, ω is a volume form on M, and ζ is a vector field on M. Since $\mathrm{d}\omega=0$, we have

$$L_\zeta\omega=\mathrm{d}\iota_\zeta\omega=f\omega$$

for some smooth function $f\colon M\longrightarrow\mathbb{R}$. We call f the divergence of ζ with respect to ω and write $f=\mathrm{Div}_\omega(f)$. This generalizes the notion of divergence from Calculus. By Stokes' theorem we have

$$\int_{\partial M}\iota_\zeta\omega=\int_M\mathrm{Div}_\omega(f)\omega. \tag{2.21}$$

The lefthand side is the average flow of ζ across ∂M with resect to $\omega|_{\partial M}$. This identity generalizes Divergence Theorem in Calculus.

2.7 Connection and curvature

Suppose $\pi\colon E\longrightarrow M$ is a vector bundle. The projection map π induces a surjective homomorphism between vector bundles over E

$$\mathrm{d}\pi\colon TE\longrightarrow\pi^*TM.$$

The kernel of this map is the subspace of "vertical tangent vectors" in TE and is naturally isomorphic to π^*E. In other words, we get an exact sequence of vector bundles

$$0\longrightarrow\pi^*E\longrightarrow TE\stackrel{\mathrm{d}\pi}{\longrightarrow}\pi^*TM\longrightarrow 0. \tag{2.22}$$

A right-inverse for $\mathrm{d}\pi$ would give us a decomposition

$$TE\cong\pi^*E\oplus\pi^*TM; \tag{2.23}$$

of (2.22); thus, it would allow us to decompose any tangent vector $v\in TE$ into a sum of a horizontal component $v^h\in T^hE\cong\pi^*TM$ and a vertical component $v^\perp\in\pi^*E$. For a trivial vector bundle $E=M\times W$, the product structure yields a canonical decomposition as in (2.23). For an arbitrary vector bundle E, a connection ∇, as we describe below, will give us a suitable decomposition (2.23) and vice versa; see Lemma 2.73.

From another perspective, let $s\colon M\longrightarrow E$ be a section. Derivative of s, as a function between manifolds, is a linear map $\mathrm{d}s\colon TM\longrightarrow TE$. If $E=M\times W$ is a trivial bundle, we have $s(x)=(x,f(x))$ for some function $f\colon M\longrightarrow W$ and

$$\mathrm{d}_xs=\mathrm{id}\oplus\mathrm{d}_xf\colon T_xM\longrightarrow T_xM\oplus W \qquad \forall\, x\in M.$$

So $\mathrm{d}_x s$ has two components, where the vertical component corresponds to the classical notion of the derivative of a function in calculus. For an arbitrary vector bundle, a connection ∇ and thus the corresponding decomposition (2.23) would enable us pick up the vertical part of $\mathrm{d}s$ which we denote by ∇s.

Definition 2.66. Suppose $\pi\colon E\longrightarrow M$ is a smooth real vector bundle. A connection ∇ is an $\mathbb{R}$-linear map

$$\nabla\colon \Gamma(M,E)\longrightarrow \Gamma(M,T^*M\otimes_{\mathbb{R}} E)$$

that satisfies the Leibniz rule

$$\nabla(f\zeta)=f\nabla\zeta+\mathrm{d}f\otimes\zeta \qquad \forall\ f\in C^\infty(M,\mathbb{R}),\ \zeta\in\Gamma(M,E). \tag{2.24}$$

Remark 2.67. Note that ∇ is a first-order operator. At any point $p\in M$, $(\nabla\zeta)|_p$ depends on the values ζ on an infinitesimal neighborhood of p. This is evident from the second term on the righthand side of (2.31). Therefore, ∇ is not a tensor; i.e. it is not a section of $\Gamma(M,T^*M\otimes_{\mathbb{R}}\mathrm{End}(E))$.

In any local trivialization $E|_{V_\alpha}\cong V_\alpha\times W$, ∇ can be written as

$$\nabla_\alpha:=\mathrm{d}+\Theta_\alpha \qquad \text{s.t.} \qquad \Theta_\alpha\in\Gamma\big(M,T^*M\otimes_{\mathbb{R}}\mathrm{End}(W)\big). \tag{2.25}$$

For $W=\mathbb{R}^n$ (resp. $\mathbb{C}^n$), we have $\mathrm{End}(W)=M_{n\times n}(\mathbb{R})$ (resp. $M_{n\times n}(\mathbb{C})$) and θ_α is an $n\times n$ matrix of real-valued (resp. complex-valued) 1-forms:

$$\Theta_\alpha=\big[\theta_{\alpha,ij}\big]_{\substack{1\le i\le n\\ 1\le j\le n}}.$$

For a different local trivialization $E|_{V_\beta}=V_\beta\times W$ with the change of trivialization map

$$V_{\alpha,\beta}\times W\longrightarrow V_{\beta,\alpha}\times W, \qquad (x,w)\longrightarrow(\varphi_{\alpha,\beta}(x),\Phi_{\alpha,\beta}(x)w),$$

the 1-formed valued endomorphisms θ_α and θ_β are related by

$$\Theta_\beta=-\mathrm{d}\Phi_{\alpha,\beta}\Phi_{\alpha,\beta}^{-1}+\Phi_{\alpha,\beta}\Theta_\alpha\Phi_{\alpha,\beta}^{-1}, \tag{2.26}$$

where the products on the righthand side are matrix products.

HW 2.68. Show that if ∇ and ∇' are two connections on E, then

$$\nabla-\nabla'=\Theta\in\Gamma(M,T^*M\otimes_{\mathbb{R}}\mathrm{End}(E));$$

i.e. the difference of every two connections is a globally defined $\mathrm{End}(E)$-valued 1-form.

Lemma 2.69. *Every vector bundle admits a connection. The space of connections on E is an affine space with tangent space $\Gamma(M,T^*M\otimes_{\mathbb{R}}\mathrm{End}(E))$.*

Proof. The second statement follows from HW 2.68. Let $\{V_\alpha\}_{\alpha\in\mathcal{I}}$ be a smooth atlas on M with a partition and unity $\{\eta_\alpha\colon V_\alpha\longrightarrow\mathbb{R}\}_{\alpha\in\mathcal{I}}$ and local trivializations $E|_{V_\alpha}=V_\alpha\times W$. Fix local connections ∇_α on $V_\alpha\times W$; e.g. you may put $\Theta_\alpha=0$ in (2.25) and just take the standard derivative map d. The summation

$$\nabla\zeta:=\sum_{\alpha\in\mathcal{I}}\nabla_\alpha(\eta_\alpha\zeta)$$

is well-defined and defines a connection on E. □

HW 2.70. In the proof above, explain why none of the terms $\zeta \longrightarrow \nabla_\alpha(\eta_\alpha \zeta)$ is a connection by itself but the summation on right defines a connection.

Remark 2.71. Replacing $\mathbb{R}$ with $\mathbb{C}$ in Definition 2.66 and the rest of the statements above gives us $\mathbb{C}$-linear connections on complex vector bundles.

Remark 2.72. A connection on E induces a connection on tensor/exterior powers of E.

Lemma 2.73. *Suppose M is a smooth manifold and $\pi\colon E \longrightarrow M$ is a vector bundle. A connection ∇ on E induces a splitting (2.23) of the exact sequence (2.22) such that*

$$(\mathrm{d}\zeta(v))^\perp = \nabla_v \zeta \in E_x \qquad \forall\ \zeta \in \Gamma(M,E),\ v \in T_xM. \tag{2.27}$$

Proof. For any $x \in M$, fix a local chart V around x and a local trivialization $E|_V \cong V \times W$. In this trivialization ∇ has the form

$$\nabla := \mathrm{d} + \Theta \qquad \text{s.t.} \qquad \Theta \in \Gamma\big(M, T^*M \otimes_{\mathbb{R}} \mathrm{End}(W)\big).$$

For any $w \in W$, define

$$T^{\Theta;h}_{(x,w)} V \times W = \big\{(v, \Theta(v)w) \in T_xV \oplus W \colon \forall\ v \in T_xM\big\}. \tag{2.28}$$

We need to show that (2.28) is invariant under the change of local trivialization. Suppose $E|_V \cong V \times W$ is another local trivialization with the corresponding endomorphism-valued 1-form Θ' and the change of trivialization map

$$\widetilde{\Phi}\colon V \times W \longrightarrow V \times W, \qquad (x,w) \longrightarrow (x, -\Phi(x)w).$$

By (2.26), Θ and Θ' are related by

$$\Theta' = -\mathrm{d}\Phi\,\Phi^{-1} + \Phi\,\Theta\,\Phi^{-1}.$$

We show that

$$\mathrm{d}\widetilde{\Phi}\colon T(V \times W) \longrightarrow T(V \times W)$$

maps $T^{\Theta;h}_{(x,w)}(V \times W)$ to $T^{\Theta';h}_{(x,w')}(V \times W)$, where $w' = \Phi(x)w$. For $(v, -\Theta(v)w) \in T^{\Theta;h}_{(x,w)} V \times W$, we have

$$\begin{aligned}
\mathrm{d}_{(x,w)}\widetilde{\Phi}\big(v, -\Theta(v)w\big) &= \Big(v, \big(\mathrm{d}_x\Phi(v)\big)w - \Phi(x)\Theta(v)w\Big)\\
&= \Big(v, \big(\mathrm{d}_x\Phi(v) - \Phi(x)\Theta(v)\big)w\Big)\\
&= \Big(v, -\big(-\mathrm{d}_x\Phi(v)\Phi(x)^{-1} + \Phi(x)\Theta(v)\Phi(x)^{-1}\big)\Phi(x)w\Big)\\
&= \big(v, -\Theta'(v)w'\big) \in T^{\Theta';h}_{(x,w')}(V \times W).
\end{aligned}$$

We can also describe the horizontal subspace $T^hE \subset TE$ globally in the following way. For every $x \in M$ and $w \in E_x$, let ζ be a section of E on neighborhood of x such that $\zeta(x) = w$. Define

$$T^h_{(x,w)}E = \big\{\mathrm{d}_x\zeta(v) - \nabla_v\zeta|_x \in T_{x,w}E \ \ \forall v \in T_xM\big\}.$$

It follows from Leibniz rule $T^h_{(x,w)}E$ is independent of the choice of the extension ζ. Since $\nabla_v\zeta|_x \in E_x$ is in the kernel of $\mathrm{d}\pi\colon TE \longrightarrow TM$ and $\pi \circ s = \mathrm{id}_M$, we have

$$\mathrm{d}\pi\big(\mathrm{d}_x\zeta(v) - \nabla_v\zeta|_x\big) = v.$$

Therefore, $\mathrm{d}\pi\colon T^h_{(x,w)}E \longrightarrow T_xM$ is an isomorphism. $\square$

Given a smooth vector bundle $\pi\colon E \longrightarrow M$ and a connection ∇ on E, for every $x_0 \in M$, $\zeta_0 \in E_x$, and a smooth path $\gamma\colon [0,\varepsilon] \longrightarrow M$ starting at x_0 ($\gamma(0) = x_0$), by parallel transport of ζ_0 along γ we mean a smooth family vectors $\zeta(t) \in E_{\gamma(t)}$ such that (i) $\zeta(0) = \zeta_0$ and (ii) $\nabla_{\dot\gamma}\zeta \equiv 0$.

Lemma 2.74. *With notation as above, the parallel translate of ζ_0 exists over the entire γ and is unique.*

Proof. It is easy to see in terms of the local description of a connection that the equation $\nabla_{\dot\gamma}\zeta \equiv 0$ is an ODE with initial condition $\zeta(0) = \zeta_0$. Therefore, the solution exists and is unique. □

For each smooth path $\gamma(t)\colon [0,1] \longrightarrow M$ connecting $x_0 = \gamma(0)$ to $x_1 = \gamma(1)$, the parallel translation map

$$\mathcal{P}_\gamma\colon E_{x_0} \longrightarrow E_{x_1}$$

is a vector space isomorphism. For different paths γ and γ' from x_0 to x_1, $\mathcal{P}_\gamma$ and $\mathcal{P}_{\gamma'}$ are often different. Thus, for a loop γ that starts and ends at x_0,

$$\mathcal{P}_\gamma\colon E_{x_0} \longrightarrow E_{x_0}$$

is often a non-trivial endomorphism of E_{x_0}; $\mathcal{P}_\gamma \in \mathrm{End}(E_{x_0})$ is called the holonomy of ∇ around γ. We say ∇ is flat connection if its holonomy on contractible loops is trivial. The holonomy map of a flat connection gives a map $\pi_1(M, x_0) \longrightarrow \mathrm{End}(E_{x_0})$. Suppose M is connected and $\mathrm{pr}\colon \widetilde{M} \longrightarrow M$ is the universal cover of M. If E admits a flat connection then

$$\widetilde{E} := \mathrm{pr}^*E \longrightarrow \widetilde{M}$$

is isomorphic to the trivial bundle $\widetilde{M}\times W$. Via parallel transport, a flat connection determines an isomorphism $\widetilde{E} \cong \widetilde{M}\times W$ such that the pull back connection is the standard derivative map d on $\widetilde{M}\times W$. In conclusion, flat vector bundles correspond to representations $\rho\colon \pi_1(M) \longrightarrow \mathrm{End}(W)$. For each ρ, we have

$$E \cong \big(\widetilde{M} \times W\big)/G \tag{2.29}$$

where G acts by the group of deck transformation on $\widetilde{M}$ and G acts on W by ρ.

Holonomy is closely to related to the curvature F^∇ of ∇, which we are going to define. In fact, ∇ is a flat connection if and only if $F^\nabla \equiv 0$.

Proposition 2.75. . *For any connection ∇, the expression*

$$F^\nabla(\zeta_1,\zeta_2)\xi = \nabla_{\zeta_1}\nabla_{\zeta_2}\xi - \nabla_{\zeta_2}\nabla_{\zeta_1}\xi - \nabla_{[\zeta_1,\zeta_2]}\xi, \qquad \forall\ \zeta_1,\zeta_2 \in \Gamma(M,TM),\ \xi \in \Gamma(M,E), \tag{2.30}$$

is $C^\infty(M,\mathbb{R})$-linear in all three inputs. Hence it defines an element

$$F^\nabla \in \Omega^2(M,\mathrm{End}(E)) := \Gamma(M, \Lambda^2T^*M \otimes_{\mathbb{R}} \mathrm{End}(E))$$

called the curvature of ∇.

In (2.30), $[\zeta_1,\zeta_2]$ is the Lie bracket of vectors fields ζ_1 and ζ_2. It is a vector field satisfying

$$[\zeta_1,\zeta_2](f) := \mathrm{d}f([\zeta_1,\zeta_2]) = \zeta_1(\zeta_2(f)) - \zeta_2(\zeta_1(f)).$$

In local coordinates, if

$$\zeta_1 = \sum_{i=1}^m a_i\partial_{x_i} \quad \text{and} \quad \zeta_1 = \sum_{i=1}^m b_i\partial_{x_i}$$

then

$$[\zeta_1,\zeta_2] = \sum_{i=1}^{m}\sum_{j=1}^{m} \big(a_j \frac{\partial b_i}{\partial_{x_j}} - b_j \frac{\partial a_i}{\partial_{x_j}}\big)\,\partial_{x_i}.$$

Both the Lie bracket and F^∇ are skew-symmetric in ζ_1,ζ_2.

HW 2.76. Replace ξ with $f\xi$ and apply Leibniz rule (mutiple times) to prove Proposition 2.75.

A connection ∇ can also be seen as a way of extending the exterior derivative d to E-valued differential forms. Let

$$\Omega^k(M,E) := \Gamma(M, \Lambda^k T^*M \otimes_{\mathbb{R}} E)$$

denote space E-valued differential k-forms on M. A connection ∇ can be seen as a derivative map

$$\mathrm{d}_\nabla \colon \Omega^0(M,E) \longrightarrow \Omega^1(M,E)$$

and d_∇ extends to a derivate map

$$\mathrm{d}_\nabla \colon \Omega^k(M,E) \longrightarrow \Omega^{k+1}(M,E)$$

for all $k \geq 0$. As in the case of the exterior derivative d, d_∇ satisfies the Leibniz identity

$$\mathrm{d}_\nabla(\omega\wedge\eta) = \mathrm{d}\omega\wedge\eta + (-1)^k\omega\wedge \mathrm{d}_\nabla\eta, \qquad \forall\omega\in\Omega^k(M,\mathbb{R}),\ \eta\in\Omega^\ell(M,E).$$

Unlike the exterior derivative d, $\mathrm{d}_\nabla\circ\mathrm{d}_\nabla \neq 0$; thinking of F^∇ as a map

$$F^\nabla \colon \Omega^k(M,E) \longrightarrow \Omega^{k+2}(M,E);$$

we have $\mathrm{d}_\nabla\circ\mathrm{d}_\nabla(\eta) = F^\nabla\wedge\eta$; so F^∇ measures how much $\mathrm{d}_\nabla\circ\mathrm{d}_\nabla$ deviates from being a cochain map.

In terms of the local connection matrices Θ in (2.25), the End(E)-valued 2-form F^∇ has the form

$$F = \mathrm{d}\Theta + \Theta\wedge\Theta.$$

If $\mathrm{rank}_{\mathbb{R}}\ E = 1$, then Θ is an honest 2-form and $\Theta\wedge\Theta = 0$; thus, F defines a global closed 2-form on M whose cohomology class $[F]\in H^2(M,\mathbb{R})$ is independent of the choice of ∇. This cohomology class will happen to be zero (why?).

If $r = \mathrm{rank}_{\mathbb{R}}E > 1$, then Θ is a matrix of 1-forms and $\Theta\wedge\Theta$ can be a non-trivial matrix of 2-forms. Nevertheless, we have

$$\text{(Bianchi Indentity)}\qquad \begin{aligned} \mathrm{d}F &= \mathrm{d}(\mathrm{d}\Theta + \Theta\wedge\Theta) \\ &= \mathrm{d}\Theta\wedge\Theta - \Theta\wedge\mathrm{d}\Theta = [F,\Theta] := F\wedge\Theta - \Theta\wedge F. \end{aligned}$$

For an honest 1-form Θ, we have $\mathrm{d}\Theta\wedge\Theta - \Theta\wedge\mathrm{d}\Theta = 0$, but this is not true for a matrix of 1-forms.

For any fixed pair of integers $1 \leq i,j \leq r$, the individual ij-th terms of F_α, where F_α is the curvature matrix with respect a local trivializations $E|_{V_\alpha} \cong V_\alpha\times\mathbb{R}^r$, do not paste together to define a 2-form on M; the change of trivialization maps $\Phi_{\alpha,\beta}$ mix these terms together by conjugation, i.e.

$$F_\beta = \Phi_{\alpha,\beta}F_\alpha\Phi_{\alpha,\beta}^{-1}.$$

However, symmetric functions of F_α are preserved by the change of trivialization maps. The basic symmetric functions are

$$\sigma_{2,\alpha} = \mathrm{trace}(F_\alpha),\ \ldots\ , \sigma_{2r,\alpha} = \det(F_\alpha)$$

in degrees 2 up to $2r$. For each $1 \leq k \leq r$, the local $2k$-forms $\{\sigma_{2k,\alpha}\}$ paste together to define well-defined global $2k$-forms σ_{2k} on M. These differential forms depend (only) on ∇.

Even though, by Bianchi Identity, $\mathrm{d}F \neq 0$, the combinations $\sigma_{2,\alpha}, \ldots, \sigma_{2r,\alpha}$ are closed forms. For example,

$$\mathrm{d}\sigma_2 = \mathrm{d}\ \mathrm{trace}(F) = \mathrm{trace}(\mathrm{d}F) = \mathrm{trace}(F \wedge \Theta - \Theta \wedge F) = 0.$$

Also, if we change ∇ to a another connection ∇', the resulting closed differential forms $\sigma'_2, \ldots, \sigma'_{2r}$ differ from $\sigma_2, \ldots, \sigma_{2r}$ by exact forms. Therefore, the cohomology class

$$[\sigma_2] \in H^2_{\mathrm{dR}}(M, \mathbb{C}), \ldots, [\sigma_{2r}] \in H^{2r}_{\mathrm{dR}}(M, \mathbb{C})$$

are invariants of the smooth vector bundle E. If E is trivial, by choosing the trivial connection $\nabla = \mathrm{d}$, we observe that $[\sigma_{2k}] = 0$ for all $1 \leq k \leq r$. The converse however is not always true. If E is a flat line bundle as in (2.29), then all these cohomology classes are 0 but E can be non-trivial.

Everything we have been talking about has a straightforward generalization to complex vector bundles in the following sense.

Definition 2.77. Suppose $\pi\colon E \longrightarrow M$ is a smooth complex vector bundle. A Chern (complex linear) connection ∇ is a $\mathbb{C}$-linear map

$$\nabla\colon \Gamma(M, E) \longrightarrow \Gamma(M, (T^*M \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} E)$$

that satisfies the Leibniz rule

$$\nabla(f\zeta) = f\nabla\zeta + \mathrm{d}f \otimes \zeta \qquad \forall\ f \in C^\infty(M, \mathbb{C}),\ \zeta \in \Gamma(M, E). \tag{2.31}$$

Note that by replacing T^*M with $T^*M \otimes_{\mathbb{R}} \mathbb{C}$ we are allowing differential forms with complex coefficients; we will denote the space of $\mathbb{C}$-valued differential k-forms by $\Omega^k(M, \mathbb{C})$. In general, if $E \longrightarrow M$ is a real vector bundle, then $E_{\mathbb{C}} := E \otimes_{\mathbb{R}} \mathbb{C}$ gives us a complex vector bundle of the same rank. The complex conjugation induces a conjugation on $E_{\mathbb{C}}$.

Similarly, the curvature of a Chern connection is the $\mathrm{End}_{\mathbb{C}}(E)$-valued 2-form

$$F^\nabla \in \Omega^2(M, \mathrm{End}_{\mathbb{C}}(E)) := \Gamma\big(M, \Lambda^2_{\mathbb{C}}(T^*M \otimes \mathbb{C}) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(E)\big).$$

If $\mathrm{rank}_{\mathbb{C}}(E) = r$, the Chern classes of E, as de Rham cohomology classes, are defined by

$$1 + tc_1(E) + t^2 c_2(E) + \ldots + t^r c_r(E) = \det(I + \frac{\mathfrak{i}t}{2\pi} F).$$

They happen to be real-valued. In particular,

$$c_1(E) = [\frac{\mathfrak{i}}{2\pi}\mathrm{trace}(F)] \in H^2_{\mathrm{dR}}(M, \mathbb{R}) \quad \text{and} \quad c_r(E) = [\Big(\frac{\mathfrak{i}}{2\pi}\Big)^r \det(F_\alpha)] \in H^{2r}_{\mathrm{dR}}(M, \mathbb{R}). \tag{2.32}$$

So far, we have been purposefully avoiding any mention of a metric in our discussion of manifolds and vector bundles. Differential Topology is mainly about the smooth structure of manifolds and vector bundles. Introduction of a metric would enable us to put more conditions on a connection and thus the curvature form. In the case of a real vector bundle $E \longrightarrow M$, we equip E with a Riemannian structure in the following sense. A Riemannian metric g on E is a smooth fiberwise positive-definite symmetric bilinear map

$$g \equiv \langle -, - \rangle : E \times E \longrightarrow \mathbb{R}.$$

Here, symmetric means

$$\langle u, v \rangle = \langle u, v \rangle \qquad \forall\, x \in M,\ u, v \in E_x$$

and positive-definite means

$$\langle u, u \rangle > 0 \qquad \forall\, x \in M,\ 0 \neq u \in E_x.$$

In any local trivialization $E|_{V_\alpha} \cong V_\alpha \times \mathbb{R}^r$, we have

$$\langle u, v \rangle_\alpha = u^T A_\alpha(x) v \qquad \forall\, x \in M,\ u, v \in \mathbb{R}^r,$$

where $A_\alpha(x)$ is positive-definite symmetric matrix and the function

$$V_\alpha \longrightarrow M_{r\times r}(\mathbb{R}), \quad x \longrightarrow A_\alpha(x)$$

is smooth. If A_β is the matrix corresponding to a different trivialization $E|_{V_\beta} \cong V_\beta \times W$, A_β and A_α are related by the formula

$$A_\alpha(x) = \Phi_{\alpha,\beta}(x)^T A_\beta(\varphi_{\alpha,\beta}(x)) \Phi_{\alpha,\beta}(x). \tag{2.33}$$

A Riemannian metric g on E also gives us an identification of E and its dual E^*:

$$E \ni v \Leftrightarrow \langle v, - \rangle \in E^*. \tag{2.34}$$

Therefore, every real vector bundle E is isomorphic to its dual. Since

$$[\sigma_k(E)] = (-1)^k [\sigma_k(E^*)] \qquad \forall\, k \geq 0,$$

it follows that $[\sigma_k] = 0$ for all odd k.

Given a Riemannian metric $g \equiv \langle -, - \rangle$ on E, we say a connection ∇ is compatible with g if

$$\mathrm{d}\, \langle \zeta, \xi \rangle = \langle \nabla\zeta, \xi \rangle + \langle \zeta, \nabla\xi \rangle\,. \tag{2.35}$$

HW 2.78. If ∇ and g are compatible, in any local trivialization $E|_{V_\alpha} \cong V_\alpha \times \mathbb{R}^r$, find the relation between A_α and Θ_α.

If $E \longrightarrow M$ is a complex vector bundle, we can equip E with a Hermitian metric. A Hermitian metric h

$$h \equiv \langle -, - \rangle : \overline{E} \times E \longrightarrow \mathbb{C}.$$

is a smooth family of Hermitian inner products on the fibers of E in the following sense:

- h is complex linear on the second input and anti-complex linear in the first input;
- $\langle \overline{u}, v \rangle = \overline{\langle \overline{v}, u \rangle} \qquad \forall\, x \in M,\ u, v \in E_x$;

- $\langle \overline{u}, u\rangle > 0 \qquad \forall\, x\in M,\ 0\neq u\in E_x$.

In any local trivialization $E|_{V_\alpha} \cong V_\alpha \times \mathbb{C}^r$, we have

$$\langle \overline{u}, v\rangle_\alpha = \overline{u}^T A_\alpha(x) v \qquad \forall\, x\in M,\ u,v\in\mathbb{C}^r,$$

where $A^*_\alpha := \overline{A_\alpha}^T = A_\alpha$, $u^* A_\alpha u > 0$ for all $u \in C^r - \{0\}$, and the function

$$V_\alpha \longrightarrow M_{r\times r}(\mathbb{C}), \quad x \longrightarrow A_\alpha(x)$$

is smooth. If A_β is the matrix corresponding to a different trivialization $E|_{V_\beta} \cong V_\beta \times \mathbb{C}^r$, A_β and A_α are related by the formula

$$A_\alpha(x) = \Phi_{\alpha,\beta}(x)^* A_\beta(\varphi_{\alpha,\beta}(x))\Phi_{\alpha,\beta}(x).$$

A Riemannian metric h on E also gives us an identification of $\overline{E}$ and its dual E^*:

$$\overline{E} \ni \overline{v} \Leftrightarrow \langle \overline{v}, -\rangle \in E^*.$$

Therefore, the dual of every complex vector bundle E is $\mathbb{C}$-linearly isomorphic to the conjugate of E.

Given a Hermitian metric $h \equiv \langle -,-\rangle$ on E, we say a Chern connection ∇ is compatible with h if

$$\mathrm{d}\left\langle \overline{\zeta}, \xi\right\rangle = \left\langle \overline{\nabla\zeta}, \xi\right\rangle + \left\langle \overline{\zeta}, \nabla\xi\right\rangle.$$

HW 2.79. If ∇ and h are compatible, in any local trivialization $E|_{V_\alpha} \cong V_\alpha \times \mathbb{C}^r$, find the relation between A_α and Θ_α.

HW 2.80. Use partition of unity to prove that every real/complex vector bundle admits a Riemannian/Hermitian metric.

Remark 2.81. Every Hermitian metric h on E has the form $h = g + \mathfrak{i}\omega$ such g is a Riemannian metric on the underlying real vector bundle of E and ω is a non-degenerate skew-symmetric bilinear form. Here, non-degenerate means that

$$\omega(u,-) = 0 \Leftrightarrow u = 0, \qquad \forall\, x \in M,\ u \in E_x.$$

By the second bullet above, ω and g are related by

$$\omega(u,v) = g(\mathfrak{i}u, v) \qquad \forall\, x \in M,\ u,v\in E_x.$$

Therefore, any of h, g, or ω, determines the rest.

We will dig more into metrics on manifolds in Section 2.8.

2.8 Riemannian manifolds

A Riemannian manifold is a smooth manifold M with a (Riemannian) metric $g = \langle -,-\rangle$ on its tangent bundle. Locally, on a chart V with local coordinates $(x_1,\ldots,x_m)$, a metric g has the form

$$g = g_{ij}(x)\ \mathrm{d}x_i \otimes \mathrm{d}x_j$$

such that $\big(g_{ij}(x)\big)$ is a smooth family of positive-definite symmetric matrices. The space of connections ∇ on TM that are compatible with g in the sense of (2.35) is non-empty and it has more than one element. There is a unique connection ∇ in this set, known as the Levi-Civita connection, which is Torsion-free in the following sense:

$$T_\nabla(\zeta,\xi) \coloneqq \nabla_\zeta \xi - \nabla_\xi \zeta - [\zeta,\xi] = 0 \qquad \forall\, \zeta,\xi \in \Gamma(M,TM).$$

In local coordinates $(x_1,\ldots,x_m)$ on M, we can expand any connection in the following way

$$\nabla_{\partial_{x_i}} \partial_{x_j} = \sum_{k=1}^{m} \Gamma_{ij}^k \partial_{x_k}.$$

For the Levi-Civita connection ∇, Γ_{ij}^k are called the Christoffel symbols of g and are given by

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{\ell=1}^{m} g^{k\ell}\Big(\frac{\partial g_{\ell i}}{\partial_{x_j}} + \frac{\partial g_{\ell j}}{\partial_{x_i}} - \frac{\partial g_{ij}}{\partial_{x_\ell}}\Big), \qquad \big(g^{ij}\big) = \big(g_{ij}\big)^{-1}. \tag{2.36}$$

The curvature tensor of ∇, usually denoted by R instead of F^∇ can also be expanded as

$$R(\partial_{x_i},\partial_{x_j})\,\partial_{x_k} = R_{k,ij}^{\ell}\,\partial_{x_\ell}, \tag{2.37}$$

where each $R_{k,ij}^\ell$ can be explicitly written in terms of the first derivatives of the Christoffel symbols. The full curvature tensor is often too complicated to work with. We often combine some of the terms to get a simpler tensor or just a function that is easier to work with. For example, the scalar curvature of g is the full trace of R given by

$$\kappa\colon M \longrightarrow \mathbb{R}, \qquad \kappa(x) = \sum_{i,j} \langle R(e_i,e_j)e_j, e_i\rangle = \sum_{i,j,k} g^{kj} R_{k,ij}^i$$

where $e_1,\ldots,e_m$ is an orthonormal basis of T_xM. Scalar curvature of a surface Σ embedded in $\mathbb{R}^3$ with respect to the metric induced by the standard metric on $\mathbb{R}^3$ is known as the Gaussian curvature.

Example 2.82. In this example, we calculate the scalar curvature of S_r^2 (2-dimensional sphere of radius r) and explain the formula for general S_r^n. The orientation-preserving parametrization of a 2-sphere of fixed radius r in spherical coordinates is given by

$$\mathbf{x}(\varphi,\theta) = r(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

where $(\varphi,\theta) \in (0,\pi)\times(0,2\pi)$. Therefore, we have

$$\begin{aligned}
\mathbf{x}_\varphi &= \frac{\partial \mathbf{x}}{\partial\varphi} = r\big(\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi\big)\\
\mathbf{x}_\theta &= \frac{\partial \mathbf{x}}{\partial\theta} = r\big(-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0\big)\\
g_{\varphi\varphi} &= \langle \mathbf{x}_\varphi, \mathbf{x}_\varphi\rangle = r^2,\\
g_{\theta\varphi} &= g_{\varphi\theta} = \langle \mathbf{x}_\theta, \mathbf{x}_\varphi\rangle = 0,\\
g_{\theta\theta} &= \langle \mathbf{x}_\theta, \mathbf{x}_\theta\rangle = r^2\sin^2\varphi;
\end{aligned}$$

i.e.

$$g = \begin{pmatrix} g_{\varphi\varphi} & g_{\theta\varphi} \\ g_{\varphi\theta} & g_{\theta\theta} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\varphi \end{pmatrix} \text{ with inverse } g^{-1} = \begin{pmatrix} \frac{1}{r^2} & 0 \\ 0 & \frac{1}{r^2\sin^2\varphi} \end{pmatrix}.$$

By (2.36), we have

$$\Gamma^\theta_{\theta\theta} = \frac{1}{2}\sum_{\ell=\theta \text{ or } \varphi}^m g^{\theta\ell}\Big(\frac{\partial g_{\ell\theta}}{\partial_\theta} + \frac{\partial g_{\ell\theta}}{\partial_\theta} - \frac{\partial g_{\theta\theta}}{\partial_\ell}\Big) = \frac{1}{2}g^{\theta\theta}\Big(2\frac{\partial g_{\theta\theta}}{\partial_\theta} - \frac{\partial g_{\theta\theta}}{\partial_\theta}\Big) = 0,$$

and

$$\Gamma^\theta_{\theta\varphi} = \frac{1}{2}\sum_{\ell=\theta \text{ or } \varphi}^m g^{\theta\ell}\Big(\frac{\partial g_{\ell\theta}}{\partial_\varphi} + \frac{\partial g_{\ell\varphi}}{\partial_\theta} - \frac{\partial g_{\theta\varphi}}{\partial_\ell}\Big) = \frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial_\varphi} = \text{cotan}(\varphi).$$

Similar calculations yield

$$\Gamma^\theta_{\varphi\theta} = \Gamma^\theta_{\theta\varphi} = \text{cotan}(\varphi), \quad \Gamma^\theta_{\varphi\varphi} = 0;$$
$$\Gamma^\varphi_{\theta\theta} = -\sin\varphi\cos\varphi, \quad \Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = 0, \quad \Gamma^\varphi_{\varphi\varphi} = 0.$$

We conclude that,

$$\nabla_{\partial_\theta}\partial_\theta = \Gamma^\varphi_{\theta\theta}\partial_\varphi + \Gamma^\theta_{\theta\theta}\partial_\theta = -\sin\varphi\cos\varphi\partial_\varphi;$$
$$\nabla_{\partial_\varphi}\partial_\theta = \nabla_{\partial_\theta}\partial_\varphi = \Gamma^\varphi_{\theta\varphi}\partial_\varphi + \Gamma^\theta_{\theta\varphi}\partial_\theta = \text{cotan}(\varphi)\partial_\theta;$$
$$\nabla_{\partial_\varphi}\partial_\varphi = \Gamma^\varphi_{\varphi\varphi}\partial_\varphi + \Gamma^\theta_{\varphi\varphi}\partial_\theta = 0.$$

In the formula (2.37), we need an orthonormal frame. The frame $(\partial_\varphi, \partial_\theta)$ is orthogonal but not normal. By dividing with their lengths, we get an orthonormal frame

$$(e_1, e_2) = \Big(\frac{1}{r}\partial_\varphi, \frac{1}{r\sin\varphi}\partial_\theta\Big).$$

Note that the only nontrivial terms in the formula (2.37) are

$$\langle R(e_1, e_2)e_2, e_1\rangle = \langle R(e_2, e_1)e_1, e_2\rangle.$$

However, this normalization is not necessary. Since curvature is a tensor, we have

$$\langle R(e_1, e_2)e_2, e_1\rangle = \frac{\langle R(\partial_\varphi, \partial_\theta)\partial_\theta, \partial_\varphi\rangle}{\text{area}(\partial_\varphi, \partial_\theta)^2} = \frac{\langle R(\partial_\varphi, \partial_\theta)\partial_\theta, \partial_\varphi\rangle}{r^4\sin\varphi^2}$$

We have

$$\begin{aligned} R(\partial_\varphi, \partial_\theta)\partial_\theta &= \nabla_{\partial_\varphi}\nabla_{\partial_\theta}\partial_\theta - \nabla_{\partial_\theta}\nabla_{\partial_\varphi}\partial_\theta - \nabla_{[\partial_\varphi, \partial_\theta]}\partial_\theta \\ &= \nabla_{\partial_\varphi}\big(-\sin\varphi\cos\varphi\partial_\varphi\big) - \nabla_{\partial_\theta}\big(\text{cotan}(\varphi)\partial_\theta\big) - 0 \\ &= \big(-\cos\varphi^2 + \sin\varphi^2\big)\partial_\varphi + \text{cotan}(\varphi)\sin\varphi\cos\varphi\partial_\varphi = \sin\varphi^2\partial_\varphi \end{aligned}$$

Therefore,

$$\kappa = 2\frac{\langle R(\partial_\varphi, \partial_\theta)\partial_\theta, \partial_\varphi\rangle}{r^4\sin\varphi^2} = \frac{2}{r^2}.$$

For n-sphere S^n_r of radius r, the scalar curvature is given by

$$\kappa = 2\sum_{i<j}\langle R(e_i, e_j)e_j, e_i\rangle.$$

Each pair (e_i, e_j) is tangent to some $S^2_r \subset S^n_r$. The corresponding term $2\langle R(e_i,e_j)e_j, e_i\rangle$ is equal to $2r^{-2}$. Therefore,

$$\kappa_{S^n_r} = \binom{n}{2} \times \frac{2}{r^2} = \frac{n(n-1)}{r^2}.$$

□

A Riemannian metric allows us to define a distance function on M (thus a metric in the topological sense). If $\gamma\colon [0,1] \longrightarrow M$ is a smooth path (satisfying $\dot\gamma \neq 0$), we define its length to be the non-negative quantity

$$|\gamma| = \int_{t=0}^{1} |\dot\gamma| \mathrm{d}t, \qquad |\dot\gamma| = \sqrt{\langle\dot\gamma,\dot\gamma\rangle}.$$

It follows from Chain Rule that $|\gamma|$ is independent of the choice of the parametrization. A Riemannian metric also allows us to define the angle between two tangent vectors by

$$\text{angle between } u,v = \cos^{-1}\left(\frac{\langle u,v\rangle}{|u||v|}\right) \in [0,\pi] \qquad \forall\, x\in M,\ u,v \in T_xM.$$

If M is connected, given two points $x, y \in M$, it is natural to seek for a path γ connecting the two points that has the minimum length. Such a path is called a geodesic between x and y. We can also consider geodesics that start at a point x and leave x at a particular direction $v \in T_xM$. Geodesics generalize the concept of straight line in Euclidean geometry. The questions are:

(Question 1) *what is the equation of a geodesic?*

(Question 2) *why do they exist and are they unique?*

The Lagrangian approach in math/physics to answer questions such as Question 1 above is to consider the space of all paths between two points and find the critical points of the length functional

$$L(\gamma) = |\gamma| = \int_{t=0}^{1} |\dot\gamma| \mathrm{d}t$$

whose minima are the desired paths. So the question reduces to: *what is the derivative of L?* At the cost of a reparametrization, we may assume $|\dot\gamma| \equiv c$ for some positive constant c. Otherwise, it would be easier to work with the functional

$$E(\gamma) = \int_{t=0}^{1} |\dot\gamma|^2 \mathrm{d}t.$$

HW 2.83. Show that a minima of L is also a minima of E.

An infinitesimal deformation of γ corresponds to a vector field $\nu \equiv \{\nu(t) \in T_{\gamma(t)}M\}_{t\in[0,1]}$ along γ that vanishes at the end points. A straightforward calculation shows that

$$\mathrm{d}_\gamma L(\nu) = \frac{1}{c}\int_{t=0}^{1} \frac{\langle \nabla_{\dot\gamma}\nu, \dot\gamma\rangle}{|\dot\gamma|}\mathrm{d}t = \frac{1}{c}\int_{t=0}^{1} \langle \nu, \nabla_{\dot\gamma}\dot\gamma\rangle \,\mathrm{d}t\,.$$

The second identity follows from Leibniz rule and the fact that $\nu(0) = \nu(1) = 0$. We conclude that γ is a critical point of L if and only if

$$\text{(Geodesic Equation)}\ \ \nabla_{\dot\gamma}\dot\gamma = 0.$$

In any local coordinate $\gamma(t) = (x_1(t), \ldots, x_m(t))$, (Geodesic Equation) is a second degree ODE

$$\ddot{x}_k + \sum_{i=1}^{m}\sum_{j=1}^{m} \Gamma_{ij}^k \; \dot{x}_i \; \dot{x}_j = 0.$$

Therefore, for every $x \in M$ and $v \in T_xM$, there is a unique geodesic $\gamma_{x,v}$ starting at $x = \gamma_{x,v}(0)$ with the initial velocity $v = \dot{\gamma}_{x,v}(0)$. If M is closed, then $\gamma_{x,v}$ is defined for over the entire $\mathbb{R}$. For different $x, y \in M$, there is a geodesic γ connecting the two points but it may not be unique. For long paths, critical points of L are not all minima and they are not necessarily isolated. In short distances, (Geodesic Equation) has a unique solution between every two points which corresponds to the minimum distance, but there are long geodesics that wind around and return to a nearby point (or the same point). In conclusion, for each $x \in M$, there is a sufficiently small ball

$$B_\epsilon(x) = \{v \in T_xM \colon |v| < \epsilon\} \subset T_xM$$

restricted to which the so-called exponentiation map

$$\exp_x \colon B_\epsilon(x) \longrightarrow M, \qquad v \longrightarrow \gamma_{x,v}(1) \in M, \tag{2.38}$$

is a diffeomorphism. One can think of $\exp_x^{-1}$ as a chart map around $x \in M$ that maps all the geodesics through x to straight line segments passing through the origin in T_xM.

Exponential identification can be extend to a neighborhood of every closed submanifold in the following way. Suppose $W \subset M$ is a submanifold. The metric g allows us to identify the normal bundle $\mathcal{N}_M W$ with the orthogonal complement of TW in $TM|_W$:

$$TW^\perp \coloneqq \{v \in TM|_W \colon \langle v, u\rangle = 0 \;\; \forall\, u \in T_{\pi(v)}W\}.$$

Extending the case $W = \{x\}$ above, there is a function

$$\varepsilon \colon W \longrightarrow \mathbb{R}_{>0}$$

and such that exponentiation map

$$\begin{aligned} &\exp_W \colon B_\epsilon(W) \longrightarrow M, \qquad T_xW^\perp \ni v \longrightarrow \gamma_{x,v}(1) \in M, \\ &B_\varepsilon(W) \coloneqq \{v \in TW^\perp \colon |v| < \varepsilon(\pi(v))\} \subset TW^\perp \end{aligned}$$

is a diffeomorphism. In other words, a neighborhood of every submanifold W in M can be identified with a neighborhood of the zero section in the normal bundle $\mathcal{N}_M W$. If W is compact, we can take ε to be a constant. If furthermore $\mathcal{N}_M W$ is isomorphic to the trivial line bundle $W \times \mathbb{R}^r$, the composition of $\exp_W$ and a trivialization shows that a neighborhood of W in M is diffeomorphic to a product. This is, for example, the case whenever M is an oriented manifold and $W = \partial M$ if compact: a neighborhood of $\partial M \subset M$ is orientably diffeomorphic to $(-1, 0] \times \partial M$.

A metric g on an oriented m-manifold M also allows us to define a volume form $\omega_g \in \Omega^m(M, \mathbb{R})$ whose integration on every open set $U \subset M$ is positive. If $(x_1, \ldots, x_m)$ are local coordinates on $V \subset M$ compatible with the orientation, define

$$\omega_{g,V} = \sqrt{\det(g_{ij})}\; \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_m.$$

Note that $\det(g^{ij}) > 0$ because (g^{ij}) is positive definite. For another such chart $V' \subset M$ with local coordinates $(y_1, \ldots, y_m)$ and transition map

$$x = (x_1, \ldots, x_m) \longrightarrow \varphi(x) = (y_1, \ldots, y_m),$$

it follows from the compatibility of the orientations that

$$\det(\mathrm{d}\varphi) > 0.$$

Therefore, by (2.33), we have

$$\sqrt{\det(g_{ij})} = \sqrt{\det(g'_{ij})}\ \det(\mathrm{d}\varphi).$$

Finally, it follows from the equation above and Chain Rule that

$$\varphi^* \omega_{g,V'} = \sqrt{\det(g_{ij})}\ \det(\mathrm{d}\varphi)^{-1}\ \det(\mathrm{d}\varphi)\ \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_m = \omega_{g,V}.$$

Therefore, the local m-forms $\{\omega_{g,V}\}$ are compatible along the overlaps and define a global nowhere-vanishing m-form ω_g on M. This volume-form can be used to define integration of functions on M. We also define the volume of M with respect to g to be $\int_M \omega_g$ (this can be infinite).

By (2.34), a Riemannian metric g on M gives an isomorphism of the tangent and cotangent bundles, and thus a metric (still denoted by g) on T^*M. In local notation of this section, we have

$$\langle \mathrm{d}x_i, \mathrm{d}x_j \rangle = g^{ij}.$$

The isomorphism $TM \cong_g T^*M$ also allows us to define the gradient vector field of a smooth function $f\colon M \longrightarrow \mathbb{R}$ by

$$\mathrm{d}f = \langle \nabla f, - \rangle$$

which generalizes the notion of the gradient in Calculus. In local coordinates, we have

$$\nabla f = \sum_{i,j=1}^{m} \frac{\partial f}{\partial_{x_i}}\ g^{ij}\ \partial_{x_j}.$$

At each $x \in M$, if $\nabla f(x) \neq 0$, since $\mathrm{d}f(v) = \langle \nabla f, v \rangle$, $\nabla f(x)$ is the direction at which f increases the most. Critical points of f correspond to zeros of the vector field ∇f. For every regular value c of f, ∇f is orthogonal to the level set $M_c = f^{-1}(c)$; thus, ∇f is a non-zero section of $TM_c^{\perp} \cong \mathcal{N}_M M_c$. In conclusion, $\mathcal{N}_M M_c$ is isomorphic to the trivial line bundle. If g is a metric on M,

$$n = \frac{\nabla f}{|\nabla f|}$$

define a normal vector to M_c. Through the isomorphism

$$TM|_{M_c} \cong \mathbb{R} \cdot n \oplus TM_c \tag{2.39}$$

the metric g induces a metric on M_c. If M is also oriented, then M_c is oriented as well and

$$\omega_{g_{M_c}} = \iota_n \omega_g. \tag{2.40}$$

Example 2.84. In this example, we calculate the area of the unit sphere S^2 with respect to the metric induced by the standard metric on $\mathbb{R}^3$ in two ways.

For the unit sphere $S^2 \subset \mathbb{R}^3$, the outward normal vector field n in (2.39) is half of the gradient vector of the defining equation of S^2; i.e.

$$n = \zeta|_{S^2}, \quad \zeta = \frac{1}{2}\nabla(x^2+y^2+z^2) = x\partial_x + y\partial_y + z\partial_z.$$

Therefore, the area form of S^2 with respect to the induced metric is the restriction of the 2-form

$$\iota_\zeta \,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z = x\,\mathrm{d}y\wedge\mathrm{d}z - y\,\mathrm{d}x\wedge\mathrm{d}z + z\,\mathrm{d}x\wedge\mathrm{d}y$$

to S^2. We can calculate the integral

$$\mathrm{area}(S^2) = \int_{S^2}\omega_{S^2}, \qquad \omega_{S^2} = \big(x\,\mathrm{d}y\wedge\mathrm{d}z - y\,\mathrm{d}x\wedge\mathrm{d}z + z\,\mathrm{d}x\wedge\mathrm{d}y\big)|_{S^2}. \tag{2.41}$$

either directly or by using Stokes' theorem.

(1) By Stokes' theorem,

$$\begin{aligned}\mathrm{area}(S^2) &= \int_{S^2}\big(x\,\mathrm{d}y\wedge\mathrm{d}z - y\,\mathrm{d}x\wedge\mathrm{d}z + z\,\mathrm{d}x\wedge\mathrm{d}y\big)\\ &= \int_B \mathrm{d}\,(x\,\mathrm{d}y\wedge\mathrm{d}z - y\,\mathrm{d}x\wedge\mathrm{d}z + z\,\mathrm{d}x\wedge\mathrm{d}y) = 3\int_B \mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z = 3\,\mathrm{vol}(B),\end{aligned}$$

where B is the unit ball in $\mathbb{R}^3$ bounded by $\partial M = S^2$; see (2.21). The last integral can be easily calculated in spherical coordinates giving us the value $\mathrm{vol}(B) = \frac{4}{3}\pi$. We conclude that $\mathrm{area}(S^2) = 4\pi$.

(2) To calculate (2.41) directly, we use the parametrization of S^2 coming from the spherical coordinates. The map

$$\Psi\colon (0,\pi)\times(0,2\pi) \longrightarrow S^2, \qquad (\varphi,\theta) \longrightarrow (\cos(\theta)\sin(\varphi), \sin(\theta)\sin(\varphi), \cos(\varphi))$$

gives an oriented identification of $(0,\pi)\times(0,2\pi)$ and a dense open set in S^2; i.e. Ψ^{-1} is an oriented chart covering the entire S^2 except an arc. By Chain Rule

$$\int_{(0,\pi)\times(0,2\pi)} \Psi^*\omega_{S^2} = \int_{S^2}\omega_{S^2} = \mathrm{area}(S^2).$$

We have

$$\begin{aligned}\Psi^*\omega_{S^2} =& \cos(\theta)\sin(\varphi)\ \mathrm{d}\big(\sin(\theta)\sin(\varphi)\big)\wedge\mathrm{d}\cos(\varphi) - \\ & \sin(\theta)\sin(\varphi)\ \mathrm{d}\big(\cos(\theta)\sin(\varphi)\big)\wedge\mathrm{d}\cos(\varphi) + \\ & \cos(\varphi)\ \mathrm{d}\big(\cos(\theta)\sin(\varphi)\big)\wedge\mathrm{d}\big(\sin(\theta)\sin(\varphi)\big)\\ =& -\cos(\theta)^2\sin(\varphi)^3\ \mathrm{d}\theta\wedge\mathrm{d}\varphi\\ & -\sin(\theta)^2\sin(\varphi)^3\ \mathrm{d}\theta\wedge\mathrm{d}\varphi\\ & -\cos(\varphi)^2\sin(\varphi)\sin(\theta)^2\ \mathrm{d}\theta\wedge\mathrm{d}\varphi\\ & -\cos(\varphi)^2\sin(\varphi)\cos(\theta)^2\ \mathrm{d}\theta\wedge\mathrm{d}\varphi\\ =& \sin(\varphi)\ \mathrm{d}\varphi\wedge\mathrm{d}\theta.\end{aligned}$$

Therefore,

$$\text{area}(S^2) = \int_{(0,\pi)\times(0,2\pi)} \Psi^*\omega_{S^2} = \int_{(0,\pi)\times(0,2\pi)} \sin(\varphi)\ \mathrm{d}\varphi\ \mathrm{d}\theta = 2\pi \int_{(0,\pi)} \sin(\varphi)\ \mathrm{d}\varphi = 4\pi.$$

Recall that at the zeros of section s of any vector bundle $E \longrightarrow M$, we have a canonical decomposition $T_xE \longrightarrow T_xM \oplus E_x$ which gives us a well-defined (independent of any connection) vertical-derivative map

$$\mathrm{d}_x^\perp s \colon T_xM \longrightarrow E_x.$$

Applying this to the section $s = \nabla f$ of $E = TM$, we get a second-derivative map

$$\mathcal{H}_x = \mathrm{d}_x^\perp \nabla f \colon T_xM \longrightarrow T_xM, \qquad \forall\ x \in M \text{ with } \nabla f(x) = 0,$$

which is called the Hessian of f at at x. Equivalently, we can work with the section $s = \mathrm{d}f$ of $E = T^*M$ to get

$$\mathcal{H}_x^* = \mathrm{d}_x^\perp \mathrm{d}f \colon T_xM \longrightarrow T_x^*M.$$

This way, we don't need any metric to define $\mathcal{H}^*$. They are related by

$$\langle H_x(v), - \rangle = \mathcal{H}_x^*(v) \qquad \forall\ v \in T_xM.$$

Nevertheless, we get a function

$$Q_x \colon T_xM \otimes T_xM \longrightarrow \mathbb{R}, \qquad Q_x(v,u) = \langle \mathcal{H}_x(v), u \rangle = \big(\mathcal{H}_x^*(v)\big)(u) \qquad \forall\ u,v \in T_xM \tag{2.42}$$

which is a (symmetric) quadratic form on T_xM. This is what we know as the second derivative matrix in Calculus. The eigenvalues of Q_x (or $\mathcal{H}_x$) tell us about the local behavior of f at a critical point.

Definition 2.85. We say x is a non-degenerate critical point of f if all the eigenvalues of Q_x are non-zero. We say f is a Morse function if all the critical points of f are non-degenerate.

If f is Morse, for every critical point x of M so that Q_x has p positive and $q = m - p$ negative eigenvalues, there are local coordinates $(y_1, \ldots, y_m)$ around x such that

$$f(y_1, \ldots, y_m) = f(x) + \sum_{i=1}^{p} y_i^2 - \sum_{i=p+1}^{m} y_j^2. \tag{2.43}$$

Definition 2.86. For every critical point x of a more function f, the integer $q = m - p$ is called the index of x.

The local description 2.43 helps us understand how the level sets M_c change as we go from a regular value $c < f(x)$ to another regular value $c > f(x)$. Different values of (p,q) correspond to so-called different handle attachments. A critical point x with $p = m$ is a local minimum and a critical point x with $p = 0$ is a local maximum. If $m = 2$, the case $(p,q) = (1,1)$ is known as a saddle point. For $m > 2$, we have different kinds of "saddle points". Therefore, starting from minimum points of f, M is built from a collection of balls by attaching handles of different types as we move toward the maxima of f.

Example 2.87. Consider a 2-torus T and a hight function $f\colon T \longrightarrow \mathbb{R}$ as in Figure 1. The function has four critical points A, B, C, D, where A is a minimum ($p{=}2, q{=}0$), D is a maximum ($p = 0, q = 2$), and B and C are saddle points ($p = 1, q = 1$). Starting at A, the construction of T starts with a disk $T_{\leq t} = f^{-1}((-\infty, t]) \subset \mathbb{R}^2$, where $\min(f) < t < b$. As we increase the value of t, the size of $T_{\leq t}$ increases but its topology remains the same. The level set T_b is not a manifold (it is singular). The change of topology from $T_{\leq b-\varepsilon}$ (a disk) to $T_{\leq b+\varepsilon}$ (a cylinder), for any sufficiently small $\varepsilon > 0$, corresponds to a handle attachment as seen in Figure 1. As t increases from b to the next critical value c, the topology of $T_{\leq t}$ remains the same. Once again, the change of topology from $T_{\leq c-\varepsilon}$ to $T_{\leq c+\varepsilon}$, for any sufficiently small $\varepsilon > 0$, corresponds to a handle attachment (this time in the reverse direction). Finally, reaching the maximum point D corresponds to closing the manifold $T_{\leq t}$, $c < t < \max(f)$, by attaching a disk (2-handle) to it along its boundary circle.

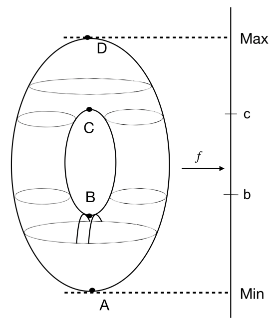


Figure 1: A Morse function on 2-torus and its level-sets.

Example 2.88. We use Theorem 2.43 to show that every manifold admits a (plethora of) Morse function. By Theorem 2.43, M can be embedded into a sufficiently large $\mathbb{R}^n$. We assume M does not include the origin. For almost every $p \in \mathbb{R}^n$, we show that the distance-square function from p

$$f_p(x) = \|x - p\|^2$$

restricts to a Morse function on M. Using the standard metric on $\mathbb{R}^n$, we have

$$\mathrm{d}f_p(v) = 2\,\langle x - p, v\rangle\,.$$

Therefore, $p \in M$ is a critical point if and only if $x - p$ is orthogonal to T_pM. Let

$$TM^{\perp} = \{(x, v) \in M \times \mathbb{R}^n \colon\ \langle u, v\rangle = 0 \quad \forall\ u \in T_xM\}.$$

Consider the map

$$\varrho\colon TM^{\perp} \longrightarrow \mathbb{R}^n, \qquad (x, v) \longrightarrow p = x + v.$$

Therefore x is a critical point of f_p if and only if $p \in \varrho(T_xM^{\perp})$. Using calculations in local coordinates on M, we can show that x is a degenerate critical point of f_p, if and only if (x, v) is a critical point of ϱ (second derivative of f_p at x equals derivative of ϱ at (x, v)). By Sard's theorem, the set of regular values of ϱ is open and dense. Therefore, for generic p, f_p is Morse.

HW 2.89. Suppose $M \subset \mathbb{R}^n$ is a closed submanifold. Show that for generic line $L \cong \mathbb{R}$, orthogonal projection to L defines a Morse function on M.

HW 2.90. Show that if $f\colon M \longrightarrow \mathbb{R}$ and $g\colon N \longrightarrow \mathbb{R}$ are Morse, then $f+g\colon M\times N \longrightarrow \mathbb{R}$ is Morse, and

$$\mathrm{Crit}(f+g) = \mathrm{Crit}(f)\times \mathrm{Crit}(g).$$

Theorem 2.91. *Let M be a closed manifold. Then the set of Morse functions on M is a dense open subset of $C^\infty(M,\mathbb{R})$.*

3 Different (co)homology theories and their interactions

3.1 Singular/simplicial/cellular homology

An m-simplex Δ is the convex hull of $m+1$ generic points in $\mathbb{R}^n$ for any $n \geq m$. The simplex is so-named because it represents the simplest possible polytope in any given space. For example, a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron. The boundary $\partial\Delta$ of any m-simplex Δ is a union of $m+1$ $(m-1)$-simplices. We call them codimension 1 faces of Δ. Inductively, we define a codimension k face of Δ to be a boundary component of a codimension $k-1$ face. We may think of an m-simplex Δ as a manifold with boundary and corner (where corners correspond to codimension 2 and higher faces). Topologically, an m-simplex Δ is homeomorphic to a ball in $\mathbb{R}^m$, and it can be smoothly identified with

$$\Delta_{\mathrm{std}} = \big\{(x_1,\ldots,x_m)\colon x_i \geq 0, \quad \sum_{i=1}^m x_i \leq 1\big\}.$$

Such an identification and the standard orientation on $\mathbb{R}^m$ fixes an orientation on Δ. Using the orientation convention in (2.18), an orientation on Δ then induces an orientation on each boundary $(m-1)$-simplex $\Delta' \subset \partial\Delta$. Every codimension 2 face Δ'' is the intersection of two boundary components Δ'_1 and Δ'_2. The boundary orientations induced by Δ'_1 and Δ'_2 on Δ'' are the opposite. This is the key observation in the definition of the singular homology below.

Definition 3.1. A singular n-simplex in a topological space M is a continuous map $\sigma\colon \Delta \longrightarrow M$ from an oriented n-simplex $\Delta = \Delta_{\mathrm{std}}$ to M. The boundary of σ, denoted by $\partial\sigma$ is the formal sum of the singular $(n-1)$-simplices represented by the restriction of σ to the faces of the standard n-simplex (with the boundary orientation).

For $n \geq 0$, let $C_n(M,\mathbb{Z})$ denote the free abelian group generated (over $\mathbb{Z}$) by all singular n-simplices. For $n<0$, define $C_n(M,\mathbb{Z})$ to be the trivial group. In $C_n(M,\mathbb{Z})$, multiplication of a singular n-simplex $\sigma\colon \Delta \longrightarrow M$ by (-1) can be exchanged with the change of orientation on Δ. An element of $C_n(M,\mathbb{Z})$ is called a singular n-chain. The boundary operator ∂ in Definition 3.1 linearly extends to all singular n-chains in $C_n(M,\mathbb{Z})$. The extension, called the boundary operator, is also denoted by

$$\partial\colon C_n(M,\mathbb{Z}) \longrightarrow C_{n-1}(M,\mathbb{Z}).$$

By the last line before Definition 3.1, we have $\partial^2=0$; therefore, the so-called singular homology groups

$$H_n^{\mathrm{sing}}(M,\mathbb{Z}) \coloneqq \frac{Z_n(M,\mathbb{Z}) \coloneqq \ker\big(\partial\colon C_n(M,\mathbb{Z}) \longrightarrow C_{n-1}(M,\mathbb{Z})\big)}{B_n(M,\mathbb{Z}) \coloneqq \mathrm{Image}\big(\partial\colon C_{n+1}(M,\mathbb{Z}) \longrightarrow C_n(M,\mathbb{Z})\big)}, \qquad \forall\, n\geq 0,$$

are well-defined. Replacing $\mathbb{Z}$ with $\mathbb{R}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{C}$, or any other ring/field, we get singular homology with values in that ring/field. An element in $Z_n(M,\mathbb{Z})$ is called singular n-cycle and an element in $B_n(M, \mathbb{Z})$ is called a singular n-boundary.

The abelian group $C_n(M, \mathbb{Z})$ is monstrous. Therefore, even if M is a nice topological space such as a smooth manifold, it is not clear why $H_n^{\text{sing}}(M, \mathbb{Z})$ should be finite. If M is path-connected, it is fairly easy to show that $H_0^{\text{sing}}(M, \mathbb{Z}) \cong \mathbb{Z}$.

Every m-manifold admits a triangulation; i.e. it can be written as a union of m-simplices glued along their codimension 1 faces. Figure 2 illustrates a (complicated) triangulation of the 2-torus; one can triangulate a 2-torus with only 2 triangles. In the literature, there are different types of

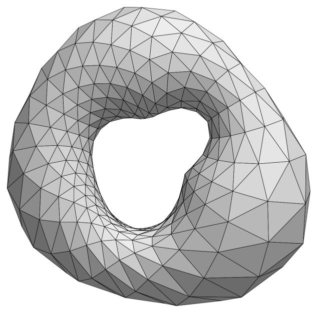


Figure 2: A triangulation of 2-torus (image courtesy of wikipedia)

triangulations on a topological space. Smooth manifolds admit the best kind of triangulation.

Theorem 3.2 (Cairns[**?**]-Whitehead [**?**])**.** *Every smooth manifold admits an (essentially unique) compatible piecewise linear structure.*

Therefore, we skip the details and focus on applications. We refer to [**?**] for a quick overview of the results.

A simplicial complex $\mathcal{K}$ is a collection of simplices that satisfies the following conditions:

(1) every face of a simplex from $\mathcal{K}$ is also in $\mathcal{K}$;

(2) the non-empty intersection of any two simplices $\Delta_1, \Delta_2 \in \mathcal{K}$ is a (union of) face of both Δ_1 and Δ_2.

A pure or homogeneous simplicial m-complex $\mathcal{K}$ is a simplicial complex where the largest dimension of any simplex in $\mathcal{K}$ equals m and every simplex of dimension $n < m$ is a codimension $m-n$ face of some m-simplex $\Delta \in \mathcal{K}$. By the discussion above (existence of a nice triangluation), every smooth m-manifold M can be given the structure of pure simplicial m-complex $\mathcal{K}_M$. If M is compact, we can choose $\mathcal{K}_M$ to have only finitely many faces. If M is oriented, each m-simplex Δ in $\mathcal{K}_M$ inherits an orientation from the orientation on M.

The definition of simiplicial homology of a simplicial complex $\mathcal{K}$ is very close to definition of singular homology. It is, however, much more useful for concrete calculations. Let $\mathcal{K}$ be a simplicial complex. A simplicial k-chain with coefficients in $\mathbb{Z}$ is a finite formal sum

$$\sum c_i \Delta_i$$

where each c_i is an integer and Σ_i is an oriented k-simplex in $\mathcal{K}$. In this definition, just like singular homology, we declare that each oriented simplex is equal to the negative of the simplex with the opposite orientation. The group of k-chains on $\mathcal{K}$ is denoted $C_k(\mathcal{K},\mathbb{Z})$. This is a free abelian group which has a basis in one-to-one correspondence with the set of k-simplices in $\mathcal{K}$. To define a basis explicitly, one has to choose an orientation of each simplex. If M is an oriented m-manifold, for m-simplices in $\mathcal{K}_M$, we usually choose the orientation induced by the orientation on M. For each oriented k-simplex Δ in $\mathcal{K}$, the boundary $\partial\Delta \in C_{k-1}(\mathcal{K},\mathbb{Z})$ is the formal sum of the codimension 1 faces with the induced boundary orientation. The boundary operator ∂ linearly extends to all simplicial k-chains in $C_k(\mathcal{K},\mathbb{Z})$:

$$\partial\colon C_k(\mathcal{K},\mathbb{Z}) \longrightarrow C_{k-1}(\mathcal{K},\mathbb{Z}).$$

By the last line before Definition 3.1, we have $\partial^2 = 0$; therefore, the so-called simplicial homology groups

$$H_k^{\text{simp}}(\mathcal{K},\mathbb{Z}) := \frac{Z_k(\mathcal{K},\mathbb{Z}) := \ker\big(\partial\colon C_k(\mathcal{K},\mathbb{Z}) \longrightarrow C_{k-1}(\mathcal{K},\mathbb{Z})\big)}{B_k(\mathcal{K},\mathbb{Z}) := \text{Image}\big(\partial\colon C_{k+1}(\mathcal{K},\mathbb{Z}) \longrightarrow C_k(\mathcal{K},\mathbb{Z})\big)}, \qquad \forall\ k \geq 0,$$

are well-defined. Replacing $\mathbb{Z}$ with $\mathbb{R}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{C}$, or any other ring/field, we get simplicial homology with values in that ring/field.

Even though singular homology is defined for an arbitrary topological space and simplicial homology is defined for simplicial complexes, for a manifold M with a triangulation $\mathcal{K}_M$, these homology groups are the same:

$$H_k^{\text{simp}}(\mathcal{K}_M,\mathbb{Z}) \cong H_k^{\text{sing}}(M,\mathbb{Z}) \qquad \forall\ k \geq 0.$$

In particular, $H_k^{\text{sing}}(M,\mathbb{Z}) = 0$ for all $k > \text{dim}M$. The isomorphism above means that instead of considering all the singular k-simplices $\sigma\colon \Delta \longrightarrow M$, we can only restrict to those k-simplices σ that are the inclusion map of some $\Delta \in \mathcal{K}_M$ into M.

Example 3.3. We can give $M = S^1$ the structure of a pure 1-complex with one 1-simplex $I = [0,1]$ and one zero-complex $p = \{1\} \cong -\{0\}$. Therefore, $C_1 = \mathbb{Z}\cdot I$, $C_0 = \mathbb{Z}\cdot p$, and

$$\partial\colon C_1 \longrightarrow C_0, \qquad I \longrightarrow \{1\} - \{0\} = p - p = 0.$$

We conclude that $H_1^{\text{simp}}(S^1,\mathbb{Z}) \cong \mathbb{Z}$ is generated by the homology class of 1-cycle I, and $H_0^{\text{simp}}(S^1,\mathbb{Z}) \cong \mathbb{Z}$ is generated by the homology class of 0-cycle p.

HW 3.4. Calculate the homology groups of the Klein bottle.

We complete the discussion of this section by reviewing the definition of CW-complexes and cellular homology. The concept of CW complex was introduced Whitehead to meet the needs of homotopy theory. This class of topological spaces is broader and has some better categorical properties than simplicial complexes, but still retains a combinatorial nature that allows for computation (often with a much smaller complex). For each $k \geq 0$, a k-cell is a k-dimensional open ball in $\mathbb{R}^k$. A 0-dimensional CW complex is a topological space with discrete topology. A k-dimensional CW complex is constructed, inductively, by gluing the boundaries of a number of k-cells to a $(k-1)$-dimensional CW complex. The topology of the resulting k-dimensional CW complex is the quotient topology defined by these gluing maps. Since any open ball in $\mathbb{R}^k$ is homeomorphic to $\mathbb{R}^k$, $\mathbb{R}^k$ is a k-dimensional CW complex with only one k-cell. The k-skeleton $\mathcal{C}^{(k)}$ of a CW complex $\mathcal{C}$ is the union of all of its k-cells. Since the interior of every k-simplex is a k-cell, every simplicial complex $\mathcal{K}$ is naturally a CW complex (which we will denote by

$\mathcal{C}_{\mathcal{K}}$). The converse is not true, the gluing maps of the inductive construction above can be quite complicated. In conclusion, we have the following hierarchy:

smooth manifolds $\subsetneq$ simplicial complexes $\subsetneq$ CW complexes $\subsetneq$ "nice" topological spaces.

Suppose $\mathcal{C}$ is a CW complex. Every k-cell $e \in \mathcal{C}^{(k)}$ comes with a gluing map

$$f\colon \partial^{\mathrm{top}} e = S^{k-1} \longrightarrow \mathcal{C}_{k-1}, \tag{3.1}$$

where $\mathcal{C}_k$ is the topological space made of all ℓ-cells with $\ell \leq k$ and $\partial^{top} e$ is the topological boundary[9] of e. Let $\mathcal{C}_{k-1}/\mathcal{C}_{k-2}$ denote the topological space obtained by collapsing $\mathcal{C}_{k-2}$ into a point (we don't go into the details of this) and $\pi\colon \mathcal{C}_{k-1} \longrightarrow \mathcal{C}_{k-1}/\mathcal{C}_{k-2}$ denote the corresponding projection map. If we collapse the boundary of an n-cell into a point we obtain S^n. Therefore, one should think of $\mathcal{C}_{k-1}/\mathcal{C}_{k-2}$ as a bucket of $(k-1)$-spheres attached to each other at a point. Each $(k-1)$-sphere $[e]$ in this bucket correspond to a $(k-1)$-cell e in $\mathcal{C}^{(k)}$. If $\mathcal{C}^{(n)} = \{e_\alpha\}_{\alpha \in \mathcal{I}_n}$, for each $\alpha \in \mathcal{I}_n$, by further collapsing all the spheres $[e_\beta]$, with $\beta \in \mathcal{I} - \{\alpha\}$, we obtain a projection map $\pi_\alpha\colon \mathcal{C}_n/\mathcal{C}_{n-1} \longrightarrow [e_\alpha] \cong S^n$.

For $n \geq 0$, let $C_n(\mathcal{C}, \mathbb{Z})$ denote the free abelian group generated (over $\mathbb{Z}$) by all n-cells in $\mathcal{C}^{(n)}$ (together with a choice of orientation on each n-cell); i.e.

$$C_n(\mathcal{C}, \mathbb{Z}) \cong \mathbb{Z}^{\mathcal{I}_n}.$$

For each oriented k-cell e, the boundary $(k-1)$-chain $\partial e \in C_{k-1}(\mathcal{C}, \mathbb{Z})$ is defined in the following way. With notation as in (3.1), for each $\alpha \in \mathcal{I}_{k-1}$, twe have a continuous map between $(k-1)$-spheres

$$\partial^{\mathrm{top}} e = S^{k-1} \xrightarrow{f} \mathcal{C}_{k-1} \xrightarrow{\pi} \mathcal{C}_{k-1}/\mathcal{C}_{k-2} \xrightarrow{\pi_\alpha} [e_\alpha] \cong S^{k-1}.$$

Let $d_\alpha \in \mathbb{Z}$ denote the degree of this map (with respect to the pre-determined orientations on the cells). We define

$$\partial e = \sum_{\alpha \in \mathcal{I}_{k-1}} d_\alpha e_\alpha.$$

The boundary operator δ linearly extends to all k-chains in $C_k(\mathcal{C}, \mathbb{Z})$:

$$\partial\colon C_k(\mathcal{C}, \mathbb{Z}) \longrightarrow C_{k-1}(\mathcal{C}, \mathbb{Z}).$$

For the same reason as before we have $\partial^2 = 0$; therefore, the so-called cellular homology groups

$$H_k^{\mathrm{cell}}(\mathcal{C}, \mathbb{Z}) := \frac{Z_k(\mathcal{C}, \mathbb{Z}) := \ker\big(\partial\colon C_k(\mathcal{C}, \mathbb{Z}) \longrightarrow C_{k-1}(\mathcal{C}, \mathbb{Z})\big)}{B_k(\mathcal{C}, \mathbb{Z}) := \mathrm{Image}\big(\partial\colon C_{k+1}(\mathcal{C}, \mathbb{Z}) \longrightarrow C_k(\mathcal{C}, \mathbb{Z})\big)}, \qquad \forall\, k \geq 0,$$

are well-defined. Replacing $\mathbb{Z}$ with $\mathbb{R}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{C}$, or any other ring/field, we get simplicial homology with values in that ring/field.

Once again, every smooth manifold M has the structure of a CW complex $\mathcal{C}_M$ (for example one arising from a triangulation $\mathcal{K}_M$) and

$$H_k^{\mathrm{cell}}(\mathcal{C}_M, \mathbb{Z}) \cong H_k^{\mathrm{simp}}(\mathcal{K}_M, \mathbb{Z}) \cong H_k^{\mathrm{sing}}(M, \mathbb{Z}) \qquad \forall\, k \geq 0.$$

For this reason, we simply denote these homology groups by $H_*(\mathcal{C}_M, \mathbb{Z})$. In some cases such as in the example bellow, cellular homology is more convenient for calculations.

[9] We are writing $\partial^{\mathrm{top}} e$ instead of just ∂e to distinguish it from the chain map defined below.

Remark 3.5. There is a different way of realizing a smooth manifold M as a CW complex using a Morse function. We will explain this in details in Section 3.4.

Example 3.6. For each $m \geq 0$, the real projective space $\mathbb{RP}^m$ has the structure of a CW complex $\mathcal{C}_{\text{std}}$ made of exactly one cell in each dimension $0 \leq k \leq m$. The gluing map of the k-th cell e_k to $\mathcal{C}_{k-1} = \mathbb{RP}^{k-1}$ is the $(2:1)$-covering map $S^{k-1} \longrightarrow \mathbb{RP}^{k-1}$. The cells can be oriented in a way that the boundary maps of the chain complex

$$0 \longrightarrow C_m(\mathcal{C}_{\text{std}}, \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(\mathcal{C}_{\text{std}}, \mathbb{Z}) \cong \mathbb{Z} \longrightarrow 0$$

are equal to

$$\partial = (1 + (-1)^k)\colon C_k(\mathcal{C}_{\text{std}}, \mathbb{Z}) \cong \mathbb{Z} \longrightarrow C_{k-1}(\mathcal{C}_{\text{std}}, \mathbb{Z}) \cong \mathbb{Z}.$$

In other words, we have

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \longrightarrow 0$$

We conclude that

$$H_i(\mathbb{RP}^m, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \mathbb{Z} & \text{if } i = m \text{ and } m \text{ is odd}, \\ \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} & \text{if } 0 \leq i \leq m,\ i = \text{odd} \\ 0 & \text{otherwise.} \end{cases}$$

For each $m \geq 0$, the complex projective space $\mathbb{CP}^m$ has the structure of a CW complex $\mathcal{C}_{\text{std}}$ made of exactly one cell in each even dimension $0 \leq 2k \leq 2m$. The gluing map of the $2k$-th cell e_{2k} to $\mathcal{C}_{2k-1} = \mathcal{C}_{2k-2} = \mathbb{CP}^{k-1}$ is the projection map $S^{2k-1} \longrightarrow \mathbb{CP}^{k-1}$. The latter is a fiber bundle with S^1-fibers. It follows that the boundary maps in

$$0 \longrightarrow C_{2m}(\mathcal{C}_{\text{std}}, \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\partial} C_{2m-1}(\mathcal{C}_{\text{std}}, \mathbb{Z}) \cong 0 \xrightarrow{\partial} C_{2m-2}(\mathcal{C}_{\text{std}}, \mathbb{Z}) \cong \mathbb{Z} \cdots$$

are all trivial and

$$H_i(\mathbb{CP}^m, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 2k,\ 0 \leq k \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.7. Suppose M is a manifold and $f\colon M' \longrightarrow M$ is a continuous map from a closed oriented n-dimensional manifold M' into M. Then f defines a homology class $[f] \in H_n(M, \mathbb{Z})$ in the following way. Fix a triangulation $M' = \bigcup_{\alpha \in \mathcal{I}} \Delta_\alpha$ of M' which gives M' the structure of a pure n-dimensional simplicial complex $\mathcal{K}_{M'}$ with oriented n-simplices $\{\Delta_\alpha\}_{\alpha \in \mathcal{I}}$. Note that $\mathcal{I}$ is finite because M' is compact. Also, because $\partial M = \emptyset$, every $(n-1)$-simplex in $\mathcal{K}_{M'}$ either connects two different Δ_α or it appears as two boundary components of the same Δ_α attached to each other. The last sentence implies that

$$\partial \sum_{\alpha \in \mathcal{I}} \sigma_\alpha = 0,$$

where $\sigma_\alpha = f|_{\Delta_\alpha}\colon \Delta_\alpha \longrightarrow M$. Therefore, $\sum_{\alpha \in \mathcal{I}} \sigma_\alpha$ defines a homology class $[f] \in H_n(M, \mathbb{Z})$. In particular, every closed oriented n-dimensional submanifold $M' \subset M$ has a well-defined homology class $[M'] \subset H_n(M, \mathbb{Z})$.

Suppose $f\colon M \longrightarrow M'$ is a continuous map between two topological spaces. For each singular n-simplex $\sigma\colon \Delta \longrightarrow M$, the composition $f\circ\sigma\colon \Delta \longrightarrow M'$ is a singular n-simplex for M'. Therefore, for each $n \geq 0$, composition with f defines a push-forward map

$$f_*\colon C_n(M,\mathbb{Z}) \longrightarrow C_n(M',\mathbb{Z}). \tag{3.2}$$

It is easy to see that f_* commutes with the boundary operators of singular homology on M and M', i.e. the following diagram

$$\begin{tikzcd} \cdots \arrow[r] \arrow[d, "f_*"] & C_n(M,\mathbb{Z}) \arrow[r, "\partial"] \arrow[d, "f_*"] & C_{n-1}(M,\mathbb{Z}) \arrow[r] \arrow[d, "f_*"] & \cdots \arrow[d, "f_*"] \\ \cdots \arrow[r] & C_n(M',\mathbb{Z}) \arrow[r, "\partial"] & C_{n-1}(M',\mathbb{Z}) \arrow[r] & \cdots \end{tikzcd} \tag{3.3}$$

is commutative. In situations like this we say the f_* is a "chain map" between the chain complexes $(C_\bullet(M,\mathbb{Z}),\partial)$ and $(C_\bullet(M',\mathbb{Z}),\partial)$. A simple diagram chasing shows that every chain map induces a (similarly denote) map between the homology groups. In our case, f induces group homomorphisms

$$f_*\colon H_n^{\text{sing}}(M,\mathbb{Z}) \longrightarrow H_n^{\text{sing}}(M',\mathbb{Z}) \quad \forall\, n\in\mathbb{Z}.$$

In passing to homology, a lot of information about f is lost. Therefore, a natural question to ask is: *when two different chain maps induce the same group homomorphisms?*

For abstract chain maps

$$f \equiv (f_n)_{n\in\mathbb{Z}},\;\; g \equiv (g_n)_{n\in\mathbb{Z}}\colon (C_*,\partial) \longrightarrow (C'_*,\partial'),$$

a chain homotopy is a collection of degree increasing maps $\{h_n\}$ as in the following diagram

$$\begin{tikzcd} \cdots \arrow[r] \arrow[d] & C_{n+1}, \arrow[r, "\partial"] \arrow[d, "g_{n+1}"', "f_{n+1}"] & C_n \arrow[r, "\partial"] \arrow[d, "g_n"', "f_n"] \arrow[dl, "h_n"] & C_{n-1} \arrow[r] \arrow[d, "g_{n-1}"', "f_{n-1}"] \arrow[dl, "h_{n-1}"] & \cdots \arrow[d] \\ \cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'"] & C'_n \arrow[r, "\partial'"] & C'_{n-1} \arrow[r] & \cdots \end{tikzcd},$$

such that

$$\partial' \circ h_n + h_{n-1}\partial = f_n - g_n \qquad \forall\, n\in\mathbb{Z}.$$

Again, a diagram chasing argument shows that chain homotopic chains maps f and g induce the same maps between homology groups.

Warning. The last diagram is not commutative!

Back to the chain maps (3.3) induced by continuous functions, we say $f, g\colon M \longrightarrow M'$ are homotopic if there exists a continuous function

$$h\colon [0,1]\times M \longrightarrow M$$

such that $h|_{\{0\}\times M} = g$ and $h|_{\{1\}\times M} = f$. A topological homotopy h as above induces an algebraic homotopy between

$$f_*,\; g_*\colon (C_\bullet(M,\mathbb{Z}),\partial) \longrightarrow (C_\bullet(M',\mathbb{Z}),\partial),$$

in the following way. For each oriented singular n-simplex $\sigma\colon \Delta \longrightarrow M$, and any natural decomposition of $[0,1]\times\Delta$ into a union of oriented (with product orientation) $(n+1)$-simplices, the map $h\circ(\mathrm{id}\times\sigma)\colon [0,1]\times\Delta \longrightarrow M'$ can be seen as singular $(n+1)$-chain in $C_{n+1}(M',\mathbb{Z})$. This linearly extends to all singular chains,

$$h_*\colon C_n(M,\mathbb{Z}) \longrightarrow C_{n+1}(M',\mathbb{Z}).$$

An oriented comparison of boundary components shows that h_* is a chain homotopy between f_* and g_*. We conclude that if $f,g\colon M\longrightarrow M'$ are homotopic, their induced maps on homology groups

$$f_*, g_*\colon H_k(M,\mathbb{Z}) \longrightarrow H_k(M',\mathbb{Z}) \qquad \forall\ k\geq 0 \tag{3.4}$$

are the same.

We say topological spaces M and M' are homotopic if there are continuous maps

$$f\colon M\longrightarrow M' \quad \text{and} \quad f'\colon M'\longrightarrow M$$

such that $g\circ f$ is homotopic to id_M and $f\circ g$ is homotopic to $\mathrm{id}_{M'}$.

HW 3.8. Show that homotopic topological spaces have identical singular homology groups.

Example 3.9. Show that $\mathbb{R}^m$ is homotopic to a point. Conclude that

$$H_i(\mathbb{R}^m,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i=0,\\ 0 & \text{otherwise.}\end{cases}$$

More generally, prove that if $E\longrightarrow M$ is a vector bundle, then

$$H_i(E,\mathbb{Z}) \cong H_i(M,\mathbb{Z}) \qquad \forall\ i\in\mathbb{N}.$$

Suppose $M=U_1\cup U_2$ is a decomposition of a topological space M into two open sets. We discuss a relation between the singular homology of M and the subsets U_1 and U_2, known as Mayer-Vietoris sequence. This enables a break down of the homology of M into simpler pieces. The Mayer-Vietoris sequence holds for a variety of (co)homology theories, including singular homology and simplicial homology, and de Rham cohomology. In general, the sequence holds for those theories satisfying the Eilenberg-Steenrod axioms. Let

$$\iota_a\colon U_a\longrightarrow M \quad \text{and} \quad j_a\colon U_{12}:=U_1\cap U_2\longrightarrow U_a, \qquad a=1,2,$$

denote the inclusion maps. Every singular n-chain in U_1 or U_2 can be seen as a singular chain in M; thus, we get push-forward inclusion maps

$$\iota_{a*}\colon C_n(U_a,\mathbb{Z}) \longrightarrow C_n(M,\mathbb{Z}) \qquad \forall\ n\geq 0,\ a=1,2.$$

Likewise, we get inclusion maps

$$j_{a*}\colon C_n(U_{12},\mathbb{Z}) \longrightarrow C_n(U_a,\mathbb{Z}) \qquad \forall\ n\geq 0,\ a=1,2.$$

The following sequence is exact

$$0\longrightarrow C_n(U_{12},\mathbb{Z}) \xrightarrow{j_*:=j_{1*}\oplus j_{2*}} C_n(U_1,\mathbb{Z})\oplus C_n(U_2,\mathbb{Z}) \xrightarrow{\iota_*:=\iota_{1*}\oplus-\iota_{2*}} C_n(M,\mathbb{Z})\longrightarrow 0; \tag{3.5}$$

i.e. j_* is injective, ι_* is surjective, and $\ker(\iota_*) = \text{image}(j_*)$. By letting n vary, we get the following commutative diagram:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & C_n(U_{12},\mathbb{Z}) & \xrightarrow{\quad\partial\quad} & C_{n-1}(U_{12},\mathbb{Z}) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} & \\
\cdots \longrightarrow & C_n(U_1,\mathbb{Z}) \oplus C_n(U_2,\mathbb{Z}) & \xrightarrow{\partial\oplus\partial} & C_{n-1}(U_1,\mathbb{Z}) \oplus C_{n-1}(U_2,\mathbb{Z}) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & \\
\cdots \longrightarrow & C_n(M,\mathbb{Z}) & \xrightarrow{\quad\partial\quad} & C_{n-1}(M,\mathbb{Z}) & \longrightarrow \cdots
\end{array}$$

which is a short exact sequence of chain complexes

$$0 \longrightarrow \Big(C_\bullet(U_{12},\mathbb{Z}),\partial\Big) \xrightarrow{j_*} \Big(C_\bullet(U_1,\mathbb{Z}) \oplus C_\bullet(U_2,\mathbb{Z}),\partial\oplus\partial\Big) \xrightarrow{\iota_*} \Big(C_\bullet(M,\mathbb{Z}),\partial\Big) \longrightarrow 0.$$

In general, a short exact sequence of chain complexes gives rise to a long exact sequence of the associated homology groups

$$\cdots H^{\text{sing}}_{n+1}(M,\mathbb{Z}) \xrightarrow{\delta_*} H^{\text{sing}}_n(U_{12},\mathbb{Z}) \xrightarrow{j_*} H^{\text{sing}}_n(U_1,\mathbb{Z}) \oplus H^{\text{sing}}_n(U_2,\mathbb{Z}) \xrightarrow{\iota_*} H^{\text{sing}}_n(M,\mathbb{Z}) \longrightarrow \cdots$$

where ι_* and j_* are the similarly denoted maps induced by (3.10) and δ_* are the connecting homomorphisms. This is known as the Mayer-Vietoris sequence for singular homology. In the case of singular homology, δ_* has the following explicit definition. Each cohomology class in $H^{\text{sing}}_{n+1}(M,\mathbb{Z})$ can be represented by a singular $(n+1)$-chain

$$\omega = \sum_{i\in\mathcal{I}} (\sigma_i \colon \Delta_i \longrightarrow M)$$

such that each σ_i has image in either U_1 or U_2. Arbitrarily divide $\mathcal{I}$ into a disjoint union of $\mathcal{I}_1$ and $\mathcal{I}_2$ such that

$$\omega_1 = \sum_{i\in\mathcal{I}_1} (\sigma_i \colon \Delta_i \longrightarrow U_1) \qquad \text{and} \qquad \omega_2 = \sum_{i\in\mathcal{I}_2} (\sigma_i \colon \Delta_i \longrightarrow U_2)$$

are singular $(n+1)$-chains in U_1 and U_2, respectively. Since $\partial\omega = 0$, we conclude that $\partial\omega_1 = -\partial\omega_2 \in C_{n-1}(U_1\cap U_2,\mathbb{Z})$. Since $\partial\circ\partial = 0$, ∂w_1 defines a homology class $[\omega_1]$ in $H_{n-1}(U_1\cap U_2,\mathbb{Z})$. We define $\delta_*([w]) = [\omega_1]$.

HW 3.10. Check that the homology class $[\omega_1]$ is independent of the choices involved.

Example 3.11. Consider the standard covering $S^2 = U_1\cup U_2$ with two open disks. The annulus $U_1 \cap U_2 \cong S^1 \times [0,1]$ deformation retracts to S^1 so it has the same homology groups as S^1. Each U_i deformation retracts to a point so it its homology is concentrated at degree 0. For this decomposition, the Mayer-Vietoris sequence

$$\begin{aligned}
0 \longrightarrow & H_2(S^1,\mathbb{Z}) \longrightarrow H_2(U_1,\mathbb{Z}) \oplus H_2(U_2,\mathbb{Z}) \longrightarrow H_2(S^2,\mathbb{Z}) \longrightarrow \\
& H_1(S^1,\mathbb{Z}) \longrightarrow H_1(U_1,\mathbb{Z}) \oplus H_1(U_2,\mathbb{Z}) \longrightarrow H_1(S^2,\mathbb{Z}) \longrightarrow \\
& H_0(S^1,\mathbb{Z}) \xrightarrow{j_*} H_0(U_1,\mathbb{Z}) \oplus H_0(U_2,\mathbb{Z}) \longrightarrow H_0(S^2,\mathbb{Z}) \longrightarrow 0\ .
\end{aligned}$$

reads

$$\begin{aligned}0 \longrightarrow 0 \longrightarrow 0\oplus 0 \longrightarrow H_2(S^2,\mathbb{Z}) \longrightarrow \\ \mathbb{Z} \longrightarrow 0\oplus 0 \longrightarrow H_1(S^2,\mathbb{Z}) \longrightarrow 0\\ \mathbb{Z} \longrightarrow \mathbb{Z}\oplus\mathbb{Z} \longrightarrow H_0(S^2,\mathbb{Z}) \longrightarrow 0\ .\end{aligned}$$

Therefore, $H_2(S^2,\mathbb{Z}) = \mathbb{Z}$. Since $H_0(S^1,\mathbb{Z}) \xrightarrow{j_*} H_0(U_1,\mathbb{Z})$ is injective, we also conclude that $H_1(S^2,\mathbb{Z}) = 0$.

Remark 3.12. Similarly, if $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$ is a decomposition of a simplicial complex $\mathcal{K}$ into two sub-simplicial complexes, we get a Mayer-Vietoris long exact sequence

$$\cdots H^{\text{simp}}_{n+1}(\mathcal{K},\mathbb{Z}) \xrightarrow{\delta_*} H^{\text{simp}}_n(\mathcal{K}_1\cap\mathcal{K}_2,\mathbb{Z}) \xrightarrow{j_*} H^{\text{simp}}_n(\mathcal{K}_1,\mathbb{Z})\oplus H^{\text{simp}}_n(\mathcal{K}_2,\mathbb{Z}) \xrightarrow{\iota_*} H^{\text{simp}}_n(\mathcal{K},\mathbb{Z}) \longrightarrow \cdots\ .$$

Remark 3.13. By taking the dual space of the chain complexes above, we can define singular, simplicial, and cellular cohomology (instead of homology). These all are the same when M is a smooth manifold. For example, the cochain complex of the singular cohomology is the cochain complex

$$\cdots \longrightarrow C^k(M,\mathbb{Z}) \xrightarrow{\partial^*} C^{k+1}(M,\mathbb{Z}) \longrightarrow$$

where

$$C^k(M,\mathbb{Z}) \coloneqq \text{Hom}(C_k(M,\mathbb{Z}),\mathbb{Z}) \qquad \forall\ k\geq 0,$$

and ∂^* is the dual of ∂. Changing $\mathbb{Z}$ with $\mathbb{R}$ or any other coefficient ring, we can defined singular/simplicial/cellular cohomology with coefficients in that ring.

3.2 de Rham cohomology; take two

We defined de Rham cohomology of a smooth m-manifold M in Section 2.5 as the cohomology of the cochain complex

$$0 \longrightarrow \Omega^0(M,\mathbb{R}) \xrightarrow{\text{d}} \cdots \xrightarrow{\text{d}} \Omega^{m-1}(M,\mathbb{R}) \xrightarrow{\text{d}} \Omega^m(M,\mathbb{R}) \longrightarrow 0.$$

If M is not compact, we can additionally consider the cochain complex

$$0 \longrightarrow \Omega^0_c(M,\mathbb{R}) \xrightarrow{\text{d}} \cdots \xrightarrow{\text{d}} \Omega^{m-1}_c(M,\mathbb{R}) \xrightarrow{\text{d}} \Omega^m_c(M,\mathbb{R}) \longrightarrow 0,$$

where $\Omega^k_c(M,\mathbb{R})$ is the space of compactly supported differential k-forms on M. We denote the corresponding cohomology groups by

$$H^k_{c,\text{dR}}(M,\mathbb{R}) \coloneqq \frac{\ker\big(\text{d}\colon \Omega^k_c(M,\mathbb{R}) \longrightarrow \Omega^{k+1}_c(M,\mathbb{R})\big)}{\text{Image}\big(\text{d}\colon \Omega^{k-1}_c(M,\mathbb{R}) \longrightarrow \Omega^k_c(M,\mathbb{R})\big)}.$$

The question is: *for a non-compact manifold M, how does $H^k_{c,\text{dR}}(M,\mathbb{R})$ compare to $H^k_{\text{dR}}(M,\mathbb{R})$?*

We answer this question under the assumption that M is oriented and does not have a boundary. Then, for each $0\leq k\leq m$, consider the bilinear map

$$\langle -,-\rangle : \Omega^k_c(M,\mathbb{R})\times\Omega^{m-k}(M,\mathbb{R}) \longrightarrow \mathbb{R}, \qquad (\alpha,\beta)\longrightarrow \langle\alpha,\beta\rangle \coloneqq \int_M \alpha\wedge\beta. \tag{3.6}$$

The integral on right is finite because $\alpha\wedge\beta$ is compactly supported. If $\alpha = \text{d}\eta$, since $\partial M = \emptyset$ and $\alpha\wedge\beta = \text{d}(\eta\wedge\beta)$, it follows from Stokes' Theorem that $\langle\alpha,\beta\rangle = 0$. Similarly, if $\beta = \text{d}\eta$, then $\langle\alpha,\beta\rangle = 0$. We conclude that (3.6) descends to a bilinear map

$$\langle -,-\rangle : H^k_{c,\text{dR}}(M,\mathbb{R})\times H^{m-k}_{\text{dR}}(M,\mathbb{R}) \longrightarrow \mathbb{R}. \tag{3.7}$$

Definition 3.14. An open cover $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ of an m-manifold M is called a *good cover* if every nonempty finite intersection $U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}$ is diffeomorphic to $\mathbb{R}^m$. A manifold which has a finite good cover is said to be of finite type.

Note that, by definition, every manifold admitting a good cover is without boundary. We can make a similar definition for manifolds with boundary.

Every smooth manifold has a good cover where U_α are sufficiently small geodetically convex balls as in (2.38). Every closed manifold has finite type. Being of finite type, for instance, implies that the (co)-homology groups of M are finite. We will encounter a similar condition in Morse homology. The goal is to avoid bad examples like Figure 3.

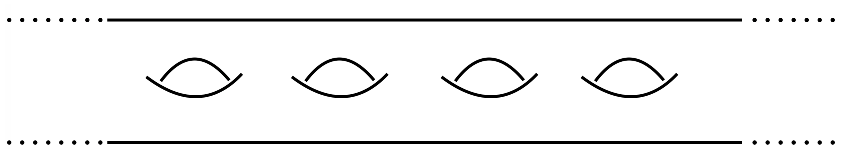


Figure 3: An open surface of infinite genus

Theorem 3.15. *Suppose M is an oriented smooth m-manifold of finite type. Then, the pairing (3.7) is non-degenerate; i.e.*

- $\langle[\alpha],-\rangle=0\Rightarrow[\alpha]=0\in H^k_{c,\mathrm{dR}}(M,\mathbb{R})$;
- $\langle-,[\beta]\rangle=0\Rightarrow[\beta]=0\in H^{m-k}_{\mathrm{dR}}(M,\mathbb{R})$.

Theorem 3.15 is equivalent to

$$H^k_{c,\mathrm{dR}}(M,\mathbb{R})\cong H^{m-k}_{\mathrm{dR}}(M,\mathbb{R})^*; \tag{3.8}$$

i.e. $H^k_{c,\mathrm{dR}}(M,\mathbb{R})$ is canonically isomorphic to the dual space

$$H^{m-k}_{\mathrm{dR}}(M,\mathbb{R})^*:=\mathrm{Hom}(H^{m-k}_{\mathrm{dR}}(M,\mathbb{R}),\mathbb{R})$$

of $H^{m-k}_{\mathrm{dR}}(M,\mathbb{R})$. Since every real vector space is (non-canonically) isomorphic to its dual, we conclude that

$$H^k_{c,\mathrm{dR}}(M,\mathbb{R})\cong H^{m-k}_{\mathrm{dR}}(M,\mathbb{R}). \tag{3.9}$$

The isomorphism (3.8) is known as Poincare duality. Proof of 3.15 uses a local statement, Mayer-Vietoris long-exact sequence, and the five-lemma. First, we first explain the Mayer-Vietoris sequences for de Rham and compactly-supported de Rham cohomologies. Then, we prove the necessary local statements and prove Theorem 3.15.

Suppose $M=U_1\cup U_2$ is a decomposition of a smooth manifold M into two open sets. The inclusion maps

$$\iota_a\colon U_a\longrightarrow M\quad\text{and}\quad j_a\colon U_{12}:=U_1\cap U_2\longrightarrow U_a,\qquad a=1,2,$$

give rise to pull-back maps

$$\iota_a^*\colon\Omega^n(M,\mathbb{R})\longrightarrow\Omega^n(U_a,\mathbb{R})\quad\text{and}\quad j_a^*\colon\Omega^n(U_a,\mathbb{R})\longrightarrow\Omega^n(U_{12},\mathbb{R}).$$

The following sequence is exact

$$0 \longrightarrow \Omega^n(M,\mathbb{R}) \xrightarrow{\iota^*:=\iota_1^*\oplus\iota_2^*} \Omega^n(U_1,\mathbb{R}) \oplus \Omega^n(U_2,\mathbb{R}) \xrightarrow{j^*:=j_1^*\oplus -j_2^*} \Omega^n(U_{12},\mathbb{R}) \longrightarrow 0. \tag{3.10}$$

By letting n vary, we get a short exact sequence of cochains

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \Omega^n(M,\mathbb{R}) & \xrightarrow{\mathrm{d}} & \Omega^{n+1}(M,\mathbb{R}) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} & & \\
\cdots & \longrightarrow & \Omega^n(U_1,\mathbb{R}) \oplus \Omega^n(U_2,\mathbb{R}) & \xrightarrow{\mathrm{d}\oplus\mathrm{d}} & \Omega^{n+1}(U_1,\mathbb{R}) \oplus \Omega^{n+1}(U_2,\mathbb{R}) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} & & \\
\cdots & \longrightarrow & \Omega^n(U_{12},\mathbb{R}) & \xrightarrow{\mathrm{d}} & \Omega^{n+1}(U_{12},\mathbb{R}) & \longrightarrow & \cdots
\end{array}$$

which we briefly write as

$$0 \longrightarrow \Big(\Omega^\bullet(M,\mathbb{R}),\mathrm{d}\Big) \xrightarrow{\iota^*} \Big(\Omega^\bullet(U_1,\mathbb{R}) \oplus \Omega^\bullet(U_2,\mathbb{R}),\mathrm{d}\oplus\mathrm{d}\Big) \xrightarrow{j^*} \Big(\Omega^\bullet(U_{12},\mathbb{R}),\mathrm{d}\Big) \longrightarrow 0.$$

Like before, this short exact sequence of cochain complexes gives us a Mayer-Vietoris long exact sequence of de Rham cohomology groups

$$\cdots H^{n-1}_{\mathrm{dr}}(U_{12},\mathbb{R}) \xrightarrow{\delta^*} H^n_{\mathrm{dr}}(M,\mathbb{R}) \xrightarrow{\iota^*} H^n_{\mathrm{dr}}(U_1,\mathbb{R}) \oplus H^n_{\mathrm{dr}}(U_2,\mathbb{R}) \xrightarrow{j^*} H^n_{\mathrm{dr}}(U_{12},\mathbb{R}) \longrightarrow \cdots .$$

For the compactly supported version, instead of the pull back maps ι^* and j^*, we have the inclusion maps

$$\iota_{a*}\colon \Omega^n_c(U_a,\mathbb{R}) \longrightarrow \Omega^n_c(M,\mathbb{R}) \quad \text{and} \quad j_{a*}\colon \Omega^n(U_{12},\mathbb{R}) \longrightarrow \Omega^n(U_a,\mathbb{R}), \quad \forall\ a=1,2,\ n\in\mathbb{N}.$$

Consequently, we get a Mayer-Vietoris sequence where the order if terms is reversed (but the sequence is still degree-increasing):

$$\cdots H^{n-1}_{c,\mathrm{dr}}(M,\mathbb{R}) \xrightarrow{\delta_*} H^n_{c,\mathrm{dr}}(U_{12},\mathbb{R}) \xrightarrow{j_*} H^n_{c,\mathrm{dr}}(U_1,\mathbb{R}) \oplus H^n_{c,\mathrm{dr}}(U_2,\mathbb{R}) \xrightarrow{\iota_*} H^n_{c,\mathrm{dr}}(M,\mathbb{R}) \longrightarrow \cdots .$$

This is compatible with (3.8) since Poincare duality reverses the degrees and arrows:

$$\begin{array}{ccccccc}
H^{n-1}_{\mathrm{dr}}(U_{12},\mathbb{R}) & \xrightarrow{\delta^*} & H^n_{\mathrm{dr}}(M,\mathbb{R}) & \xrightarrow{\iota^*} & H^n_{\mathrm{dr}}(U_1,\mathbb{R}) \oplus H^n_{\mathrm{dr}}(U_2,\mathbb{R}) & \xrightarrow{j^*} & H^n_{\mathrm{dr}}(U_{12},\mathbb{R})^* \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
H^{m-n+1}_{c,\mathrm{dr}}(U_{12},\mathbb{R})^* & \xrightarrow{\delta^*} & H^{m-n}_{c,\mathrm{dr}}(M,\mathbb{R})^* & \xrightarrow{\iota^*} & H^{m-n}_{c,\mathrm{dr}}(U_1,\mathbb{R})^* \oplus H^{m-n}_{\mathrm{dr}}(U_2,\mathbb{R})^* & \xrightarrow{j^*} & H^{m-n}_{c,\mathrm{dr}}(U_{12},\mathbb{R})^* .
\end{array}$$

Suppose $E \longrightarrow M$ is a smooth vector bundle and $U \subset E$ is an open neighborhood of M in E. We say U is start-shaped if

$$\forall\ v\in U,\ r\in[0,1] \Rightarrow rv \in U.$$

Proposition 3.16. *Suppose $\pi\colon E \longrightarrow M$ is a smooth rank r real vector bundle and $U \subset E$ is a star-shaped open neighborhood of M in E. Let π_U denote the restriction of π to U. Then for all $p\geq 0$, the pull-back map*

$$\pi_U^*\colon \Omega^p(M,\mathbb{R}) \longrightarrow \Omega^p(U,\mathbb{R})$$

induces an isomorphism

$$H^i_{\mathrm{dR}}(U,\mathbb{R}) \cong H^i_{\mathrm{dR}}(M,\mathbb{R}). \tag{3.11}$$

In particular, for any star-shaped domain in $\mathbb{R}^m$ *we have*

$$H^i_{\mathrm{dR}}(U,\mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } i=0, \\ 0 & \text{otherwise} \end{cases} \tag{3.12}$$

Proposition 3.17. *Suppose* M *is a manifold of finite-type,* $\pi\colon E \longrightarrow M$ *is an oriented smooth rank* r *real vector bundle, and* $U \subset E$ *is a star-shaped open neighborhood of* M *in* E*. Then for all* $p \geq 0$ *there is a natural isomorphism*

$$H^{i+r}_{\mathrm{c,dR}}(U,\mathbb{R}) \cong H^i_{\mathrm{c,dR}}(M,\mathbb{R}) \qquad \forall\, i \in \mathbb{Z}. \tag{3.13}$$

In particular, for any star-shaped domain in $\mathbb{R}^m$ *we have*

$$H^i_{\mathrm{c,dR}}(U,\mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } i=m, \\ 0 & \text{otherwise} \end{cases} \tag{3.14}$$

The identities in (3.12) are known as Poincare Lemma: *any closed* p*-form* η *defined on an star-shaped domain* $U \subset \mathbb{R}^m$ *is exact, for any integer* p *with* $1 \leq p \leq m$*.*

The isomorphism (3.13) is known as Thom isomorphism. There is a class of so-called Thom forms on U such that

(1) e is a closed r-form supported in U,

(2) for every compact set $K \subset U$ the restriction $e|_{\pi^{-1}(K)}$ has compact support,

(3) for every $x \in M$, we have $\int_{\pi^{-1}(x)} e = 1$,

(4) for every $p \in \mathbb{Z}$ the map

$$e \wedge \pi^*\colon \Omega^p_c(M,\mathbb{R}) \longrightarrow \Omega^{p+r}_c(U,\mathbb{R}), \qquad \eta \longrightarrow e \wedge \pi^*\eta \tag{3.15}$$

induces the isomorphism (3.13).

If M is closed, e defines a unique compactly supported cohomology class $[e] \in H^r_{c,\mathrm{dR}}(U,\mathbb{R})$, known as the Thom class, that under the isomorphism (3.13) corresponds to $[1] \in H^0_{\mathrm{dR}}(M,\mathbb{R})$. If M is oriented, we can indirectly prove (3.13) in the following way:

- first, we prove (3.14) (or equally (3.13) when the bundle is trivial);
- then, we use (3.14), (3.12), and Mayer-Vietoris (induction), to prove Theorem 3.15;
- finally, assuming that M is also oriented, we deduce (3.13).

In any case, (3.13) can be proved using Mayer-Vietoris and induction. We will explain these in more details after the proof of (3.15). Here is a short homological description of (3.13) by A. Hatcher using cellular cohomology. The m-manifold M has the structure of an m-dimensional CW complex. One can assume, without loss of generality, that M is a CW complex with a single 0-cell (this can be achieved using an appropriate Morse function).The argument below holds for any such reasonable CW complex M. The Thom space $T(E)$ is the quotient $D(E)/S(E)$ of the unit disk bundle of E by the unit sphere bundle. In other words, $S(E)$ is being collapsed into a point. One can give $T(E)$ a CW structure with $S(E)/S(E)$ as the only 0-cell and with an $(r+k)$-cell for each k-cell of M. These cells in $T(E)$ arise from pulling back k-cells on M to $D(E)$. In particular, $T(E)$ has a single r-cell (corresponding to the 0-cell of M) and an $(r+1)$-cell for each 1-cell of M. There are no cells in $T(E)$ between dimension 0 and r. The cellular boundary of an $(r+1)$-cell is 0 if E is orientable over the corresponding 1-cell of B, and it is twice the n-cell in the opposite case. Thus $H_r^{\text{cell}}(T(E),\mathbb{Z}) \cong \mathbb{Z}$ if E is orientable and 0 if E is non-orientable. In the orientable case a generator of $H_r^{\text{cell}}(T(E),\mathbb{Z})$ restricts to a generator of $H_n(S^r,\mathbb{Z})$ in the "fiber" S^r of $T(E)$ over the 0-cell of M, hence the same is true for all the "fibers" S^r and obtains a homology Thom class. Under a version of Poincare duality between homology and cohomology that we will prove later, this gives the Thom class $[e]$ in $H^r_{c,\text{dR}}(E,\mathbb{R})$.

HW 3.18. Show that the compact cohomology of the Möbius strip is identically zero. Thus, viewing the the Möbius strip as a real line bundle over S^1 gives an un-orientable example where (3.13) fails.

First, we give a direct proof of Poincare Lemma to illustrate the main idea. We then generalize the proof and prove (3.11) using the fact that π is a homotopy equivalence.

Proof of Poincare Lemma. Suppose η is a closed p-form. Consider the family of diffeomorphisms

$$\varphi\colon U\times\mathbb{R}_{\geq 0}\longrightarrow U,\qquad (x,t)\longrightarrow \varphi_t(x)=e^{-t}x.$$

This the non-negative flow of the ODE corresponding to the vector field $\zeta=-(\sum_{i=1}^m x_i\partial_{x_i})$. Let

$$\Omega^p(U,\mathbb{R})\ni \eta_t=(\varphi_t^*\eta)|_U\qquad \forall\ t\geq 0.$$

Note that $\eta_0=\eta$. For $p\geq 1$, check that η_t converges to 0 exponentially fast. By the Fundamental Theorem of Calculus, we have

$$\eta=\eta_0=-\int_0^\infty \dot{\eta}_t|_{t=s}\ \mathrm{d}s.$$

By (2.20), we have

$$\varphi_s^*L_\zeta\eta=\dot{\eta}_t|_{t=s}.$$

Since η is closed, we have $L_\zeta\eta=\mathrm{d}\iota_\zeta\eta$. Therefore, since d and pull-back commutes, we have

$$\eta=-\int_0^\infty \varphi_s^*\mathrm{d}\iota_\zeta\eta\ \mathrm{d}s=-\int_0^\infty \mathrm{d}\varphi_s^*\iota_\zeta\eta\ \mathrm{d}s=\mathrm{d}\Big(-\int_0^\infty \varphi_s^*\iota_\zeta\eta\ \mathrm{d}s\Big).$$

We conclude that η is exact. □

Proof of (3.11). The following proof is a straightforward generalization of the argument above. We aim to show that the co-chain map

$$\pi^*\colon \Big(\Omega^\bullet(M,\mathbb{R}),\mathrm{d}\Big)\longrightarrow\Big(\Omega^\bullet(E,\mathbb{R}),\mathrm{d}\Big)$$

induced by $\pi\colon E\longrightarrow M$ induces the identity map on cohomology. Let $\iota\colon M\longrightarrow E$ denote the inclusion map as the zero section. We already have $\pi\circ\iota=\mathrm{id}_M$. To prove the statement above on cohomology, it is enough to show that

$$\pi^*\circ\iota^*\colon \Big(\Omega^\bullet(E,\mathbb{R}),\mathrm{d}\Big)\longrightarrow\Big(\Omega^\bullet(E,\mathbb{R}),\mathrm{d}\Big)$$

is chain homotopic to the identity map. Suppose η is a p-form on E. Consider the family of diffeomorphisms

$$\varphi\colon U \times \mathbb{R}_{\geq 0} \longrightarrow U, \qquad (v,t) \longrightarrow \varphi_t(v) = e^{-t}v.$$

In a local trivialization $E|_V \cong V \times \mathbb{R}^r$, we have

$$\varphi_t(x,y) = (x, e^{-t}y) \qquad \forall\ (x,y) \in V \times \mathbb{R}^r. \tag{3.16}$$

This the non-negative flow of the ODE corresponding to the vector field

$$\zeta(v) = v \in \pi^*(E) \cong T^{\perp}E \subset TE.$$

Let

$$\Omega^p(U,\mathbb{R}) \ni \eta_t = (\varphi_t^*\eta)|_U \qquad \forall\ t \geq 0.$$

Note that $\eta_0 = \eta$. It easy follows from (3.16) that η_t converges to $\pi^* \circ \iota^*(\eta)$ exponentially fast. By the Fundamental Theorem of Calculus, we have

$$\eta - \pi^* \circ \iota^*(\eta) = -\int_0^\infty \dot{\eta}_t|_{t=s}\ \mathrm{d}s.$$

By (2.20), we have

$$\varphi_s^* L_\zeta \eta = \dot{\eta}_t|_{t=s}.$$

We have $L_\zeta \eta = \mathrm{d}\iota_\zeta \eta + \iota_\zeta \mathrm{d}\eta$. Define

$$K(\eta) = -\int_0^\infty \varphi_s^* \iota_\zeta \eta\ \mathrm{d}s \in \Omega^{p-1}(U,\mathbb{R}) \qquad \forall\ \eta \in \Omega^p(U,\mathbb{R}).$$

Therefore, since d and pull-back commutes, we have

$$\eta - \pi^* \circ \iota^*(\eta) = -\int_0^\infty \varphi_s^*\ \big(\mathrm{d}\iota_\zeta\eta + \iota_\zeta \mathrm{d}\eta\big)\ \mathrm{d}s = -\int_0^\infty \mathrm{d}\varphi_s^*\iota_\zeta\eta\ \mathrm{d}s = \mathrm{d}\,K(\eta) + K\mathrm{d}\eta.$$

Therefore,

$$\mathrm{id} - \pi^* \circ \iota^* = \mathrm{d} \circ K + K \circ \mathrm{d};$$

i.e. $\pi^* \circ \iota^*$ and the identity map are chain homotopic. □

Similarly to (3.2), every smooth map $f\colon M \longrightarrow M'$ induces a cochain map

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \Omega^n(M',\mathbb{R}) & \xrightarrow{\mathrm{d}} & \Omega^{n+1}(M',\mathbb{R}) & \longrightarrow & \cdots \\
\downarrow f^* & & \downarrow f^* & & \downarrow f^* & & \downarrow f^* \\
\cdots & \longrightarrow & \Omega^n(M,\mathbb{R}) & \xrightarrow{\mathrm{d}} & \Omega^{n+1}(M,\mathbb{Z}) & \longrightarrow & \cdots,
\end{array}$$

and, consequently, a collection of pull-back homomorphisms between the corresponding de Rham cohomology groups

$$f^*\colon H^n_{\mathrm{dR}}(M',\mathbb{R}) \longrightarrow H^n_{\mathrm{dR}}(M,\mathbb{R}) \quad \forall\ n \in \mathbb{N}.$$

The following analogue of (3.4) is a corollary of (3.13).

Corollary 3.19. *Suppose $f, g\colon M \longrightarrow M'$ are smoothly homotopic; i.e. there exists*

$$h\colon [0,1]\times M \longrightarrow M'$$

such that $h(0,x)=g(x)$ and $h(1,x)=f(x)$ for all $x\in M$. Then the cochain maps

$$f^*, g^*\colon \big(\Omega^\bullet(M,\mathbb{R}),\mathrm{d}\big) \longrightarrow \big(\Omega^\bullet(M',\mathbb{R}),\mathrm{d}\big)$$

are chain homotopic. Therefore, their induced maps on cohomology are the same.

HW 3.20. Deduce the corollary above from (3.13).

With the same reasoning as in the homological case, we conclude that

- two smooth manifolds M and M' with the same homotopy type have the same de Rham cohomology;
- if the submanifold $A\subset M$ is a deformation retract of M, then A and M have the same de Rham cohomology.

Proof of (3.14). For $r\geq 0$, let

$$\pi\colon \mathbb{R}^{r+1}\longrightarrow \mathbb{R}^r, \qquad (t,x_1,\ldots,x_r)\longrightarrow (x_1,\ldots,x_r).$$

For any star-shaped domain $U\subset\mathbb{R}^{r+1}$ and every $p\in\mathbb{Z}$, we prove

$$H^{p+1}_{c,\mathrm{dR}}(U,\mathbb{R})\cong H^{p}_{c,\mathrm{dR}}(\pi(U),\mathbb{R}).$$

For $p<0$, both sides are trivial. Thus, we may assume $p\geq 0$. Every $\eta\in\Omega^{p+1}_c(U,\mathbb{R})$ has the form

$$\eta = \mathrm{d}t\wedge\alpha+\beta \tag{3.17}$$

where

$$\alpha=\sum_{a:=a_1<\ldots<a_p} f_a(t,x)\;\mathrm{d}x_{a_1}\wedge\ldots\wedge\mathrm{d}x_{a_p} \quad\text{and}\quad \beta=\sum_{b:=b_0<\ldots<b_p} g_b(t,x)\;\mathrm{d}x_{b_0}\wedge\ldots\wedge\mathrm{d}x_{b_p}$$

only involve $\{\mathrm{d}x_i\}_{i=1}^r$. The equation $\mathrm{d}\eta=0$ is equivalent to

$$\mathrm{d}_x\alpha=\dot\beta \qquad\text{and}\qquad \mathrm{d}_x\beta=0,$$

where d_x means exterior derivative with respect to x-coordinates and

$$\dot\beta=\sum_{b:=b_0<\ldots<b_p}\frac{\partial g_b(t,x)}{\partial t}\;\mathrm{d}x_{b_0}\wedge\ldots\wedge\mathrm{d}x_{b_p}$$

In particular,

$$\beta(t,x)=\int_{-\infty}^{t}\mathrm{d}_x\alpha(s,x)\;\mathrm{d}s\,. \tag{3.18}$$

Define

$$\pi_*\colon \Omega^{p+1}_c(U,\mathbb{R})\longrightarrow\Omega^p_c(\pi(U),\mathbb{R}), \qquad \pi_*(\eta)=\overline{\alpha}:=\sum_{a:=a_1<\ldots<a_p}\overline{f_a}(x)\;\mathrm{d}x_{a_1}\wedge\ldots\wedge\mathrm{d}x_{a_p}$$

where

$$\overline{f_a}(x) = \int_{-\infty}^{\infty} f_a(t,x)\mathrm{d}t \qquad \forall\, x \in \pi(U).$$

We aim to show π_* induces an isomorphism at the cohomology level. It is straightforward to check that π_* commutes with d. Therefore, it descends to a homomorphism

$$\pi_*\colon H^{p+1}_{c,\mathrm{dR}}(U,\mathbb{R}) \longrightarrow H^{p}_{c,\mathrm{dR}}(\pi(U),\mathbb{R}). \tag{3.19}$$

Given a closed form $\overline{\alpha}$ with compact support K in $\pi(U)$, choose a sufficiently small $\varepsilon = \varepsilon(\overline{\alpha}) > 0$ such that $(-\varepsilon,\varepsilon)\times K \subset U$. Choose $h = h_{\overline{\alpha}}\colon \mathbb{R} \longrightarrow \mathbb{R}$ with compact support in $(-\varepsilon,\varepsilon)$ such that $\int_{\mathbb{R}} h(t)\mathrm{d}t = 1$. Let

$$\eta = h(t)\mathrm{d}t \wedge \pi^*\overline{\alpha}.$$

Then, η is closed and belongs to $\Omega^{p+1}_c(U,\mathbb{R})$, and $\overline{\alpha} = \pi_*\eta$. Therefore, (3.19) is surjective.

Next, we show that (3.19) is injective. Suppose $\overline{\alpha} = \mathrm{d}\overline{\gamma}$ for some $\overline{\gamma}\in\Omega^{p-1}_c(\pi(U),\mathbb{R})$. Then,

$$\mathrm{d}(h(t)\mathrm{d}t \wedge \pi^*\overline{\gamma}) = h(t)\mathrm{d}t \wedge \pi^*\overline{\alpha}.$$

Therefore, η and $\eta - h(t)\mathrm{d}t\wedge\pi^*\overline{\alpha}$ have the same cohomology class in $H^{p+1}_{c,\mathrm{dR}}(U,\mathbb{R})$. Furthermore,

$$\pi_*(\eta - h(t)\mathrm{d}t \wedge \pi^*\overline{\alpha}) = 0.$$

Therefore, by replacing η with $\eta - h(t)\mathrm{d}t \wedge \pi^*\overline{\alpha}$, we may assume $0 \neq [\eta] \in H^{p+1}_{c,\mathrm{dR}}(U,\mathbb{R})$ and $\overline{\alpha} = \pi_*(\eta) = 0$.

For $\eta = \mathrm{d}t \wedge \alpha + \beta \in \Omega^{p+1}_c(U,\mathbb{R})$ as in (3.17) with $\mathrm{d}\eta = 0$ and $\overline{\alpha} = 0$, define

$$\gamma = \sum_{a:=a_1<\ldots<a_p} F_a(t,x)\; \mathrm{d}x_{a_1} \wedge \ldots \wedge \mathrm{d}x_{a_p} \tag{3.20}$$

where

$$F_a(t,x) = \int_{-\infty}^{t} f_a(s,x)\mathrm{d}s.$$

Since $\overline{\alpha} = 0$, we conclude that γ is compactly supported; i.e $\gamma \in \Omega^p_c(U,\mathbb{R})$. It follows from (3.18) that

$$\mathrm{d}\gamma = \mathrm{d}t \wedge \alpha + \int_{-\infty}^{t} \mathrm{d}_x\alpha = \eta.$$

□

Remark 3.21. If there is $\varepsilon > 0$ such that $(-\varepsilon,\varepsilon)\times\pi(U) \subset U$, then h can be chosen to be independent of $\overline{\alpha}$. Then, the proof above can be seen as showing that the operator $(h(t)\mathrm{d}t \wedge \pi^*)\circ\pi_*$ is chain homotopic to the identity map; i.e.

$$\mathrm{id} - (h(t)\mathrm{d}t \wedge \pi^*)\circ\pi_* = \mathrm{d}K + K\mathrm{d}.$$

Here $K\colon \Omega^{p+1}_c(U,\mathbb{R}) \longrightarrow \Omega^p_c(U,\mathbb{R})$ is given by $\eta \longrightarrow \gamma$ in (3.20). The proof above readily extends to a proof of (3.13), whenever E is a trivial vector bundle. If $E = M\times\mathbb{R} \longrightarrow M$ and M is closed, $[e(t)] = [h(t)\mathrm{d}t]$ is the Thom-class mentioned after (3.15).

Proof of Theorem 3.15. If M is diffeomorphic to $\mathbb{R}^m$, the result follows from (3.12), and (3.14), and the observation below. Suppose η is a compactly-supported $(m-1)$-form on $\mathbb{R}^m$. Choose $R > 0$ sufficiently large so that $\mathrm{supp}(\eta) \subset B_R(0)$. Then, by Stokes' Theorem, we have

$$0 = \int_{\partial B_R(0)} \eta = \int_{B_R(0)} \mathrm{d}\eta.$$

We conclude that the homomorphism

$$\Omega^m_c(\mathbb{R}^m, \mathbb{R}) \longrightarrow \mathbb{R}, \qquad \omega \longrightarrow \int_{\mathbb{R}^m} \omega,$$

descends to a surjective homomorphism

$$H^m_{c,\mathrm{dR}}(\mathbb{R}^m, \mathbb{R}) \longrightarrow \mathbb{R}.$$

By (3.14), the latter is indeed an isomorphism. It follows that the isomorphism

$$H^m_{c,\mathrm{dR}}(\mathbb{R}^m, \mathbb{R}) \cong H^0_{\mathrm{dR}}(\mathbb{R}^m, \mathbb{R})^* \cong \mathbb{R}$$

in (3.8) comes from the pairing (3.7).

Next, support $M = U_1 \cap U_2$, and the statement of Theorem 3.15 is true for U_1, U_2, and $U_{12} = U_1 \cap U_2$, we conclude that the statement is also true for M. For every $n \in \mathbb{Z}$, consider the commutative diagram

$$\begin{array}{ccccccccc}
H^{n-1}_{\mathrm{dr}}(U_1,\mathbb{R}) \oplus H^{n-1}_{\mathrm{dr}}(U_2,\mathbb{R}) & \xrightarrow{j^*} & H^{n-1}_{\mathrm{dr}}(U_{12},\mathbb{R}) & \xrightarrow{\delta^*} & H^{n}_{\mathrm{dr}}(M,\mathbb{R}) & \xrightarrow{\iota^*} & H^{n}_{\mathrm{dr}}(U_1,\mathbb{R}) \oplus H^{n}_{\mathrm{dr}}(U_2,\mathbb{R}) & \xrightarrow{j^*} & H^{n}_{\mathrm{dr}}(U_{12},\mathbb{R})^* \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{m-n+1}_{c,\mathrm{dr}}(U_1,\mathbb{R})^* \oplus H^{m-n+1}_{\mathrm{dr}}(U_2,\mathbb{R})^* & \xrightarrow{j^*} & H^{m-n+1}_{c,\mathrm{dr}}(U_{12},\mathbb{R})^* & \xrightarrow{\delta^*} & H^{m-n}_{c,\mathrm{dr}}(M,\mathbb{R})^* & \xrightarrow{\iota^*} & H^{m-n}_{c,\mathrm{dr}}(U_1,\mathbb{R})^* \oplus H^{m-n}_{\mathrm{dr}}(U_2,\mathbb{R})^* & \xrightarrow{j^*} & H^{m-n}_{c,\mathrm{dr}}(U_{12},\mathbb{R})^*
\end{array}$$

where the first row is the Mayer-Vietoris sequence of the standard de Rham cohomology, the second row is the dual of the Mayer-Vietoris sequence of the compactly supported de Rham cohomology, and the vertical maps are the homomorphisms corresponding to (3.7). By assumption, the first and last two vertical maps are isomorphisms. By the so-called five-lemma below, the middle one has to be an isomorphism as well.

Lemma 3.22. *(five-Lemma) Let*

$$\begin{array}{ccccccccc}
A & \xrightarrow{a} & B & \xrightarrow{b} & C & \xrightarrow{c} & D & \xrightarrow{d} & E \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
A' & \xrightarrow{a'} & B' & \xrightarrow{b'} & C' & \xrightarrow{c'} & D' & \xrightarrow{d'} & E'
\end{array}$$

be a commutative diagram in any abelian category (such as the category of abelian groups or the category of vector spaces over a given field). If the rows are exact, f_2 and f_4 are isomorphisms, f_1 is an epimorphism, and f_5 is a monomorphism, then f_3 is also an isomorphism.

Suppose M be a smooth oriented manifold of finite-type. Therefore, there is an open covering $M = \bigcup_{\alpha=1}^N U_\alpha$ such that every nonempty finite intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ is diffeomorphic to $\mathbb{R}^m$. We prove Theorem 3.15, by induction on N. By the induction assumption, the statement of Theorem 3.15 is true for U_1, $V = \bigcup_{\alpha=2}^N U_\alpha$, and

$$U_1 \cap V = \bigcup_{\alpha=2}^{N} (U_\alpha \cap U_1).$$

Therefore, by the argument before five-lemma, it also holds for M. $\square$

Proof of (3.13) (assuming M is orientable). Applying Theorem 3.15 to the manifold E, (3.11) to $E \longrightarrow M$, and Theorem 3.15 to M we get

$$H^{p+r}_{c,\mathrm{d}R}(E,\mathbb{R})^* \cong H^{(m+r)-(p+r)}_{\mathrm{d}R}(E,\mathbb{R}) \cong H^{m-p}_{\mathrm{d}R}(M,\mathbb{R}) \cong H^{p}_{c,\mathrm{d}R}(M,\mathbb{R})^*\,.$$

Therefore,

$$H^{p+r}_{c,\mathrm{d}R}(E,\mathbb{R}) \cong H^{p}_{c,\mathrm{d}R}(M,\mathbb{R})\,.$$

$\square$

The map π_* in (3.19) that comes from integration on fibers of E extends to arbitrary oriented vector bundle $\pi\colon E \longrightarrow M$ in the following way.

Fix an orientation-preserving local trivialization $\Phi\colon V \times \mathbb{R}^r \longrightarrow E|_V$. Let $(x_1,\ldots,x_m)$ denote the local coordinates on V and $(t_1,\ldots,t_r)$ denote the fiber coordinates. For every $\eta \in \Omega^{p+r}(E,\mathbb{R})$, $\Phi^*\eta$ has the form

$$\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_r \wedge \alpha + \beta \tag{3.21}$$

where

$$\alpha = \sum_{a:=a_1<\ldots<a_p} f_a(t,x)\ \mathrm{d}x_{a_1} \wedge \ldots \wedge \mathrm{d}x_{a_p}$$

and each term in β misses at least one of $\{\mathrm{d}t_i\}_{i=1}^r$.

Define

$$\pi_* \circ \Phi^*\colon \Omega^{p+r}_c(E,\mathbb{R}) \longrightarrow \Omega^p(V,\mathbb{R}), \qquad \pi_* \circ \Phi^*(\eta) = \overline{\alpha} := \sum_{a:=a_1<\ldots<a_p} \overline{f_a}(x)\ \mathrm{d}x_{a_1} \wedge \ldots \wedge \mathrm{d}x_{a_p}$$

where

$$\overline{f_a}(x) = \int_{\mathbb{R}^r} f_a(t,x)\ \mathrm{d}t_1 \cdots \mathrm{d}t_r \qquad \forall\ x \in V.$$

It is straightforward to check that π_* commutes with d.

Lemma 3.23. *The map $\pi_* \circ \Phi^*$ is independent of the choice of local trivialization Φ.*

Proof. Suppose Φ_1 and Φ_2 are two different trivializations. Let

$$\Phi_{12} = \Phi_2^{-1}\Phi_1\colon V \times \mathbb{R}^r \longrightarrow V \times \mathbb{R}^r, \qquad (x,t) \longrightarrow (x,s) = (x,\varphi(x)t)$$

denote the change of trivialization map for some $\varphi\colon V \longrightarrow GL_+(\mathbb{R}^r)$. Here, $GL_+(\mathbb{R}^r)$ is the group of orientation preserving linear isomorphisms of $\mathbb{R}^r$ (i.e. matrices with positive determinant). With notation as in (3.21), if

$$\Phi_1^*\eta = \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_r \wedge \alpha_1 + \beta_1 \quad \text{and} \quad \Phi_2^*\eta = \mathrm{d}s_1 \wedge \cdots \wedge \mathrm{d}s_r \wedge \alpha_2 + \beta_2,$$

with

$$\alpha_1 = \sum_{a:=a_1<\ldots<a_p} f_{1,a}(t,x)\ \mathrm{d}x_{a_1} \wedge \ldots \wedge \mathrm{d}x_{a_p} \quad \text{and} \quad \alpha_2 = \sum_{a:=a_1<\ldots<a_p} f_{2,a}(s,x)\ \mathrm{d}x_{a_1} \wedge \ldots \wedge \mathrm{d}x_{a_p},$$

it follows from Chain Rule that

$$f_{1,a}(t,x) = \det(\varphi(x)) f_{2,a}(\varphi(x)t,x) \qquad \forall\, x \in V.$$

Since $\det(\varphi(x)) > 0$, by the change of variables formula, we conclude that

$$\begin{aligned}\overline{f_{1,a}}(x) &= \int_{\mathbb{R}^r} f_{1,a}(t,x)\ \mathrm{d}t_1 \cdots \mathrm{d}t_r = \int_{\mathbb{R}^r} \det(\varphi(x)) f_{2,a}(\varphi(x)t,x)\ \mathrm{d}t_1 \cdots \mathrm{d}t_r \\ &= \int_{\mathbb{R}^r} f_{2,a}(s,x)\ \mathrm{d}s_1 \cdots \mathrm{d}s_r = \overline{f_{2,a}}(x) \qquad \forall\, x \in V.\end{aligned}$$

□

Corollary 3.24. *For every $p\in\mathbb{Z}$, the maps $\pi_* \circ \Phi^*$ are compatible on the overlaps and define a canonical push-forward map*

$$\pi_*\colon \Omega_c^{p+r}(E,\mathbb{R}) \longrightarrow \Omega_c^p(M,\mathbb{R}). \tag{3.22}$$

that commutes with the exterior derivative map.

HW 3.25. For a trivial vector bundle E over a finite-type manifold M, follow the same reasoning as in the beginning of the proof of Theorem 3.15 to show that (3.22) induces the isomorphism (3.13).

HW 3.26. For an arbitrary oriented vector bundle E over a finite-type manifold M, use local trivializations of E, Mayer-Vietoris, and induction to prove (3.13) (to avoid the orientability condition of M used in the proof above). Also, deduce that (3.22) induces the isomorphism (3.13).

If M is a connected closed manifold, we have $H^\bullet_{\mathrm{dR}}(M,\mathbb{R}) = H^\bullet_{c,\mathrm{dR}}(M,\mathbb{R})$ and $H^0_{\mathrm{dR}}(M,\mathbb{R}) \cong \mathbb{R}$. Therefore, under the isomorphism (3.13), $[1] \in H^0_{\mathrm{dR}}(M,\mathbb{R})$ corresponds to a cohomology class $[e] = [e(E)] \in H^r_{c,\mathrm{dR}}(E,\mathbb{R})$ known as the Thom-class. A compactly supported r-form e, known as a Thom form, representing $[e]$ has the properties listed after Proposition 3.17:

- properties (1)-(2) obviously hold,
- properties (3) and (4) follow from the fact that (3.22) induces the isomorphism (3.13) and

$$\pi_*(e \wedge \pi^*\alpha) = \alpha \qquad \forall \alpha \in \Omega^p(M,\mathbb{R}).$$

The Thom form e can be chosen to have support in a sufficiently small neighborhood of M. An explicit construction of e using a metric and connection on E is feasible but complicated. It is easy to check that if $E_1 \longrightarrow M$ and $E_2 \longrightarrow M$ are two oriented vector bundles then

$$e(E_1 \oplus E_2) = p_1^* e(E_1) \wedge p_2^* e(E_2).$$

where

$$p_a\colon E_1 \oplus E_2 \longrightarrow E_a \qquad a = 1,2,$$

are the natural projection maps.

If M is closed, for the inclusion map $\iota\colon M \longrightarrow E$ (as the zero-section), the cohomology class

$$\iota^*[e(E)] \in H_{\mathrm{dR}}^{\mathrm{rank}(E)}(M,\mathbb{R})$$

is called the Euler class of E. If E is trivial, then $\iota^*[e(E)] = 0$. Therefore, $\iota^*[e(E)]$ measures the non-triviality of E. Since every real vector bundle is isomorphic to its dual, we have

$$\iota^*[e(E)] = \iota^*[e(E^*)].$$

Proposition 3.27. *Suppose M is an oriented closed manifold. Then,*

$$\int_M [e(TM)] = \chi(M),$$

where $\chi(M)$ is the Euler characteristic of M defined in (2.14).

Proof. Let s be a transverse section of TM (i.e. a vector field). The family of section

$$[0,1]\times M \longrightarrow TM, \qquad (t,x) \longrightarrow s_t(x) := ts(x)$$

defines a homotopy between the zero section $\iota\colon M \longrightarrow TM$ and $s\colon M \longrightarrow TM$. Therefore,

$$\int_M s^*[e(TM)] = \int_M \iota^*[e(TM)] \qquad \forall\ s \in \Gamma(M,TM).$$

Fix a Thom form $e = e(TM)$. For every $t > 0$,

$$K_t = s_t^{-1}(\mathrm{Supp}(e)) \subset M$$

is a compact subset of M. Furthermore,

$$\lim_{t\longrightarrow\infty} K_t = \bigcap_{t\geq 0} K_t = s^{-1}(0) = \{p_1,\dots,p_k\}.$$

Therefore, for t sufficiently large, $K_t = \bigcup_{a=1}^k K_a$ where each K_a is a sufficiently small compact neighborhood around p_a. Furthermore, the graph of s_t over K_a is C^1-close to the $\mathrm{Supp}(e)\cap T_{p_a}M$. Therefore, we can smoothly deform the embedding s_t to a an embedding $\widetilde{s}_t\colon M \longrightarrow TM$ such that

- $\widetilde{s}_t^{-1}(\mathrm{Supp}(e)) = \bigcup_{a=1}^k \widetilde{K}_a$ is a disjoint union of sufficiently small compact neighborhoods of the points $s^{-1}(0)$;
- for every $a = 1,\dots,k$, there is a neighborhood V_a of $\widetilde{K}_a$ for which $\widetilde{s}_t(V_a)$ is an open subset of $T_{p_a}M$ including $\mathrm{Supp}(e)\cap T_{p_a}M$.

It is easy to see from the definition of the signs $\varepsilon(p_a)$ in (2.14) that

$$\widetilde{s}_t|_{V_a}\colon V_a \longrightarrow T_{p_a}M$$

is an orientation preserving embedding if and only if $\varepsilon(p_a) = +1$. Since $\int_{T_xM} e = 1$ for all $x \in M$, we conclude that

$$\int_M \iota^*[e(TM)] = \int_M \widetilde{s}_t^{\,*}[e(TM)] = \sum_{a=1}^k \varepsilon(p_a)\int_{\widetilde{s}_t(V_a)} e = \sum_{a=1}^k \varepsilon(p_a) = \chi(M).$$

□

HW 3.28. Suppose $E \longrightarrow M$ is an oriented vector bundle over a closed manifold. Show that if E admits a nowhere-zero section then the Euler class of E is zero.

In the same way that manifolds are generalization of euclidean (vector) spaces, fiber bundles are generalization of vector bundles.

Definition 3.29. Suppose N, M, and F are smooth manifolds and $\pi\colon F \longrightarrow M$ is a submersion. We say $\pi\colon F \longrightarrow M$ is a fiber bundle with fiber N if there is an open covering $M = \bigcup_{\alpha\in\mathcal{I}} U_\alpha$ and a family of local trivializations

$$\big\{h_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times N\big\}_{\alpha\in\mathcal{I}}$$

such that $\pi = \pi_\alpha \circ h_\alpha$ for all $\alpha \in \mathcal{I}$, where $\pi_\alpha\colon U_\alpha \times N \longrightarrow U_\alpha$ is the natural projection map.

If N is compact, by Tubular Neighborhood Theorem, a submersion $\pi\colon F \longrightarrow M$ is a fiber bundle with fibers N if $\pi^{-1}(x)$ is diffeomorphic to N for all $x \in M$. A trivial fiber bundle is a product $F = M \times N$. Similarly to vector bundles, a fiber bundle is described by a collection of local trivializations $\{U_\alpha \times N\big\}_{\alpha\in\mathcal{I}}$ that are attached by transition maps of the form

$$\Phi_{\alpha,\beta}\colon (U_\alpha \cap U_\beta) \times N \longrightarrow (U_\alpha \cap U_\beta) \times N, \qquad (x, y) \longrightarrow (x, \varphi_{\alpha,\beta}(x)(y)),$$

such that

$$\varphi_{\alpha,\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{Diff}(N)$$

are smooth maps into the group of diffeomorphisms of N satisfying the cocycle condition

$$\varphi_{\beta,\gamma}(x) \circ \varphi_{\alpha,\beta}(x) = \varphi_{\alpha,\gamma}(x) \qquad \forall\, x \in U_\alpha \cap U_\beta \cap U_\gamma.$$

The product maps

$$\begin{aligned} H^*_{\mathrm{dR}}(F) \otimes H^*_{\mathrm{dR}}(M) &\longrightarrow H^*_{\mathrm{dR}}(F), \qquad \alpha \otimes \beta \longrightarrow \alpha \wedge \pi^*\beta,\\ H^*_{\mathrm{dR}}(M) \otimes H^*_{\mathrm{dR}}(F) &\longrightarrow H^*_{\mathrm{dR}}(F), \qquad \beta \otimes \alpha \longrightarrow \pi^*\beta \wedge \alpha \end{aligned}$$

realize $H^*_{\mathrm{dR}}(F)$ as right and left $H^*_{\mathrm{dR}}(M)$-modules, respectively. Here, we are thinking of $H^*_{\mathrm{dR}}(M)$ as a ring (algebra) with $\wedge$ as its product structure.

Theorem 3.30 (Leray-Hirsch)**.** *Suppose $\pi\colon F \longrightarrow M$ is a fiber bundle with fiber N and M has finite type. Suppose there are cohomology classes $c_1, \dots, c_r$ on F which restricted to each fiber $\pi^{-1}(x) \cong N$ of F make a basis for the vector space $H^*_{\mathrm{dR}}(\pi^{-1}(x), \mathbb{R})$. Then, $H^*_{\mathrm{dR}}(F)$ is a free module over $H^*_{\mathrm{dR}}(M)$ with basis $c_1, \dots, c_r$; i.e.*

$$H^*_{\mathrm{dR}}(F, \mathbb{R}) \cong H^*_{\mathrm{dR}}(M, \mathbb{R}) \otimes H^*_{\mathrm{dR}}(N, \mathbb{R}).$$

If N also has finite type, the same holds for cohomology with compact support.

HW 3.31. Prove Theorem 3.30 using Mayer-Vietoris and induction on the size of a finite good cover.

Corollary 3.32 (Künneth formula)**.** *If M and N have finite type, then*

$$H^*_{\mathrm{dR}}(M \times N) \cong H^*_{\mathrm{dR}}(M) \otimes H^*_{\mathrm{dR}}(N).$$

Here is an interesting case of Theorem 3.30. Suppose $E \longrightarrow M$ is complex vector bundle of rank r. By projectivizing each fiber of E, we obtain a fiber bundle

$$F = \mathbb{P}(E) \longrightarrow M$$

whose fibers are isomorphic to $\mathbb{CP}^{r-1}$. The definition of tautological line bundle γ in Example 2.30 generalizes to F and yields a similarly denoted complex line bundle

$$\pi\colon \gamma_F \longrightarrow F.$$

Let $H = c_1(\gamma_F^*) \in H^2_{\mathrm{dR}}(F,\mathbb{R})$ denote the first chern class of γ_F^*. The restriction of H to each fiber $\pi^{-1}(x) \cong \mathbb{CP}^{r-1}$ is the first chern class h of γ^*. In Example 3.6, we showed that

$$H_i(\mathbb{CP}^r,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 2k,\ 0 \le k \le r, \\ 0 & \text{otherwise.} \end{cases}$$

Using Poincare duality between homology and cohomology, we will show that

$$H^i_{\mathrm{dR}}(\mathbb{CP}^r,\mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } i = 2k,\ 0 \le k \le r, \\ 0 & \text{otherwise;} \end{cases}$$

moreover, $H^i_{\mathrm{dR}}(\mathbb{CP}^r,\mathbb{R}) \cong \mathbb{R}$ is generated by h. Therefore, as ring, $H^*_{\mathrm{dR}}(\mathbb{CP}^r,\mathbb{R})$ is generated by h satisfying $h^r = 0$. By Theorem 3.30 and the observation above, $H^*_{\mathrm{dR}}(F,\mathbb{R})$ is generated, as a ring, by H and $H^*_{\mathrm{dR}}(M,\mathbb{R})$. Therefore, there are unique cohomology classes

$$c_1,\ldots,c_r, \qquad c_i \in H^{2i}_{\mathrm{dR}}(M,\mathbb{R})$$

such that

$$H^{r+1} + H^r c_1 + H^{r-1}c_2 + \cdots + Hc_{r-1} + c_r = 0 \tag{3.23}$$

We will show that c_1, ..., c_r are indeed the de Rham Chern classes of E.

HW 3.33. What is the $r = 1$ case of the discussion above.

Remark 3.34. A refined version of Theorem 3.30 is also true for singular cohomology in place of $H^*_{\mathrm{dR}}(F)$. If there are singular cohomology classes $c_1,\ldots,c_r$ on F which restricted to each fiber $\pi^{-1}(x) \cong N$ of F make a $\mathbb{Z}$-basis for the $\mathbb{Z}$-module $H^*_{\mathrm{dR}}(\pi^{-1}(x),\mathbb{Z})$, then

$$H^*(F,\mathbb{Z}) \cong H^*(M,\mathbb{Z}) \otimes H^*(N,\mathbb{Z}).$$

Using this refined version, the formula (3.23) can be used to define chern classes as integral singular cohomology classes in $H^*(M,\mathbb{Z})$.

We finish this section with a duality result between singular (=simplicial) homology (with coefficients in $\mathbb{R}$) and de Rham cohomology of a smooth manifold M. Let M be an oriented smooth manifold. Fix a triangulation $\mathcal{K}$ of M. For each oriented k-simplex Δ in $\mathcal{K}$ and every k-form η the pairing

$$\langle \eta, \Delta\rangle \longrightarrow \int_\Delta \eta$$

is defined. Since $C_k(\mathcal{K},\mathbb{R})$ contains finite linear combinations of simplices in $\mathcal{K}$, the pairing above linearly extends to a pairing

$$\langle -,-\rangle : \Omega^k(M,\mathbb{R}) \times C_k(\mathcal{K},\mathbb{R}) \longrightarrow \mathbb{R}.$$

By Stoke's theorem, this pairing descends to a pairing

$$\langle -, - \rangle : H^k(M, \mathbb{R}) \times H_k(M, \mathbb{R}) \longrightarrow \mathbb{R}. \tag{3.24}$$

The proof of the following theorem follows the same steps as in the proof of Theorem 3.15.

Theorem 3.35. *Suppose M is an oriented smooth m-manifold of finite type. Then, the pairing (3.24) is non-degenerate; i.e.*

- $\langle [\alpha], - \rangle = 0 \Rightarrow [\alpha] = 0 \in H^k_{\mathrm{dR}}(M, \mathbb{R})$;
- $\langle -, [\beta] \rangle = 0 \Rightarrow [\beta] = 0 \in H_k(M, \mathbb{R})$.

Therefore,

$$H^k_{\mathrm{dR}}(M, \mathbb{R}) \cong H_k(M, \mathbb{R})^*. \tag{3.25}$$

If M is closed, there is also a non-degenerate pairing

$$H_{m-k}(M, \mathbb{R}) \times H_k(M, \mathbb{R}) \longrightarrow \mathbb{R} \tag{3.26}$$

which results in a Poincare duality isomorphism

$$H_{m-k}(M, \mathbb{R}) \cong H_k(M, \mathbb{R})^*.$$

The pairing (3.26) is given by counting intersection points of k and $(m-k)$-cycles. Combined with (3.25), if M is oriented and closed, we get

$$H^k_{\mathrm{dR}}(M, \mathbb{R}) \cong H_{m-k}(M, \mathbb{R}). \tag{3.27}$$

If $S \subset M$ is a submanifold of codimension k, it defines a homology class $[S] \in H_{m-k}(M, \mathbb{R})$. Under the isomorphism (3.27), the Thom form of $S \subset M$ is the cohomology class corresponding to $[S]$ in $H^k_{\mathrm{dR}}(M, \mathbb{R})$.

Example 3.36. Suppose $E \longrightarrow M$ is a complex vector bundle of rank r over a closed manifold, and $s\colon M \longrightarrow E$ is a transverse section of E. Then $s^{-1}(0)$ is a closed submanifold of M that represent a homology class in $H_{m-2r}(M, \mathbb{Z})$. Under the isomorphism (3.27), the homology class $[s^{-1}(0)]$ (as an element in $H_{m-2r}(M, \mathbb{R})$) corresponds to the de Rham top chern class $c_{2r}(E)$.

HW 3.37. Use Example 3.36 and generalize the proof of Proposition 3.27 to prove the following:

Theorem 3.38. *If $E \longrightarrow M$ is a complex vector bundle over a closed manifold, then, $\iota^*[e(E)]$ coincides with the top-chern de Rham cohomology class of E defined in (2.32).*

3.3 Cech cohomology

In the previous section, we repeatedly used the combination of induction (over the size of a good cover) and Mayer-Vietoris sequence to prove various statements such as Poincare duality. A natural question is:

- *What is the generalization of Mayer-Vietoris for a decomposition $M = \bigcup_{a=1}^{n} U_a$ of M into $n > 2$ open sets?*

The result of this discussion, which is a different kind of structure compared to what Mayer-Vietoris provides, would lead us into Cech cohomology.

Let

$$[n] := \{1, \dots, n\}, \qquad \mathcal{P}_q(n) = \{I \subset [n] \colon |I| = q\} \quad \forall\, q \geq 0.$$

For every $I \subset [n]$, let

$$U_I = \bigcap_{a \in I} U_a.$$

For every $I' \subset I$, let

$$\iota_{I;I'} \colon U_I \longrightarrow U_I'$$

denote the inclusion map. These inclusion maps give rise to restriction maps

$$\mathcal{R}_{I;I'} := \iota^*_{I;I'} \colon \Omega^n(U_{I'}, \mathbb{R}) \longrightarrow \Omega^n(U_I, \mathbb{R}).$$

For all $p \geq 0$ and $q \geq -1$, let

$$E^{p,q} = \bigoplus_{I \in \mathcal{P}_{q+1}(n)} \Omega^p(U_I, \mathbb{R}).$$

Using the restriction maps $R_{I;I'}$, we get the maps

$$\mathcal{R} \colon E^{p,q-1} \longrightarrow E^{p,q}, \qquad \mathcal{R}\big(\oplus_{I \in \mathcal{P}_q(n)} \eta_I\big) = \oplus_{J \in \mathcal{P}_{q+1}(n)} \eta'_J$$

where

$$\begin{aligned} &\eta_I \in \Omega^p(U_I, \mathbb{R}) \quad \forall\, I \in \mathcal{P}_q(n), \\ &\eta'_J = \sum_{a=0}^{q} (-1)^a\, \mathcal{R}_{J-j_a;J}(\eta_{J-j_a}) \quad \forall\, J = \{j_0 < \cdots < j_q\} \in \mathcal{P}_{q+1}(n). \end{aligned}$$

For example, $\mathcal{R} \colon E^{p,-1} \longrightarrow E^{p,0}$ is simply the restriction map

$$\Omega^p(M, \mathbb{R}) \longrightarrow \bigoplus_{a \in [n]} \Omega^p(U_a, \mathbb{R}). \tag{3.28}$$

Remark 3.39. Instead of labeling the components η_I of η by subsets $I \subset [n]$, we may label them as $\eta_{i_1,\dots,i_q}$ where $I = \{i_1, \dots, i_q\} \subset [n]$ with the convention that

$$\eta_{i_1 \dots i_q} = (-1)^{\varepsilon(\sigma)} \eta_{i_{\sigma(1)} \dots i_{\sigma(q)}} \qquad \forall\, \sigma \in \mathbb{S}_q.$$

Then we can write

$$(\mathcal{R}(\eta))_{i_0 i_1 \dots i_q} = \sum_{a=0}^{q} (-1)^a\, \eta_{i_0, i_1, \dots i_{a-1} i_{a+1} \dots i_q}|_{U_{i_0 \dots i_q}} \tag{3.29}$$

without ordering the indices increasingly.

Using the exterior derivative d, instead, we get the maps

$$\mathrm{d} \colon E^{p,q} \longrightarrow E^{p+1,q}, \qquad \mathrm{d}\big(\oplus_{I \in \mathcal{P}_q(n)} \eta_I\big) = \oplus_{I \in \mathcal{P}_q(n)} \mathrm{d}\eta_I \qquad \forall\, I \in \mathcal{P}_q(n).$$

We already know that $\mathrm{d} \circ \mathrm{d} = 0$. It is easy to check that $\mathcal{R} \circ \mathcal{R} = 0$. Putting these two maps together, we obtain a double cochain complex $(E^{\bullet,\bullet}, \mathrm{d}, \mathcal{R})$ which is the content of the dotted-square in Figure 4. This double complex somehow replaces the de Rham cochain complex which

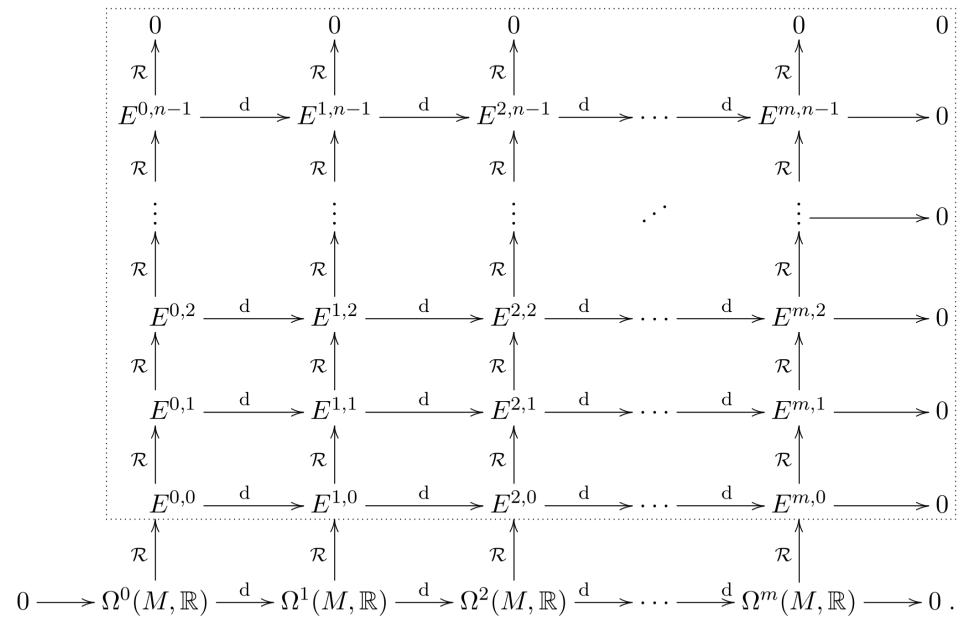


Figure 4: The double cochain complex $(E^{\bullet,\bullet}, \mathrm{d}, \mathcal{R})$.

is fills the row below the double complex at hight -1. From the double complex $(E^{\bullet,\bullet}, \mathrm{d}, \mathcal{R})$, we build a cochain complex $(C^\bullet, D)$ in the following way. Let

$$C^n = \bigoplus_{p+q=n} E^{p,q} \qquad \forall\, n \geq 0,$$
$$D\colon C^n \longrightarrow C^{n+1} \quad \text{s.t.} \quad D|_{E^{p,q}} = \mathrm{d} + (-1)^p \mathcal{R} \qquad \forall\, p+q=n.$$

Since $\mathcal{R} \circ \mathcal{R} = 0$, $\mathrm{d} \circ \mathrm{d} = 0$, and $\mathcal{R} \circ \mathrm{d} = \mathrm{d} \circ \mathcal{R}$, we get

$$D \circ D|_{E^{p,q}} = \mathrm{d} \circ \mathrm{d} + \mathcal{R} \circ \mathcal{R} + (-1)^p \mathrm{d} \circ \mathcal{R} + (-1)^{p+1} \mathcal{R} \circ \mathrm{d} = 0 \qquad \forall\, p+q=n$$

The following lemma generalizes the exactness statement of (3.10).

Lemma 3.40. *For each $p \geq 0$, the p-column*

$$0 \longrightarrow \Omega^p(M) \xrightarrow{\mathcal{R}} E^{p,0} \xrightarrow{\mathcal{R}} \cdots \xrightarrow{\mathcal{R}} E^{p,n-1} \longrightarrow 0$$

is exact.

Proof. Clearly, $\Omega^p(M)$ is the kernel of the first $\mathcal{R}$: if

$$\eta = \bigoplus_{a \in [n]} \eta_a \in E^{p,0} = \bigoplus_{a \in [n]} \Omega^p(U_a, \mathbb{R}),$$

then, with notation as in (3.29),

$$\mathcal{R}(\eta)_{ab} = \eta_a - \eta_b.$$

Therefore, $\mathcal{R}(\eta) = 0$ implies that the differential p-forms η_a and η_b match on the overall $U_{ab} = U_a \cap U_b$; hence $\{\eta_a\}$ can be glued to define a differential p-form on M. For the next ones, we use a partition of unity $\{\theta_a \colon U_a \longrightarrow [0,1]\}_{a\in[n]}$ subordinate to the covering $\{U_a\}_{a\in[n]}$ to prove the exactness. Suppose

$$\eta = \bigoplus_{i_0<\ldots<i_q} \eta_{i_0\ldots i_q} \in E^{p,q} = \bigoplus_{i_0<\ldots<i_q} \Omega^p(U_{i_0\ldots i_q})$$

is $\mathcal{R}$-closed. By definition,

$$(\mathcal{R}(\eta))_{i_0\ldots i_{q+1}} = \sum_{a=0}^{q+1}(-1)^a\, \alpha_{i_0,i_1,\ldots i_{a-1}i_{a+1}\ldots i_{q+1}}|_{U_{i_0,\ldots,i_q}} = 0 \qquad \forall\, \{i_0,\ldots,i_{q+1}\} \subset [n]. \tag{3.30}$$

Define

$$\alpha = \bigoplus_{i_1<\ldots<i_q} \alpha_{i_1\ldots i_q} \in E^{p,q-1} = \bigoplus_{i_1<\ldots<i_q} \Omega^p(U_{i_1\ldots i_q})$$

by

$$\alpha_{i_1\ldots i_q} = \sum_{i_0\in[n]} \theta_{i_0}\eta_{i_0 i_1\ldots i_q}.$$

Note that, by the convention in Remark 3.39,

- if $i_0 = i_a$ for some $a = 1,\ldots,q$, then $\eta_{i_0 i_1\ldots i_q}$ is automatically defined to be zero;
- if $i_0 > i_a$ for some $a = 1,\ldots,q$, then we can change the order to increasing at cost of a permutation sign;
- the righthand side is a p-form over $U_{i_0,\ldots,i_q}$ because $\eta_{i_0 i_1\ldots i_q}$ is a p-form over $U_{i_1\ldots i_q}$ and θ_{i_0} is supported in U_{i_0}.

We have,

$$\begin{aligned}
\mathcal{R}(\alpha)_{i_0\ldots i_q} &= \sum_{a=0}^{q}(-1)^a\, \alpha_{i_0 i_1,\ldots i_{a-1}i_{a+1}\ldots i_q}|_{U_{i_0\ldots i_q}}\\
&= \sum_{a=0}^{q}\sum_{i_{q+1}\in[n]} (-1)^a\, \theta_{i_{q+1}}\; \eta_{i_{q+1}i_0 i_1\ldots i_{a-1}i_{a+1}\ldots i_q}|_{U_{i_0\ldots i_q}}\\
&= \sum_{i_{q+1}\in[n]} \theta_{i_{q+1}}(-1)^q \sum_{a=0}^{q}(-1)^a \eta_{i_0,i_1,\ldots i_{a-1}i_{a+1}\ldots i_q i_{q+1}}|_{U_{i_0\ldots i_q}}.
\end{aligned}$$

By (3.30), the last line is equal to

$$\sum_{i_{q+1}\in[n]} \theta_{i_{q+1}}(-1)^q(-1)^q\eta_{i_0 i_1\ldots i_q}|_{U_{i_0\ldots i_q}} = \eta_{i_0 i_1\ldots i_q}|_{U_{i_0\ldots i_q}}.$$

We conclude that $\mathcal{R}(\alpha) = \eta$.

□

The main result is the following.

Lemma 3.41. *The cohomology groups of $(C^\bullet, D)$ and $(\Omega^\bullet(M,\mathbb{R}),\mathrm{d})$ are the same.*

Proof. The maps (3.28) that send row -1 to row 0 in Figure 4 can be seen as cochain maps

$$\mathcal{R}\colon (\Omega^\bullet(M,\mathbb{R}),\mathrm{d}) \longrightarrow (C^\bullet, D). \tag{3.31}$$

More precisely, if $\eta \in \Omega^p(M,\mathbb{R})$, then

$$D\mathcal{R}\eta = (\mathrm{d} + (-1)^p\mathcal{R})\mathcal{R}\eta = \mathrm{d}\mathcal{R}\eta = \mathcal{R}\mathrm{d}\eta.$$

We show that (3.31) induces an isomorphism of cohomology groups. Every $\alpha \in C^p$ can be decompsoed as

$$\alpha = \bigoplus_{r=0}^{p} \alpha_{p-r;r}, \qquad \text{s.t.} \quad \alpha_{p-r;r} \in E^{p-r,r} \quad \forall r = 0,\dots,p.$$

Then $D\alpha = 0$ if and only if

$$\mathrm{d}\alpha_{p;0} = 0, \quad \mathcal{R}\alpha_{p-1;1} = (-1)^p\mathrm{d}\alpha_{p-2;2}, \quad \dots \quad , \quad \mathcal{R}\alpha_{0;p} = 0.$$

By Lemma 3.40, we can find $\beta \in E^{0;p-1}$ such that $\mathcal{R}(\beta) = \alpha_{0;p}$. Therefore, α and $\alpha - D\beta$ define the same cohomology class and the $(0,p)$-term of $\alpha - D\beta$ is zero. Going down inductively, we can show that every D-cohomology class can be represented by some α such that

$$\alpha_{p-i;i} = 0 \qquad \forall\ i > 0;$$

i.e. $\alpha = \alpha_{p;0}$, $\mathrm{d}\alpha_{p;0} = 0$, and $\mathcal{R}(\alpha_{p;0}) = 0$. As in the beginning of the proof of Lemma 3.40, $\mathcal{R}(\alpha_{p;0}) = 0$ implies that α comes from a p-form on M which is closed because $\mathrm{d}\alpha_{p;0} = 0$. Therefore, (3.31) is surjective at the cohomological level. The injectivity is proved similarly. □

Diagram of Figure 4 is missing a column on left that maps to the first column $(E^{\bullet,q})_{q=0^n}$ of the double complex. For each $q \geq 0$, the kernel of

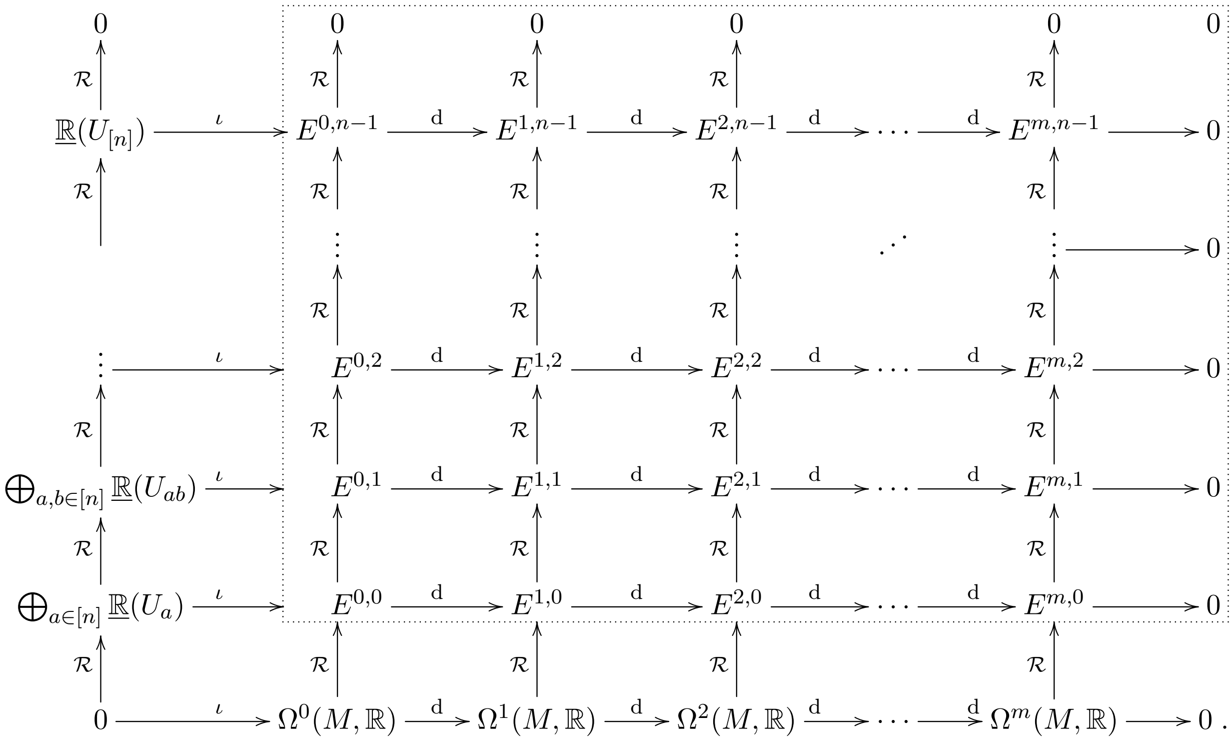
$$\mathrm{d}\colon E^{0,q} = \bigoplus_{I\in\mathcal{P}_{q+1}(n)} \Omega^0(U_I) \longrightarrow E^{1,q} = \bigoplus_{I\in\mathcal{P}_{q+1}(n)} \Omega^1(U_I)$$

is the space of locally constant functions

$$E^{-1,q} = \bigoplus_{I\in\mathcal{P}_{q+1}(n)} \underline{\mathbb{R}}(U_I).$$

Here, for each $I \in \mathcal{P}_{q+1}(n)$, $\mathbb{R}(U_I)$ is the space of locally constant functions on U_I. If U_I is connected, then $\underline{\mathbb{R}}(U_I) \cong \mathbb{R}$; otherwise, we will have one copy of $\mathbb{R}$ for each connected component.

Adding this column to the left of Figure 4 we get the following diagram.



In this diagram the maps ι are the inclusion maps. The first column is itself a cochain complex

$$0 \longrightarrow \bigoplus_{a\in[n]} \underline{\mathbb{R}}(U_a) \xrightarrow{\mathcal{R}} \bigoplus_{a,b\in[n]} \underline{\mathbb{R}}(U_{ab}) \longrightarrow \cdots \underline{\mathbb{R}}(U_{[n]}) \longrightarrow 0. \tag{3.32}$$

The cohomology groups of this sequence a priori depend on the open covering $\mathcal{A} \equiv \{U_a\}_{a\in[n]}$ and are denoted by

$$\check{H}^k(\mathcal{A},\underline{\mathbb{R}}) = \frac{\mathrm{Ker}\Big(C^k(\mathcal{A},\underline{\mathbb{R}}) \xrightarrow{\mathcal{R}} C^{k+1}(\mathcal{A},\underline{\mathbb{R}})\Big)}{\mathrm{Image}\Big(C^{k-1}(\mathcal{A},\underline{\mathbb{R}}) \xrightarrow{\mathcal{R}} C^k(\mathcal{A},\underline{\mathbb{R}})\Big)} \tag{3.33}$$

where

$$C^k(\mathcal{A},\underline{\mathbb{R}}) = \bigoplus_{I\in\mathcal{P}_{k+1}(n)} \underline{\mathbb{R}}(U_I) \qquad \forall\ k \geq 0.$$

The assignment

$$U \longrightarrow \underline{\mathbb{R}}(U) \tag{3.34}$$

is an example of the concept of sheaf (defined below) and the cohomology groups $\check{H}^k(\mathcal{A},\underline{\mathbb{R}})$ are the cech cohomology groups of this sheaf with respect to the covering $\mathcal{A}$. A natural question is:

- *How does $\check{H}^k(\mathcal{A},\underline{\mathbb{R}})$ depend on $\mathcal{A}$?*

First, note that $\check{H}^k(\mathcal{A}, \underline{\mathbb{R}})$ does depend on $\mathcal{A}$. If we take the trivial covering $\{U_1 = M\}$, then

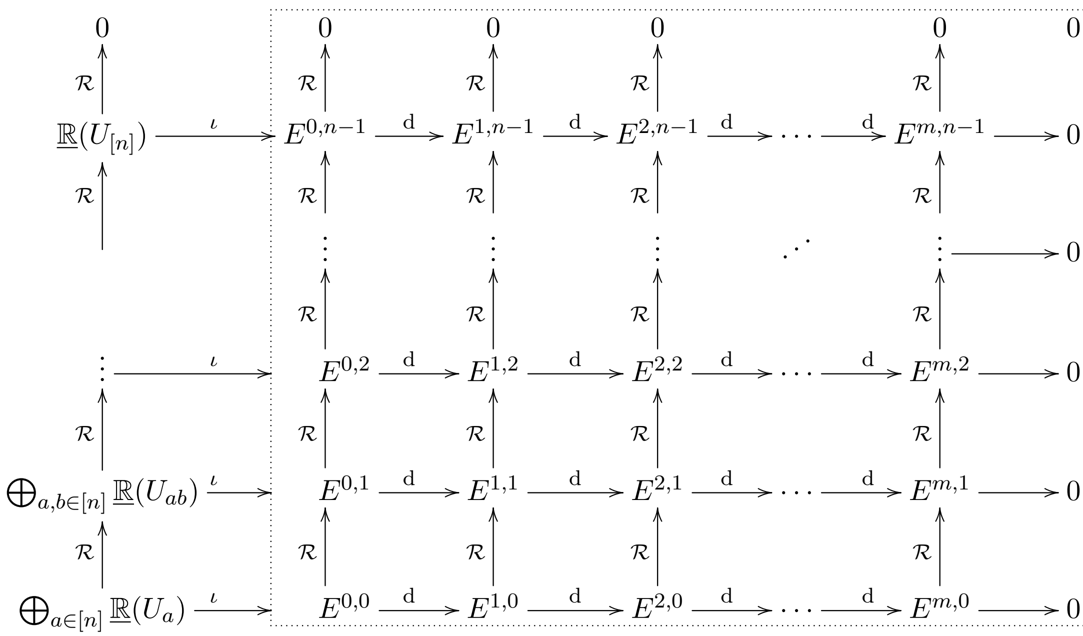
$$\check{H}^k(\mathcal{A}, \underline{\mathbb{R}}) = \begin{cases} \mathbb{R}^{\#\pi_0(M)} & \text{if} \quad k = 0; \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, we have the following result.

Proposition 3.42. *Suppose $\mathcal{A} = \{U_a\}_{a\in[n]}$ is a good covering of M in the sense of Definition 3.14. Then,*

$$\check{H}^k(\mathcal{A}, \underline{\mathbb{R}}) \cong H^k_{\mathrm{dR}}(M, \mathbb{R}) \qquad \forall\ k \in \mathbb{Z}. \tag{3.35}$$

Proof. In the proof Lemma 3.41, we mainly used the fact that the columns of Figure 4 are exact. If $\mathcal{A} = \{U_a\}_{a\in[n]}$ is a good covering, then each U_I is diffeomorphic to $\mathbb{R}^n$; therefore, by Poincare Lemma, the rows of



are exact. The same proof as in Lemma 3.41 shows that the cohomology groups of $(C^\bullet, D)$ and $(C^\bullet(\mathcal{A}, \underline{\mathbb{R}}), \mathcal{R})$ are also the same. $\square$

For $\mathcal{A}$ as in Proposition 3.35, since the cech cohomology groups are invariants of the pair $(M, \underline{\mathbb{R}})$, we simply denote them by $\check{H}^k(M, \underline{\mathbb{R}})$. They are called cech cohomology groups of the sheaf $\underline{\mathbb{R}}$ on M.

HW 3.43. Cover S^1 with n open sets (n intervals) $U_1, \ldots, U_n$ such that $U_i \cap U_j \neq \emptyset$ if and only if $j \equiv i \pm 1$ modulo n. Compute the cech cohomology of the locally constant sheaf $\underline{\mathbb{R}}$ on S^1 with respect to such open covering.

Example 3.44. We use a good covering of S^2 with 4 open disks to calculate the cohomology of S^2. Topologically, S^2 is homeomorphic to the boundary of a 3-simplex as in Figure 5. For each vertex A_i, define U_i to be the interior of the union of the three triangles containing A_i. Therefore,

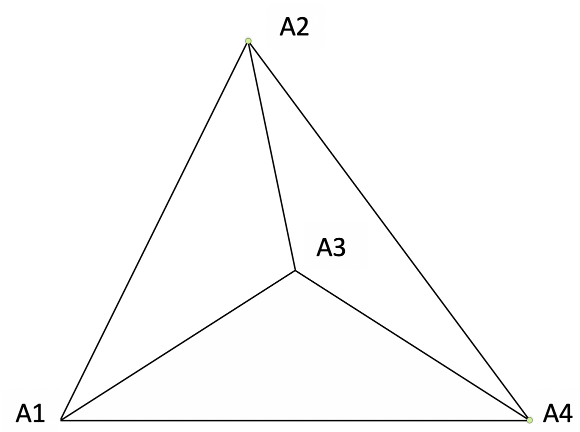


Figure 5: A 3-simplex

- for different i and j, U_{ij} is the interior of the union of the two triangles that contain the line A_iA_j;
- for different i, j, k, U_{ijk} is the interior of the triangle $A_iA_jA_k$;
- and $U_{1234} = \emptyset$.

We conclude that $\mathcal{A} = \{U_1, U_2, U_3, U_4\}$ is a good cover of S^2. The cochain complex (3.32) is equal to

$$0 \longrightarrow \mathbb{R}^4 = \bigoplus_{i=1}^{4} \mathbb{R}_{U_i} \longrightarrow \mathbb{R}^6 = \bigoplus_{i,j} \mathbb{R}_{U_{ij}} \longrightarrow \mathbb{R}^4 = \bigoplus_{i,j,k} \mathbb{R}_{U_{ijk}} \longrightarrow 0,$$

where the first map is

$$(a_1, a_2, a_3, a_4) \longrightarrow (a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}) = (a_2 - a_1, a_3 - a_1, a_4 - a_1, a_3 - a_2, a_4 - a_2, a_4 - a_3)$$

and the second map is

$$(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}) \longrightarrow$$
$$\big(a_{234}, a_{134}, a_{124}, a_{123}\big) = \big((a_{34} - a_{24} + a_{23}), (a_{34} - a_{14} + a_{13}), (a_{24} - a_{14} + a_{12}), (a_{23} - a_{13} + a_{12})\big).$$

The kernel of the first map which is H^0 is isomorphic to $\mathbb{R}$ (generated by (1,1,1,1)). Check with a computer program that the image of the second map is a 3-dimensional subspace of $\mathbb{R}^4$. Therefore, H^2 which is the cokernel of this map is isomorphic to $\mathbb{R}$. For dimensional reasons, the sequence must then be exact in the middle; therefore $H^1 = 0$.

HW 3.45. Suppose $\mathcal{A} = \{U_a\}_{a \in [n]}$ is a good covering of M. By following the proof of Lemma 3.41, starting with any cech cocycle $\eta \in C^1(\mathcal{A}, \underline{\mathbb{R}})$, find a closed 1-form $\eta' \in \Omega^1(M, \mathbb{R})$ such that $[\eta] \longrightarrow [\eta']$ realizes the isomorphism

$$\check{H}^1(\mathcal{A}, \underline{\mathbb{R}}) \xrightarrow{\cong} H^1_{\mathrm{dR}}(M, \mathbb{R}).$$

In what follows we define a pre-sheaf, sheaf, cech cohomology with respect to an open covering, and cech cohomology groups of the sheaf as a limit of the covering-dependent cohomology groups. We provide a few examples to highlight the important ideas and applications.

Definition 3.46. Let M be a topological space. A pre-sheaf $\mathcal{F}$ on M is a way of assigning a set/group/ring/etc. $\mathcal{F}(U)$ to each open set $U \subset M$ with the following properties.

- for every $V \subset U$, there is a "restriction" map/homomorphism $\mathcal{R}_{U,V} : \mathcal{F} \longrightarrow \mathcal{F}(V)$;
- for every U, we have $\mathcal{R}_{U,U} = \mathrm{id}$;
- for every $W \subset V \subset U$, we have $\mathcal{R}_{V,W} \circ \mathcal{R}_{U,V} = \mathcal{R}_{U,W}$.

Often, for $V \subset U$ and $\eta \in \mathcal{F}(U)$, we will write $\eta|_V$ instead of $\mathcal{R}_{U,V}(\eta)$.

Remark 3.47. From the categorical point of view, the collection of open sets of M are objects of a category $\mathrm{Open}(M)$ with the morphisms spaces

$$\mathrm{Hom}(U,V) = \begin{cases} \{\iota : U \longrightarrow V\} & \text{if} \quad U \subset V \\ \emptyset & \text{otherwise.} \end{cases}$$

For every category $\mathcal{C}$, there is an opposite category $\mathcal{C}^o$ that has the same objects as in $\mathcal{C}$ but its morphisms spaces are the opposite of the morphism spaces in $\mathcal{C}$; i.e.

$$Hom_{\mathcal{C}^o}(A,B) \coloneqq Hom_{\mathcal{C}}(B,A) \qquad \forall\ A,B \in \mathrm{Obj}(\mathcal{C}) = \mathrm{Obj}(\mathcal{C}^o).$$

A sheaf can be seen as a functor $\mathcal{F}$ from the category $\mathrm{Open}(M)^\circ$ to another category (e.g. category of $\mathbb{R}$-vector spaces, $\mathbb{Z}$-modules, etc.).

Example 3.48. For every manifold M and every $p \in \mathbb{N}$, the assignment

$$U \longrightarrow \Omega^p(U,\mathbb{R})$$

is a pre-sheaf where the restriction maps $\mathcal{R}$ are the obvious restriction of differential forms to open subsets. The assignment (3.34) is also a pre-sheaf. Given any group G, we can generalize (3.34) by assigning to each open set U the group $\underline{G}(U)$ of locally-constant functions with values in G. Given every vector bundle, the assignment

$$U \longrightarrow \Gamma(U, E|_U)$$

is also a pre-sheaf.

All the pre-sheaves $\mathcal{F}$ in Example 3.48 have a common feature: if $U = \bigcup_{\alpha \in \mathcal{I}} V_\alpha$, then

(1) (gluing property) if $(\eta_\alpha)_{\alpha \in \mathcal{I}} \in \bigoplus_\alpha \mathcal{F}(V_\alpha)$ such that $\eta_\alpha|_{V_{\alpha\beta}} = \eta_\beta|_{V_{\alpha\beta}}$ for all $\alpha, \beta \in \mathcal{I}$, then there is $\eta \in \mathcal{F}(U)$ such that $\eta_\alpha = \eta|_{U_\alpha}$ for all $\alpha \in \mathcal{I}$. In other words, if for all $\alpha, \beta \in \mathcal{I}$, η_α and η_β math on the overlap $V_{\alpha\beta}$, then they can be glued to each other to define an element of $\eta \in \mathcal{F}(U)$.

(2) (uniqueness) the element $\eta \in \mathcal{F}(U)$ in (1) above is unique.

Definition 3.49. Let M be a topological space. A sheaf $\mathcal{F}$ on M is a pre-sheaf that satisfies the two conditions above.

Example 3.50. To construct a pre-sheaf that is not a sheaf, we can consider the pre-sheaf that assigns to each U the group of constant functions on U with values in a fixed group G. If $U = V_1 \sqcup V_2$, and $g_1, g_2 \in G$ with $g_1 \neq g_2$, then g_1 and g_2 define constant functions on V_1 and V_2, respectively. However, the union of them is not a constant function on U! Therefore, this is not a sheaf. To fix the issue above, we consider locally-constant functions instead of constant functions.

We will see more sophisticated examples of a pre-sheaf that is not a sheaf below.

HW 3.51. Given a fiber bundle $F \longrightarrow M$, the assignment $U \longrightarrow H^*_{\rm dR}(\pi^{-1}(U),\mathbb{R})$ is a pre-sheaf. Is this also a sheaf?

Next, suppose $\mathcal{F}$ is a pre-sheaf on a topological space M that takes values in the category of abelian groups. We will denote the group structure on each $\mathcal{F}(U)$ by addition, but in practice the group structure may indeed be a product. For instance, in Example 3.48, with $G=\mathbb{C}^*$, we get the sheaf of locally constant functions with values in $\mathbb{C}^*$ where the group structure is the product. We define the cech cohomology of a pre-sheaf $\mathcal{A}$ with respect to an open cover $\mathcal{A}=\{U_\alpha\}_{\alpha\in\mathcal{I}}$ following the same recipe as in (3.33).

Define

$$C^q(\mathcal{A},\mathcal{F}) = \bigoplus_{I\in\mathcal{P}_{q+1}(\mathcal{I})} \mathcal{F}(U_I).$$

Like before, it will be convenient to write an element $\eta\in C^q(\mathcal{A},\mathcal{F})$ as

$$\eta = \bigoplus_{i_0,\dots,i_q\in\mathcal{I}} \eta_{i_0 i_1\dots i_q}$$

with the convention that

$$\eta_{i_0 i_1\dots i_q} = (-1)^{\varepsilon(\sigma)}\eta_{i_{\sigma(0)} i_{\sigma(1)}\dots i_{\sigma(q)}}$$

for every permutation σ of $(i_0,\dots,i_q)$. In particular, just like differential forms,

$$\eta_{i_0 i_1\dots i_q}=0$$

whenever $i_a=i_b$ for some $a\neq b$; otherwise,

$$\eta_{i_0 i_1\dots i_q}\in\mathcal{F}(U_{i_0 i_1\dots i_q}).$$

The co-boundary map $\mathcal{R}\colon C^{q-1}(\mathcal{A},\mathcal{F})\longrightarrow C^q(\mathcal{A},\mathcal{F})$ is defined by

$$\big(\mathcal{R}(\eta)\big)_{i_0 i_1\dots i_q} = \sum_{a=0}^{q}(-1)^a\,\eta_{i_0\dots i_{a-1}i_{a+1}\dots i_q}|_{U_{i_0\dots i_q}}.$$

It is easy to check that $\mathcal{R}\circ\mathcal{R}=0$ (again, remember that 0 means the trivial homomorphism of the category in the question. If the group structure is a product, this will be the trivial homomorphism the maps everything to 1). Therefore $\big(C^\bullet(\mathcal{A},\mathcal{F}),\mathcal{R}\big)$ is a cochain complex. The cech cohomology groups of $\mathcal{F}$ with respect to $\mathcal{A}$ are the cohomology groups of this complex:

$$\check{H}^k(\mathcal{A},\mathcal{F}) = \frac{\mathrm{Ker}\Big(C^k(\mathcal{A},\mathcal{F})\xrightarrow{\mathcal{R}} C^{k+1}(\mathcal{A},\mathcal{F})\Big)}{\mathrm{Image}\Big(C^{k-1}(\mathcal{A},\mathcal{F})\xrightarrow{\mathcal{R}} C^{k}(\mathcal{A},\mathcal{F})\Big)}. \tag{3.36}$$

The natural questions are:

(1) How does this group depend on $\mathcal{A}$?
(2) How can we obtain cohomology groups that are invariants of $\mathcal{F}$ only?

We discuss a few examples before answering these questions.

Example 3.52. Suppose M is a smooth m-manifold; the assignment

$$U \longrightarrow \mathcal{O}^*_{\mathrm{sm},M}(U) := C^\infty(U, \mathbb{C}^*)$$

defines a sheaf of abelian groups where the group structure is the point-wise multiplication of $\mathbb{C}^*$-valued smooth functions. Suppose $\mathcal{A} = \{U_\alpha\}$ is an atlas on M where each U_α is a ball in $\mathbb{R}^m$. By definition, a 1-cocycle

$$\varphi \in C^1(\mathcal{A}, \mathcal{O}^*_{\mathrm{sm},M})$$

is a collection of $\mathbb{C}^*$-valued smooth function

$$\varphi_{\alpha\beta} \colon U_{\alpha\beta} = U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*$$

such that

$$(\mathcal{R}\varphi)_{\alpha\beta\gamma} = \varphi_{\beta\gamma}\varphi^{-1}_{\alpha\gamma}\varphi_{\alpha\beta}|_{U_{\alpha\beta\gamma}} = 1.$$

Note that this is the multiplicative version of the definition of the cochain map of the cech cohomology. So $\mathcal{R}\phi = 1$ if and only if

$$\varphi_{\alpha\gamma}(x) = \varphi_{\beta\gamma}(x)\varphi_{\alpha\beta}(x) \qquad \forall\ x \in U_{\alpha\beta\gamma}.$$

This is exactly the cocycle condition of the transition maps

$$(U_\alpha \times \mathbb{C})|_{U_{\alpha\beta}} \longrightarrow (U_\beta) \times \mathbb{C}|_{U_{\alpha\beta}}, \qquad (x, c) \longrightarrow (x, \varphi_{\alpha\beta}(x)c)$$

of a complex line $E \longrightarrow M$ bundle. Conversely, if $E \longrightarrow M$ is a complex line bundle, for every $\alpha \in \mathcal{I}$, since U_α is a ball in $\mathbb{R}^m$, the restriction $E|_{U_\alpha}$ is isomorphic to the trivial bundle $U_\alpha \times \mathbb{C}$. Therefore, the transition maps $\{\varphi_{\alpha\beta} \colon U_{\alpha\beta} \longrightarrow \mathrm{End}(\mathbb{C}) = \mathbb{C}^*\}$ define a cech 1-coccyle φ. Two cech 1-cocyles $\varphi = \{\varphi_{\alpha\beta}\}$ and $\varphi' = \{\varphi'_{\alpha\beta}\}$ define the same cohomology group if and only if they differ by a coboundary; i.e. if

$$\varphi'_{\alpha\beta}\varphi^{-1}_{\alpha\beta} = (\mathcal{R}\theta)_{\alpha\beta} = \theta_\beta\theta^{-1}_\alpha \tag{3.37}$$

for some

$$\theta = (\theta_\alpha)_{\alpha\in\mathcal{I}} \in C^0(\mathcal{A}, \mathcal{O}^*_{\mathrm{sm},M}) = \prod_{\alpha\in\mathcal{I}} C^\infty(U_\alpha, \mathbb{C}^*).$$

Let E and E' denote the complex line bundles corresponding to φ and φ', respectively. By (3.37), the following diagram commutes

$$\begin{array}{ccc} U_\alpha \times \mathbb{C}|_{U_{\alpha\beta}} & \xrightarrow{\theta_\alpha} & U_\alpha \times \mathbb{C}|_{U_{\alpha\beta}} \\ \Big\downarrow \varphi_{\alpha\beta} & & \Big\downarrow \varphi'_{\alpha\beta} \\ U_\beta \times \mathbb{C}|_{U_{\alpha\beta}} & \xrightarrow{\theta_\beta} & U_\beta \times \mathbb{C}|_{U_{\alpha\beta}}\ . \end{array}$$

Which means the local isomorphisms

$$E|_{U_\alpha} \cong U_\alpha \times \mathbb{C} \longrightarrow E'|_{U_\alpha} \cong U_\alpha \times \mathbb{C}, \qquad (x, c) \longrightarrow (x, \theta_\alpha(x)c)$$

are compatible on the overlaps and define a global isomorphism $E \xrightarrow{\theta} E'$. We conclude that there is a one-to-one correspondence between the elements of the cech cohomology group $\check{H}^1(\mathcal{A}, \mathcal{O}^*_{\mathrm{sm},M})$ and the isomorphism classes of smooth complex line bundles on M. Furthermore, $\check{H}^1(\mathcal{A}, \mathcal{O}^*_{\mathrm{sm},M})$ does not depend on such an atlas $\mathcal{A}$ and therefore we denote it by $\check{H}^1(M, \mathcal{O}^*_{\mathrm{sm},M})$.

The set of isomorphism classes of complex line bundles on any smooth manifold M is a group whose identity element is the trivial bundle $M \times \mathbb{C}$ and whose product structure is the tensor product

$$E, E' \longrightarrow E \otimes E'.$$

This group structure coincides with the group structure on $\check{H}^1(M, \mathcal{O}^*_{\mathrm{sm},M})$.

HW 3.53. • Use a covering of S^2 with two disks to show that $\check{H}^1(S^2, \mathcal{O}^*_{\mathrm{sm},S^2}) \cong \mathbb{Z}$.

- Consequently, show that the group of isomorphism classes of smooth complex line bundles on $S^2 \cong \mathbb{P}^1$ is generated by γ or γ^*, where γ is the tautological complex line bundle of $\mathbb{P}^1$.

- Show that

$$\int_{\mathbb{P}^1} c_1(\gamma^*) = 1.$$

  Conclude that the isomorphism $\check{H}^1(\mathbb{P}^1, \mathcal{O}^*_{\mathbb{P}^1}) \cong \mathbb{Z}$ is also given by

$$E \longrightarrow \deg(E) := \int_{\mathbb{P}^1} c_1(E).$$

- Use HW 2.48 to show that $T\mathbb{P}^1 = (\gamma^*)^{\otimes 2}$.

HW 3.54. Suppose M is a complex m-manifold; the assignment

$$U \longrightarrow \mathcal{O}^*_M(U) = \{\text{nowhere-zero holomoprhic functions on } U\}$$

defines a sheaf of abelian groups where the group structure is the point-wise multiplication of $\mathbb{C}^*$-valued holomorphic functions. Suppose $\mathcal{A} = \{U_\alpha\}$ is an atlas on M where each U_α is a ball in $\mathbb{C}^m$. Repeat Example 3.52 to show that there is a one-to-one correspondence between the elements of the cech cohomology group $\check{H}^1(\mathcal{A}, \mathcal{O}^*_M)$ and the isomorphism classes of holomorphic line bundles on M. Repeat HW 3.53 by showing $\check{H}^1(\mathbb{P}^1, \mathcal{O}^*_{\mathbb{P}^1}) \cong \mathbb{Z}$.

HW 3.55. Unlike for $\mathbb{P}^1 = S^2$, where

$$\check{H}^1(\mathbb{P}^1, \mathcal{O}^*_{\mathrm{sm},\mathbb{P}^1}) \cong \check{H}^1(\mathbb{P}^1, \mathcal{O}^*_{\mathbb{P}^1}) \cong \mathbb{Z},$$

for $M = \mathbb{T}^2 = \mathbb{C}/\mathbb{Z}^2$, show that

$$\check{H}^1(\mathbb{T}^2, \mathcal{O}^*_{\mathrm{sm},\mathbb{T}^2}) \cong \mathbb{Z} \quad \text{and} \quad \check{H}^1(\mathbb{T}^2, \mathcal{O}^*_{\mathbb{T}^2}) \cong \mathbb{Z} \times \mathbb{T}^2.$$

This means there is a $\mathbb{T}^2$-family of different holomorphic line bundles on $\mathbb{T}^2$ that are all smoothly isomorphic to the trivial complex line bundle $\mathbb{T}^2 \times \mathbb{C}$.

Given two open coverings $\mathcal{A} = \{U_i\}_{i \in \mathcal{I}}$ and $\mathcal{B} = \{V_j\}_{j \in \mathcal{J}}$ of a topological space M, we say $\mathcal{B}$ is a refinement of $\mathcal{A}$ if for every $j \in \mathcal{J}$, there is $i \in \mathcal{I}$ such that $V_j \in U_i$. Every two open coverings of M admit a common refinement.

Example 3.56. Suppose M is a smooth m-manifold and fix $p \geq 0$. The assignment

$$U \longrightarrow \Omega^p(U)$$

defines a sheaf which we denote it by Ω^p. For every open covering $\mathcal{A}$, by Lemma 3.40, we have

$$\check{H}^k(\mathcal{A},\Omega^p)=\begin{cases}\Omega^p(M) & \text{if} \quad k=0;\\ 0 & \text{otherwise.}\end{cases}$$

In general, any sheaf $\mathcal{F}$ that admits partition of unity has this property; i.e.

$$\check{H}^k(\mathcal{A},\mathcal{F})=\begin{cases}\mathcal{F}(M) & \text{if} \quad k=0;\\ 0 & \text{otherwise.}\end{cases}$$

Locally constant sheaves $\underline{G}$, sheaves of holomorphic nature, and $\mathcal{O}^*_{\mathrm{sm,M}}$ do not admit partition of unity. On the other hand, sheaves of smooth sections of a vector bundle admit partition of unity and have trivial cech cohomology in positive degree.

Suppose $\mathcal{B}$ is a refinement of $\mathcal{A}$, and fix a map

$$\varrho\colon \mathcal{J}\longrightarrow\mathcal{I} \quad \text{s.t} \quad V_j\subset U_{\varrho(j)} \qquad \forall\, j\in\mathcal{J}. \tag{3.38}$$

If $\mathcal{F}$ is a pre-sheaf on M, the induced maps

$$\begin{aligned}&\varrho\colon C^q(\mathcal{A},\mathcal{F})=\bigoplus_{I\in\mathcal{P}_{q+1}(\mathcal{I})}\mathcal{F}(U_I)\longrightarrow C^q(\mathcal{B},\mathcal{F})=\bigoplus_{J\in\mathcal{P}_{q+1}(\mathcal{J})}\mathcal{F}(V_J),\\ &\big(\varrho(\eta)\big)_{j_0\dots j_q}=\eta_{\varrho(j_0)\dots\varrho(j_q)}|_{V_{j_0\dots j_q}} \qquad \forall\, q\geq 0,\end{aligned}$$

define a cochain map

$$\varrho\colon \big(C^\bullet(\mathcal{A},\mathcal{F}),\mathcal{R}\big)\longrightarrow\big(C^\bullet(\mathcal{B},\mathcal{F}),\mathcal{R}\big); \tag{3.39}$$

i.e. ϱ commutes with $\mathcal{R}$ on both sides. Let

$$\varrho\colon \check{H}^k(\mathcal{A},\mathcal{F})\longrightarrow\check{H}^k(\mathcal{B},\mathcal{F}) \qquad \forall\, k\geq 0$$

denote the induced maps between the cech cohomology groups.

Lemma 3.57. *For different maps ϱ and ϱ' as in (3.38), the induced cochain maps ϱ and ϱ' in (3.39) are chain homotopic. Therefore,*

$$\varrho=\varrho'\colon \check{H}^k(\mathcal{A},\mathcal{F})\longrightarrow\check{H}^k(\mathcal{B},\mathcal{F}) \qquad \forall\, k\geq 0;$$

i.e. there are well-defined group homomorphisms

$$\check{H}^k(\mathcal{A},\mathcal{F})\longrightarrow\check{H}^k(\mathcal{B},\mathcal{F}) \qquad \forall\, k\geq 0$$

that only depend on $\mathcal{A}$ and $\mathcal{B}$ (this partially answers Question (1) above).

Proof. For

$$\begin{aligned}&K\colon C^{q+1}(\mathcal{A},\mathcal{F})\longrightarrow C^q(\mathcal{B},\mathcal{F}),\\ &\big(K(\eta)\big)_{j_0\dots j_q}=\sum_{a=0}^{q}(-1)^a\eta_{\varrho(j_0)\dots\varrho(j_a)\varrho'(j_a)\dots\varrho'(j_q)}|_{V_{j_0\dots j_q}}\end{aligned}$$

show that

$$\varrho-\varrho'=\mathcal{R}K+K\mathcal{R}\colon \big(C^\bullet(\mathcal{A},\mathcal{F}),\mathcal{R}\big)\longrightarrow\big(C^\bullet(\mathcal{B},\mathcal{F}),\mathcal{R}\big).$$

□

The cech cohomology groups of M with values in $\mathcal{F}$ are defined to be the direct limits of $\check{H}^k(\mathcal{A},\mathcal{F})$ with respect to the refinement ordering $\mathcal{A}<\mathcal{B}$; i.e.

$$\check{H}^k(M,\mathcal{F}) = \varinjlim_{\mathcal{A}} \check{H}^k(\mathcal{A},\mathcal{F}) \qquad \forall\, k\geq 0.$$

This answers Question (1) above. In practice, there is often a "good class" of open coverings $\mathcal{A}$ on which the limit is achieved; i.e.

$$\check{H}^k(M,\mathcal{F}) = \check{H}^k(\mathcal{A},\mathcal{F}) \qquad \forall\, k\geq 0$$

whenever $\mathcal{A}$ belongs to this good class of open coverings. For example, in the case of the locally constant sheaf $\underline{\mathbb{R}}$ on a smooth manifold M, by Proposition 3.35, the limit is achieved on the class of good coverings.

A map of pre-sheaves $f\colon \mathcal{F}\longrightarrow\mathcal{G}$ on M is a collection of maps

$$\{f_U\colon \mathcal{F}(U)\longrightarrow \mathcal{G}(U)\}_{U\in \mathrm{Obj}(\mathrm{Open}(M))}$$

that commute with the restriction maps; i.e. $\mathcal{R}_{U,V}\circ f_U = f_V\circ \mathcal{R}_{U,V}$ for all $V\subset U$. If $\mathcal{F}$ and $\mathcal{G}$ take in (abelian) groups, the kernel of f is the per-sheaf

$$\mathrm{Ker}(f)(U) = \mathrm{Ker}\big(f_U\colon \mathcal{F}(U)\longrightarrow\mathcal{G}(U)\big) \qquad \forall\, U\in \mathrm{Obj}(\mathrm{Open}(M)).$$

HW 3.58. Show that if $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then $\mathrm{Ker}(f)$ is also a sheaf.

Similarly, the cokernel pre-sheaf is defined by

$$\mathrm{CoKer}(f)(U) = \frac{\mathcal{G}(U)}{f_U(\mathcal{F}(U))} \qquad \forall\, U\in \mathrm{Obj}(\mathrm{Open}(M)).$$

Unlike kernel, even if $\mathcal{F}$ and $\mathcal{G}$ are sheaves, the cokernel need not be to be a sheaf. Here is an example.

Example 3.59. Let $M=\mathbb{C}-\{0\}$, $\mathcal{G}=\mathcal{O}^*_M$, $\mathcal{F}=\mathcal{O}_M$, where $\mathcal{O}_M$ is the sheaf of holomorphic functions. Consider the sheaf map $\exp\colon \mathcal{O}_M\longrightarrow\mathcal{O}^*_M$, where

$$\exp_U\colon \mathcal{O}_M(U)\longrightarrow \mathcal{O}^*_M(U), \qquad \alpha(z)\longrightarrow\beta(z)=\mathrm{e}^{2\pi\mathrm{i}\alpha(z)}.$$

is the exponential map for all $U\in\mathrm{Obj}(\mathrm{Open}(M))$. Since $\log(z)$ is not defined over $U=\mathbb{C}-\{0\}$, the function $z\in\underline{\mathbb{C}}^*(\mathbb{C}-\{0\})$ gives a non-trivial element

$$[1]\neq[z]\in \frac{\mathcal{O}^*_M(\mathbb{C}-\{0\})}{\exp_{\mathbb{C}-\{0\}}(\mathcal{O}_M(\mathbb{C}-\{0\}))}.$$

Suppose $U=V_1\cup V_2$ such that V_1 and V_2 are simply connected open subsets. Then, $\log(z|_{V_1})$ and $\log(z|_{V_2})$ are defined; therefore, the restrictions of the $\mathbb{C}^*$-valued function z to V_1 and V_2 are trivial in

$$\frac{\mathcal{O}^*_M(V_1)}{\exp_U(\mathcal{O}_M(V_1))} \quad\text{and}\quad \frac{\mathcal{O}^*_M(V_2)}{\exp_U(\mathcal{O}_M(V_2))},$$

respectively. This implies that, at least, the "uniqueness" condition in Definition 3.49 is not satisfied. The problem here is that $\mathrm{Coker}(\exp_{\mathbb{C}-\{0\}})$ is bigger than what it should be. Note that $\mathrm{Ker}(\exp)$ is the sheaf $\underline{\mathbb{Z}}$ of locally constant functions with values in $\mathbb{Z}$.

As the previous example illustrates, in order to build a sheaf $\mathcal{F}$ from a pre-sheaf $\widetilde{\mathcal{F}}$, we need to break large open sets U into smaller pieces in the following way.

Proposition 3.60. *Suppose $\widetilde{\mathcal{F}}$ is a pre-sheaf on M that takes values in abelian groups. For every pair of open sets $V \subset U$, there is a canonical restriction map*

$$\mathcal{R}_{U,V} \colon \check{H}^0(U, \widetilde{\mathcal{F}}|_U) \longrightarrow \check{H}^0(V, \widetilde{\mathcal{F}}|_V) \tag{3.40}$$

such that the assignment

$$U \longrightarrow \mathcal{F}(U) = \check{H}^0(U, \widetilde{\mathcal{F}}|_U) \tag{3.41}$$

defines a sheaf on M.

The process $\widetilde{\mathcal{F}} \longrightarrow \mathcal{F}$ is known as sheafification. Before, we go over the proof, let's elaborate on the meaning of (3.41) a little bit. The two defining conditions of a sheaf $\mathcal{F}$ are equivalent to the exactness of the sequence

$$\begin{gathered} 0 \longrightarrow \mathcal{F}(U) \xrightarrow{\mathcal{R}} \bigoplus_{\alpha} \mathcal{F}(U_\alpha) \xrightarrow{\mathcal{R}} \bigoplus_{\alpha,\beta} \mathcal{F}(U_{\alpha\beta}), \\ \mathcal{F}(U) \ni \eta \longrightarrow \bigoplus_{\alpha} \eta|_{U_\alpha}, \qquad \bigoplus_{\alpha} \eta|_{U_\alpha} \longrightarrow \bigoplus_{\alpha,\beta} \big(\eta_\beta - \eta_\alpha\big)|_{U_{\alpha,\beta}}, \end{gathered}$$

for every decomposition $U = \bigcup_{\alpha \in \mathcal{I}} U_\alpha$ of an open set U into open subsets. On the other hand, for $\mathcal{A} = \{U_\alpha\}_{\alpha \in \mathcal{I}}$, by definition we have

$$\check{H}^0(\mathcal{A}, \mathcal{F}|_U) = \mathrm{Ker}\Big(\bigoplus_{\alpha} \mathcal{F}(U_\alpha) \xrightarrow{\mathcal{R}} \bigoplus_{\alpha,\beta} \mathcal{F}(U_{\alpha\beta}) \Big).$$

Therefore, if $\mathcal{F}$ is a sheaf then $\mathcal{F}(U) = H^0(\mathcal{A}, \mathcal{F}|_U)$; i.e. $\mathcal{F}(U)$ is uniquely determined by its restrictions to U_α. This explains the motivation behind (3.41): for a sufficiently refined decomposition $\mathcal{A}$, an element of $\mathcal{F}(U)$ is a collection of sections of $\widetilde{\mathcal{F}}$ on U_α that match along the overlaps. Note that $\check{H}^0(U, \widetilde{\mathcal{F}}|_U)$ involves taking limit on $\mathcal{A}$, and often, this limit is achieved for certain class of open coverings.

Proof of Proposition 3.60. First, we describe the restriction maps (3.40). For every decomposition $U = \bigcup_{\alpha \in \mathcal{I}} U_\alpha$ of an open set U into open subsets, and every open subset $V \subset U$, we get an induced open decomposition $V = \bigcup_{\alpha \in \mathcal{I}} V_\alpha$ such that $V_\alpha = V \cap U_\alpha$ for $\alpha \in \mathcal{I}$. Let $\mathcal{A} = \{U_\alpha\}_{\alpha \in \mathcal{I}}$ and $\mathcal{A} \cap V := \{V_\alpha\}_{\alpha \in \mathcal{I}}$. Since the diagram

$$\begin{array}{ccc} \bigoplus_\alpha \widetilde{\mathcal{F}}(U_\alpha) & \xrightarrow{\quad \mathcal{R} \quad} & \bigoplus_{\alpha,\beta} \widetilde{\mathcal{F}}(U_{\alpha\beta}) \\ {\scriptstyle \bigoplus_{\alpha \in \mathcal{I}} \mathcal{R}_{U_\alpha, V_\alpha}} \downarrow & & \downarrow {\scriptstyle \bigoplus_{\alpha,\beta \in \mathcal{I}} \mathcal{R}_{U_{\alpha\beta}, V_{\alpha\beta}}} \\ \bigoplus_\alpha \widetilde{\mathcal{F}}(V_\alpha) & \xrightarrow{\quad \mathcal{R} \quad} & \bigoplus_{\alpha,\beta} \widetilde{\mathcal{F}}(V_{\alpha\beta}) \end{array}$$

commutes, we get a map from the kernel of the first row to the kernel of the second row:

$$\check{H}^0(\mathcal{A}, \widetilde{\mathcal{F}}|_U) \longrightarrow \check{H}^0(\mathcal{A} \cap V, \widetilde{\mathcal{F}}|_V).$$

Taking the limit of these maps with respect to the refinements of $\mathcal{A}$, we obtain the canonical restriction map in (3.40). Using a similar commutative diagram for $W \subset V \subset U$, it is straightforward to check that (3.40) satisfies the third condition of Definition 3.46.

Suppose $U=\bigcup_{i\in\mathcal{I}}U_i$ as well as $U=\bigcup_{j\in\mathcal{J}}V_j$ are arbitrary open decompositions of U. Let

$$\mathcal{A}_j=\{U_i\cap V_j\}_{i\in\mathcal{I}}\quad\forall\, j\in\mathcal{J},\qquad \mathcal{A}=\bigcup_{j\in\mathcal{J}}\mathcal{A}_j,$$
$$\mathcal{A}_{jj'}=\{U_i\cap V_j\cap V_{j'}\}_{i\in\mathcal{I}}\quad\forall\, j\neq j'\in\mathcal{J}.$$

Suppose

$$\eta_j\in\check{H}^0(\mathcal{A}_j,\widetilde{\mathcal{F}}|_{V_j}).$$

By definition $\eta_j=\bigoplus_{i\in\mathcal{I}}\eta_{j;i}$ such that

$$\big(\eta_{j;i}-\eta_{j;i'}\big)|_{V_{j;ii'}=V_j\cap U_i\cap U_{i'}}=0\qquad i,i'\in\mathcal{I}. \tag{3.42}$$

By the definition of (3.40) above we have

$$\eta_j|_{V_{jj'}}=\bigoplus_{i\in\mathcal{I}}\eta_{j;i}|_{U_i\cap V_j\cap V_{j'}}.$$

Therefore,

$$\eta_j|_{V_{jj'}}=\eta_{j'}|_{V_{jj'}}\qquad\forall\, j,j'\in\mathcal{J}$$

if and only if

$$\eta_{j;i}|_{U_i\cap V_j\cap V_{j'}}=\eta_{j';i}|_{U_i\cap V_j\cap V_{j'}}\qquad\forall\, i\in\mathcal{I},\ j,j'\in\mathcal{J}. \tag{3.43}$$

Therefore, by (3.42) and (3.42), for $i,i'\in\mathcal{I}$ and $j,j'\in\mathcal{J}$,

$$\eta_{j;i}\big|_{(U_i\cap V_j)\cap(U_i'\cap V_{j'})}=\eta_{j';i'}\big|_{(U_i\cap V_j)\cap(U_i'\cap V_{j'})};$$

i.e. $\eta=\bigoplus_{j\in\mathcal{J},i\in\mathcal{I}}\eta_{j;i}$ defines an element of $\check{H}^0(\mathcal{A},\widetilde{\mathcal{F}}|_U)$. This implies that the sequence

$$\check{H}^0(\mathcal{A},\widetilde{\mathcal{F}}|_U)\longrightarrow\bigoplus_{j\in\mathcal{J}}\check{H}^0(\mathcal{A}_j,\widetilde{\mathcal{F}}|_{V_j})\longrightarrow\bigoplus_{j,j'\in\mathcal{J}}\check{H}^0(\mathcal{A}_{jj'},\widetilde{\mathcal{F}}|_{V_{jj'}})$$

is exact. The first map is clearly an inclusion; therefore,

$$0\longrightarrow\check{H}^0(\mathcal{A},\widetilde{\mathcal{F}}|_U)\longrightarrow\bigoplus_{j\in\mathcal{J}}\check{H}^0(\mathcal{A}_j,\widetilde{\mathcal{F}}|_{V_j})\longrightarrow\bigoplus_{j,j'\in\mathcal{J}}\check{H}^0(\mathcal{A}_{jj'},\widetilde{\mathcal{F}}|_{V_{jj'}})$$

is exact. Taking limit, we conclude that

$$0\longrightarrow\check{H}^0(U,\widetilde{\mathcal{F}}|_U)\longrightarrow\bigoplus_{j\in\mathcal{J}}\check{H}^0(V_j,\widetilde{\mathcal{F}}|_{V_j})\longrightarrow\bigoplus_{j,j'\in\mathcal{J}}\check{H}^0(V_{jj'},\widetilde{\mathcal{F}}|_{V_{jj'}})$$

is exact; i.e. $\mathcal{F}$ is a sheaf.

□

Theorem 3.61. *A pre-sheaf $\widetilde{\mathcal{F}}$ and its sheafification $\mathcal{F}$ have the same cech cohomology groups. For every sheaf $\mathcal{G}$ and a map of pre-sheaves $\widetilde{f}\colon\widetilde{\mathcal{F}}\longrightarrow\mathcal{G}$, there is a map of sheaves*

$$f\colon\mathcal{F}\longrightarrow\mathcal{G}$$

such that

$$\widetilde{f}_U=f_U\circ\iota_U\colon\widetilde{\mathcal{F}}(U)\longrightarrow\mathcal{G}(U).$$

Here, $\iota_U\colon\widetilde{\mathcal{F}}(U)\longrightarrow\mathcal{F}(U)$ is the natural restriction map.

We leave the proof of this theorem to the reader. The key point of the first statement is that cech cohomology involves taking limit with respect to refinements. The second statement follows from the commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \check{H}^0(\mathcal{A}, \widetilde{\mathcal{F}}|_U) & \longrightarrow & \bigoplus_\alpha \widetilde{\mathcal{F}}(U_\alpha) & \xrightarrow{\mathcal{R}} & \bigoplus_{\alpha,\beta} \widetilde{\mathcal{F}}(U_{\alpha\beta}) \\ & & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathcal{G}(U) & \xrightarrow{\mathcal{R}} & \bigoplus_\alpha \mathcal{G}(U_\alpha) & \xrightarrow{\mathcal{R}} & \bigoplus_{\alpha,\beta} \mathcal{G}(U_{\alpha\beta}) \,. \end{array}$$

We say that a sequence of sheaf maps

$$0 \longrightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{E} \longrightarrow 0$$

is exact if $\mathcal{F}$ is $\operatorname{Ker}(g)$ and $\mathcal{E}$ is the sheafification of $\operatorname{Coker}(f)$. In this situation, we also say $\mathcal{F}$ is a sub-sheaf of $\mathcal{G}$ and $\mathcal{E}$ is a quotient of $\mathcal{G}$. Note that the exactness above does not mean that

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{f_U} \mathcal{G}(U) \xrightarrow{g_U} \mathcal{E}(U) \longrightarrow 0$$

is exact for all open subsets $U \subset M$; i.e. g_U may not be surjective. We write

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{f_U} \mathcal{G}(U) \xrightarrow{g_U} \mathcal{E}(U)$$

to indicate that $\mathcal{F}(U) = \operatorname{Ker}(f_U)$ but $\operatorname{Coker}(g_U)$ might be non-trivial. More generally, we say

$$\cdots \longrightarrow \mathcal{F}_{k-1} \xrightarrow{f_{k-1}} \mathcal{F}_k \xrightarrow{f_k} \mathcal{F}_{k+1} \longrightarrow \cdots$$

is exact if $\operatorname{Ker}(f_k)$ is equal to the sheafification of $\operatorname{Coker}(f_{k-1})$.

Example 3.62. Here are some important examples of exact sequences.
(1) If M is a holomorphic manifold, the sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_M^* \longrightarrow 0,$$

where the first map is the inclusion map, is exact.
(2) Similarly, for every smooth manifold M, let $\mathcal{O}_{\mathrm{sm},M}$ denote the sheaf of smooth $\mathbb{C}$-valued functions. Then the sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_{\mathrm{sm},M} \longrightarrow \mathcal{O}_{\mathrm{sm},M}^* \longrightarrow 0,$$

(3) For every smooth manifold M, by Poincare Lemma, the sequence of sheaves

$$0 \longrightarrow \underline{\mathbb{R}} \xrightarrow{\iota} \Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \xrightarrow{\mathrm{d}} \Omega^2 \longrightarrow \ldots,$$

where Ω^p is the sheaf of p-forms in Example 3.48, is exact.
(4) If M is a holomorphic manifold, let Ω^p_{hol} denote the sheaf of holomorphic p-forms on M, and $\Omega^{p,q}$ denote the sheaf of smooth $(p+q)$-forms on type (p,q) on M. By $\bar{\partial}$-Poincare Lemma, the sequence

$$0 \longrightarrow \Omega^p_{\mathrm{hol}} \xrightarrow{\iota} \Omega^{p,0} \xrightarrow{\bar{\partial}} \Omega^{p,1} \xrightarrow{\bar{\partial}} \Omega^{p,2} \longrightarrow \ldots$$

is exact.

Theorem 3.63. *Corresponding to every short exact sequence of sheaves*

$$0 \longrightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{E} \longrightarrow 0$$

there is a long exact sequence of cech cohomology groups

$$\begin{aligned} 0 &\longrightarrow \check{H}^0(M,\mathcal{F}) \xrightarrow{f_*} \check{H}^0(M,\mathcal{G}) \xrightarrow{g_*} \check{H}^0(M,\mathcal{E}) \\ &\xrightarrow{\delta_*} \check{H}^1(M,\mathcal{F}) \xrightarrow{f_*} \check{H}^1(M,\mathcal{G}) \xrightarrow{g_*} \check{H}^1(M,\mathcal{E}) \\ &\longrightarrow \cdots . \end{aligned}$$

The proof is by some standard diagram chasing argument.

Example 3.64. (1) Suppose M is a smooth manifold. The short exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_{\mathrm{sm},M} \longrightarrow \mathcal{O}^*_{\mathrm{sm},M} \longrightarrow 0,$$

results in the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \check{H}^0(M,\underline{\mathbb{Z}}) \longrightarrow \check{H}^0(M,\mathcal{O}_{\mathrm{sm},M}) \longrightarrow \check{H}^0(M,\mathcal{O}^*_{\mathrm{sm},M}) \\ &\xrightarrow{\delta_*} \check{H}^1(M,\underline{\mathbb{Z}}) \longrightarrow \check{H}^1(M,\mathcal{O}_{\mathrm{sm},M}) \longrightarrow \check{H}^1(M,\mathcal{O}^*_{\mathrm{sm},M}) \\ &\xrightarrow{\delta_*} \check{H}^2(M,\underline{\mathbb{Z}}) \longrightarrow \check{H}^2(M,\mathcal{O}_{\mathrm{sm},M}) \longrightarrow \check{H}^1(M,\mathcal{O}^*_{\mathrm{sm},M}) \\ &\longrightarrow \cdots . \end{aligned}$$

Since the sheaf $\mathcal{O}_{\mathrm{sm},M}$ admits partition of unity, all of its higher degree cech cohomology groups are trivial. Therefore,

$$\begin{aligned} 0 &\longrightarrow \check{H}^0(M,\underline{\mathbb{Z}}) \longrightarrow \check{H}^0(M,\mathcal{O}_{\mathrm{sm},M}) \longrightarrow \check{H}^0(M,\mathcal{O}^*_{\mathrm{sm},M}) \\ &\xrightarrow{\delta_*} \check{H}^1(M,\underline{\mathbb{Z}}) \longrightarrow 0 \longrightarrow \check{H}^1(M,\mathcal{O}^*_{\mathrm{sm},M}) \\ &\xrightarrow{\delta_*} \check{H}^2(M,\underline{\mathbb{Z}}) \longrightarrow 0 \longrightarrow \check{H}^2(M,\mathcal{O}^*_{\mathrm{sm},M}) \\ &\longrightarrow \cdots . \end{aligned}$$

We conclude that

$$\check{H}^1(M,\mathcal{O}^*_{\mathrm{sm},M}) \cong \check{H}^2(M,\underline{\mathbb{Z}}).$$

By Example 3.52, $\check{H}^1(M,\mathcal{O}^*_{\mathrm{sm},M})$ is the group of isomorphism classes of the complex line bundles on M. The map above send each complex line bundle to its first chern class in

$$\check{H}^2(M,\underline{\mathbb{Z}}) \cong H^2_{\mathrm{sing}}(M,\underline{\mathbb{Z}}).$$

Therefore, every smooth complex line bundle is uniquely determined by its (integral) chern class. Note that, first chern class as a cohomology class in $H^2_{\mathrm{dR}}(M,\mathbb{R})$ does not see the torsion part of the integral first chern class in singular cohomology.

(2) Suppose M is holomorphic. The short exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}^*_M \longrightarrow 0,$$

results in the long exact sequence

$$\begin{aligned}
0 &\longrightarrow \check{H}^0(M,\underline{\mathbb{Z}}) \longrightarrow \check{H}^0(M,\mathcal{O}_M) \longrightarrow \check{H}^0(M,\mathcal{O}_M^*)\\
&\xrightarrow{\delta_*} \check{H}^1(M,\underline{\mathbb{Z}}) \longrightarrow \check{H}^1(M,\mathcal{O}_M) \longrightarrow \check{H}^1(M,\mathcal{O}_M^*)\\
&\xrightarrow{\delta_*} \check{H}^2(M,\underline{\mathbb{Z}}) \longrightarrow \check{H}^2(M,\mathcal{O}_M) \longrightarrow \check{H}^1(M,\mathcal{O}_M^*)\\
&\longrightarrow \cdots .
\end{aligned}$$

The first row of this diagram is exact and can be ignored. This time, however,

$$\check{H}^1(M,\mathcal{O}_M) \cong H^{0,1}_{\bar{\partial}}(M,\mathbb{C})$$

where the Dolbeault cohomology group $H^{0,1}_{\bar{\partial}}(M,\mathbb{C})$ is the first cohomology of the cochain complex

$$0 \longrightarrow \Omega^{0,0}(M) \xrightarrow{\bar{\partial}} \Omega^{0,1}(M) \xrightarrow{\bar{\partial}} \Omega^{0,2}(M) \longrightarrow \ldots .$$

For example, if M is a genus g Riemann surface, then $\check{H}^1(M,\mathcal{O}_M) \cong \mathbb{C}^g$. Therefore, we get

$$\begin{aligned}
0 &\longrightarrow \check{H}^1(M,\underline{\mathbb{Z}}) \longrightarrow \check{H}^1(M,\mathcal{O}_M) \longrightarrow \check{H}^1(M,\mathcal{O}_M^*)\\
&\xrightarrow{\delta_*} \check{H}^2(M,\underline{\mathbb{Z}}) \longrightarrow \ldots
\end{aligned}$$

The quotient

$$\frac{\check{H}^1(M,\mathcal{O}_M)}{\check{H}^1(M,\underline{\mathbb{Z}})}$$

is a torus known as $\mathrm{Pic}^0(M)$ or Jacobian of M. It is the group of holomorphic line bundles on M that are smoothly trivial; see HW 3.55. The group $\check{H}^1(M,\mathcal{O}_M^*)$ corresponds to the isomorphism classes of holomorphic line bundles on M and is denoted by $\mathrm{Pic}(M)$ (Picard group of M). As above, the map

$$\mathrm{Pic}(M) \longrightarrow \check{H}^2(M,\underline{\mathbb{Z}}) \cong H^2_{\mathrm{sing}}(M,\mathbb{Z})$$

sends each complex line bundle to its first chern class. Also, we have $\check{H}^2(M,\mathcal{O}_M) \cong H^{0,2}_{\bar{\partial}}(M,\mathbb{C})$. Therefore, we have an exact sequence.

$$0 \longrightarrow \mathrm{Pic}^0(M) \longrightarrow \mathrm{Pic}(M) \longrightarrow H^2_{\mathrm{sing}}(M,\mathbb{Z}) \longrightarrow H^{0,2}_{\bar{\partial}}(M,\mathbb{C})$$

If $H^{0,2}_{\bar{\partial}}(M,\mathbb{C}) = 0$, every element of $H^2_{\mathrm{sing}}(M,\mathbb{Z})$ is the first chern class of some holomorphic line bundle. Otherwise, there are (smooth) complex line bundles on M that do not admit any holomorphic structure. If M is a Riemann-surface, for dimensional reasons, $H^{0,2}_{\bar{\partial}}(M,\mathbb{C})=0$. In complex dimension 2, a K3 surface is a simply connected (Kahler) holomorphic surface M with $c_1(TM) = 0$. If M is a K3 surface, we have $H^{0,2}_{\bar{\partial}}(M,\mathbb{C}) \cong \mathbb{C}$. All K3 surfaces are smoothly identical (diffeomorphic). The complex line

$$\mathbb{C} \cong H^{0,2}_{\bar{\partial}}(M,\mathbb{C}) \subset H^2_{\mathrm{dR}}(M,\mathbb{C}) \cong \mathbb{C}^{22}$$

defines a point in the projective space $\mathbb{P}^{21} = \mathbb{P}(H^2_{\mathrm{dR}}(M,\mathbb{C}))$. This gives us a map

$$\{\text{The space of all K3 surfaces}\} \longrightarrow \mathbb{P}^{21}$$

that helps us understand the "moduli" space of K3 surfaces.

We finish this section with the following result.

Theorem 3.65. *Suppose M is a smooth m-manifold, then*

$$\check{H}^k(M,\underline{\mathbb{Z}}) \cong H^k_{\text{sing}}(M,\mathbb{Z}).$$

Proof. Fix a triangulation $\mathcal{K}$ of M. For each k-simplex $\tau \in \mathcal{K}$, let U_τ denote the interior of the union of m-simplices containing τ (in their boundary); c.f. Example 3.44. For a sufficiently refined $\mathcal{K}$, each U_τ is homeomorphic to $\mathbb{R}^m$. For every $k \geq 0$, let $\mathcal{K}^{(k)}$ denote the set of k-simplices in $\mathcal{K}$. Then,

$$\mathcal{A} = \{U_p\}_{p\in\mathcal{K}^{(0)}}$$

is a good covering of M such that

$$U_{p_0\cdots p_k} = \begin{cases} U_\tau & \text{if} \quad p_0,\dots,p_k \text{ are vetrices of } \tau \in \mathcal{K}^{(k)} \\ \emptyset & \text{otherwise.} \end{cases}$$

Therefore, the correspondence $\tau \leftrightarrow U_\tau$ gives a natural isomorphism

$$\text{Hom}(C_k(\mathcal{K},\mathbb{Z}),\mathbb{Z}) \cong C^k(\mathcal{A},\underline{\mathbb{Z}})$$

that matches the coboudary maps ∂^* and $\mathcal{R}$. Since $\mathcal{A}$ is a good covering, $\check{H}^k(M,\underline{\mathbb{Z}}) = \check{H}^k(\mathcal{A},\underline{\mathbb{Z}})$. Therefore, $\check{H}^k(M,\underline{\mathbb{Z}}) \cong H^k_{\text{sing}}(M,\mathbb{Z})$, for all $k \geq 0$. □

If M is an oriented smooth manifold, combining all the previous duality results over $\mathbb{R}$ we have

$$H^{m-k}_{c,\text{dR}}(M,\mathbb{R})^* \cong H^k_{\text{dR}}(M,\mathbb{R}) \cong \check{H}^k(M,\underline{\mathbb{R}}) \cong H^k_{\text{sing}}(M,\mathbb{R}) \cong H_k(M,\mathbb{R})^*$$

If M is closed, the first term above is just $H^{m-k}_{\text{dR}}(M,\underline{\mathbb{R}})$ and we can add one more term to right to get

$$H^{m-k}_{\text{dR}}(M,\mathbb{R})^* \cong H^k_{\text{dR}}(M,\mathbb{R}) \cong \check{H}^k(M,\underline{\mathbb{R}}) \cong H^k_{\text{sing}}(M,\mathbb{R}) \cong H_k(M,\mathbb{R})^* = H_{m-k}(M,\mathbb{R}).$$

If M is the interior of a compact manifold with boundary N, i.e. $M = N - \partial N$, then, there are relative singular homology groups $H_k(N/\partial N,\mathbb{Z})$ such that

$$H^{m-k}_{c,\text{dR}}(M,\mathbb{R})^* \cong H^k_{\text{dR}}(M,\mathbb{R}) \cong \check{H}^k(M,\underline{\mathbb{R}}) \cong H^k_{\text{sing}}(M,\mathbb{R}) \cong H_k(M,\mathbb{R})^* \cong H_{m-k}(N/\partial N,\mathbb{R}).$$

The direct duality between the first and the last term

$$H^{m-k}_{c,\text{dR}}(M,\underline{\mathbb{R}})^* \cong H_{m-k}(N/\partial N,\mathbb{R})$$

comes from integration of compactly supported forms on simplices.

3.4 Morse (co-)homology

Let M be a smooth oriented closed m-manifold, and $f\colon M \longrightarrow \mathbb{R}$ be a smooth Morse function; see Definition 2.85. Fix a Riemannian metric g on M and let ∇f denote the gradient vector field of f. Let

$$\phi\colon \mathbb{R}\times M \longrightarrow M, \qquad (x,t) \longrightarrow \phi_t(x),$$

denote the ODE of the vector field ∇f (If we replace ∇f with $-\nabla f$, we will get a homology theory instead of a cohomology theory). For each critical point $x\in\mathrm{Crit}(f)$, the unstable manifold U_x is the set

$$U_x = \{y \in M\colon \lim_{t\longrightarrow -\infty} \phi_t(y) = x\}$$

and the stable manifold S_x is the set

$$S_x = \{y \in M\colon \lim_{t\longrightarrow \infty} \phi_t(y) = x\}.$$

They are submanifolds by the following result.

Proposition 3.66. *Suppose M is a closed manifold. The stable and unstable sets of a critical point x are submanifolds. Moreover, they are diffeomorphic to open disks.*

Let $0 \leq \mathrm{index}(x) \leq m$ denote the index of x as in Definition 2.86. By definition, we have decomposition

$$T_xM = T_x^+M \oplus T_x^-M \tag{3.44}$$

where $T_x^{\pm}M$ are the $(\pm)$-eigen spaces of the second-derivative map Q_x in (2.42). By (2.43), the subspaces T_x^+M and T_x^-M are the tangent spaces at x to the unstable and stable submanifolds of ∇f. Therefore,

$$\dim_{\mathbb{R}} U_x = m - \mathrm{index}(x) \quad \text{and} \quad \dim_{\mathbb{R}} S_x = \mathrm{index}(x).$$

If $x, y \in \mathrm{Crit}(f)$, we define

$$\widetilde{\mathcal{M}}(x,y) = \{z \in M\colon \lim_{t\longrightarrow \infty} \phi_t(z) = y, \quad \lim_{t\longrightarrow -\infty} \phi_t(z) = x\} = U_x \cap S_y. \tag{3.45}$$

The ODE flow of ∇f gives an action of $\mathbb{R}$ on $\widetilde{\mathcal{M}}(x,y)$; the quotient space

$$\mathcal{M}(x,y) = \widetilde{\mathcal{M}}(x,y)/\mathbb{R} \tag{3.46}$$

is the set of orbits connecting x to y.

Definition 3.67. The pair (f,g) of a function $f\colon M \longrightarrow \mathbb{R}$ and a metric g is called Morse-Smale if f is Morse, and for every pair of critical points x and y, the intersection (3.45) is transverse.

Theorem 3.68. *Given a Morse function f, there is a metric g such that the pair (f,g) is Morse-Smale.*

Example 3.69. The function $f\colon \mathbb{T}^2 \longrightarrow \mathbb{R}$ in Figure 1 is Morse but not Morse-Smale. If we tilt the torus in Figure 1 a little bit so that it is not standing upright, the hight function f will become Morse-Smale; see Figure 6. In this Picture, x has index 0, y and z have index 1, and w has index 2. Furthermore, each of $M(x,y)$, $M(x,z)$, $M(y,w)$, and $M(z,w)$ is a set of size 2, while $M(x,w)$ is an open 1-dimensional manifold isomorphic to the disjoint union of 4 copies of $(0,1)$.

HW 3.70. Thinking of a torus as a rectangle with its opposite sides attached (with x corresponding to the corner points), draw a picture which illustrates the (pre-image of the) Morse flow lines of Example 3.69 on the rectangle.

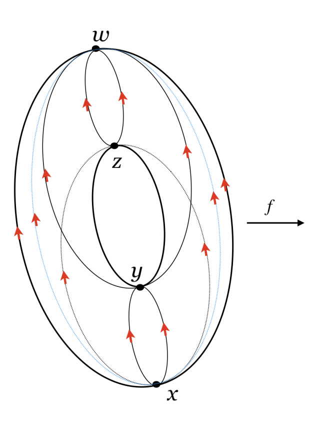


Figure 6: Gradient trajectories of a Morse-Smale function on torus.

If f is Morse-Smale, then for every $x, y \in \mathrm{Crit}(f)$, $M(x,y)$ is a smooth (but usually open) manifold of dimension

$$\dim_{\mathbb{R}} \mathcal{M}(x,y) = \mathrm{index}(y) - \mathrm{index}(x) - 1. \tag{3.47}$$

In particular, (we will prove that)

(1) if $\mathrm{index}(y) = \mathrm{index}(x) - 1$, then $\mathcal{M}(x,y)$ is a finite set of orbits

$$\mathcal{M}(x,y) = \{\gamma^1_{x,y}, \dots, \gamma^k_{x,y}\};$$

(2) and if $\mathrm{index}(y) = \mathrm{index}(x) - 2$, then $\mathcal{M}(x,y)$ has a finite number of components that are diffeomorphic to $(0,1)$ or S^1.

For each $x \in \mathrm{Crit}(f)$, fix an orientation on $T^+_x M$. Since M is oriented, by (3.44), we also get an orientation on $T^-_x M$. These orientations extend to orientations on U_x and S_x. Then, in the case (1), for each $\gamma^i_{x,y}$, we obtain an intersection orientation on the 1-dimensional manifold

$$\mathbb{R} \cong \gamma^i_{x,y} \subset U_x \cap S_y.$$

We say $\gamma^i_{x,y}$ is oriented positively, and write $\varepsilon(\gamma^i_{x,y}) = +1$, if and only if the intersection orientation on $\gamma^i_{x,y}$ coincides with the orientation of the flow of ∇f. We write

$$n_{x,y} = \sum_{i=1}^{k} \varepsilon(\gamma^i_{x,y}) \in \mathbb{Z}.$$

The Morse cochain complex is defined in the following way:

$$\begin{aligned} C^q(f) &= \bigoplus_{\substack{x\in \mathrm{Crit}(f)\\ \mathrm{index}(x)=q}} \mathbb{Z}\cdot x, \\ \partial^* \colon C^q(f) &\longrightarrow C^{q+1}(f), \qquad \partial^* x = \sum_{\substack{y\in \mathrm{Crit}(f)\\ \mathrm{index}(y)=q+1}} n_{x,y}\, y\,. \end{aligned} \tag{3.48}$$

Theorem 3.71. *The coboundary operator ∂^* above is well-defined (i.e. the coefficients $n_{x,y}$ are finite) and satisfies $\partial^* \circ \partial^* = 0$.*

Proof. Here is a sketch of the proof. For every $x, z \in \mathrm{Crit}(f)$ with $\mathrm{index}(z) - \mathrm{index}(x) = 2$, the coefficient of z in $\partial^* \circ \partial^*(x)$ is the quantity

$$\sum_{\substack{y \in \mathrm{Crit}(f) \\ \mathrm{index}(y)=\mathrm{index}(x)+1}} n_{x,y} \cdot n_{y,z}. \tag{3.49}$$

This quantity is the signed number of broken trajectories from x to y. By (3.47), the orbit space $\mathcal{M}(x, z)$ is 1-dimensional. Therefore, it is made of (a finite number of) components that are diffeomorphic to either S^1 or $(0, 1)$. For each component that is diffeomorphic to $(0, 1)$, we have a 1-parameter family of orbits $\{\gamma_t\}_{t\in(0,1)}$. As $t \longrightarrow 0, 1$, γ_t converges to a broken trajectory. Therefore, broken trajectories counted in (3.49) come in pairs. Furthermore, in each pair, the signs are different; therefore, the contribution of each pair is zero. We conclude that $\partial^* \circ \partial^*(x) = 0$. □

Example 3.72. In the example of Figure 6, for suitable choice of orientation on $T^+_x M$, $T^+_y M$, $T^+_z M$, and $T^+_w M$, the cochain complex is

$$0 \longrightarrow C^0(f) = \mathbb{Z} \cdot x \longrightarrow C^1(f) = \mathbb{Z} \cdot y \oplus \mathbb{Z} \cdot z \longrightarrow C^2(f) = \mathbb{Z} \cdot w \longrightarrow 0$$

where all the coboundary maps are trivial because for every $\mathbb{R}$-orbit there is a mirror orbit with the opposite sign. Therefore, the Morse cohomology groups of the cochain complex are the same as the singular cohomology groups of the 2-torus.

Remark 3.73. The metric plays little role in the definition of Morse homology. It only allows us to define a vector field from a function f and study its flow. More generally, we can consider the flow of any gradient-like vector field. A vector field is called gradient-like for f if (1) $\mathrm{d}_x f(\zeta(x)) \geq 0$ for all $x \in M$, and equality holds if and only if x is a critical point of f; and (2) in a Morse chart around a critical point x, ζ agrees with ∇f (w.r.t. the canonical metric of $\mathbb{R}^m$).

Remark 3.74. One may work with $\mathbb{Z}_2$ instead of $\mathbb{Z}$ to avoid the orientation problem and extend the construction of Morse cohomology to non-oriented manifolds.

Remark 3.75. The Morse homology is defined by following the flow of $-\nabla f$, instead. Therefore, the chain complex is defined in the following way:

$$\begin{aligned} &C_q(f) = \bigoplus_{\substack{x \in \mathrm{Crit}(f) \\ \mathrm{index}(x)=q}} \mathbb{Z} \cdot x, \\ &\partial\colon C_q(f) \longrightarrow C_{q-1}(f), \qquad \partial y = \sum_{\substack{x \in \mathrm{Crit}(f) \\ \mathrm{index}(x)=q-1}} n_{x,y}\, x\,. \end{aligned} \tag{3.50}$$

It is easy to see that cochain complex (3.48) is the dual of the chain complex (3.50).

Theorem 3.76. *Every Morse-Smale function $f\colon M \longrightarrow \mathbb{R}$ gives M the structure of a CW complex $\mathcal{C}$ such that the Morse homology of f coincides with the cellular homology of $\mathcal{C}$.*

Proof. By Theorem 3.66, take $\mathcal{C}$ to be the CW complex whose k-cells are the stable manifolds of the index k critical points. □

Theorem 3.76 shows that Morse (co-)homology is independent of the particular choice of f and metric. This can also be shown directly in the following way.

HW 3.77. Suppose (f,g) and (f',g') are two different pairs. Choose a function $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ that is Morse with two critical points at 0 and 1 (the minimum and maximum respectively), such that h is increasing sufficiently fast between 0 and 1 so that

$$\frac{\partial F(t,x)}{\partial t} + h'(t) > 0 \qquad \forall\ (t,x) \in [0,1] \times M.$$

Equip $[0,1] \times M$ with a metric G and a Morse-Smale function F such that

- $F(t,x) = f(x) + h(t)$ on $[0,1/3] \times M$ and $F(t,x) = f'(x) + h(t)$ for $[2/3,1] \times M$;
- G is the product metric $\mathrm{d}t^2 + g$ on $[0,1/3] \times M$ and $\mathrm{d}t^2 + g'$ on $[2/3,1] \times M$.

Then, show that

$$C^k(F) = C^k(f) \oplus C^{k-1}(f')$$

and

$$\partial_F^*\colon C^k(F) \longrightarrow C^{k+1}(F)$$

has the form

$$\partial_F^* = \begin{bmatrix} \partial_f^* & \Phi_F \\ 0 & \partial_{f'}^* \end{bmatrix}.$$

Show that the (cochain) map

$$\Phi\colon C^\bullet(f) \longrightarrow C^\bullet(f')$$

descends to an isomorphism of Morse cohomology groups.

Theorem 3.76 has some interesting consequences/applications. First, it gives an upper bound for the Betti numbers

$$b_k(M) = \text{rank } H_k(M,\mathbb{Z})$$

which is known as the "Weak Morse inequalities": if M is an oriented closed manifold, then $b_k(M)$ is smaller than the number n_k of index k critical points of any Morse function on M. Therefore, f has at least as many critical points as the sum of the ranks of the homology groups of M. If we compare the rank of the homology groups of (3.50) and that of finite dimensional spaces $C_k(f)$ more carefully, we obtain the following result.

Theorem 3.78. *(Strong Morse inequalities) Let M be a closed oriented manifold, and f a Morse function on M. Then for every $\ell \geq 0$ we have*

$$\sum_{k=0}^{\ell} (-1)^k n_k \geq \sum_{k=0}^{\ell} (-1)^k b_k(M).$$

Also, a careful comparison of Definition 2.14 and the numbers n_k shows that

$$\chi(M) = \sum_{k=0}^{\dim(M)} (-1)^k n_k.$$

Since the chain complex (3.50) and its homology have the same euler characteristic, it follows that

$$\chi(M) = \sum_{k=0}^{\dim(M)} (-1)^k b_k(M).$$

4 Classifying spaces

4.1 Classification spaces of vector bundles

If $\pi\colon E \longrightarrow M$ is a (smooth or topological) vector bundle and $f\colon N \longrightarrow M$ is a (smooth or continuous) map, the pullback vector bundle f^*E is a vector bundle over N whose fiber at $x \in N$ is $E|_{f(x)}$. If E is a complex vector bundle, using the de Rham cohomology description of chern classes using the curvature (2.32), it is easy to confirm the functorial property that

$$c_i(f^*E) = f^*c_i(E) \quad \forall\, k \in \mathbb{N}.$$

For every $k \in \mathbb{Z}^{>0}$, it is natural to ask whether:

there exists a topological space $\mathcal{P}_k$ and a (real/complex) rank k vector bundle $\mathcal{E}_k$ such that every other rank k vector bundle $E \longrightarrow M$ is of the form $E = f^*\mathcal{E}_k$ for some $f\colon M \longrightarrow \mathcal{P}_k$. Below, first, we answer this question for complex line bundles. We then extend it to higher ranks.

For every $n \geq 0$, consider the tautological line bundle

$$\gamma_n \longrightarrow \mathbb{CP}^n$$

defined in Example 2.30. For every $m \geq n$, the natural inclusion $\mathbb{C}^n = \mathbb{C}^n \times \{0\}^{m-n} \subset \mathbb{C}^m$ results in a natural embedding $\mathbb{CP}^n \subset \mathbb{CP}^m$ such that $\gamma_m|_{\mathbb{CP}^n} = \gamma_n$. Therefore, increasing n from 0 to ∞, we obtained a sequence of embeddings

$$\begin{array}{ccccccc} \gamma_0 & \hookrightarrow & \gamma_1 & \hookrightarrow & \gamma_2 & \hookrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{CP}^0 & \hookrightarrow & \mathbb{CP}^1 & \hookrightarrow & \mathbb{CP}^2 & \hookrightarrow & \cdots \end{array}$$

We define $\gamma_\infty \longrightarrow \mathbb{CP}^\infty$ to be the limiting complex line bundle. More explicitly, $\mathbb{C}^\infty$ is the infinite dimensional complex vector space of non-trivial sequences $x = (x_0, x_1, x_2, \ldots)$ such that all but finitely many x_i are 0, $\mathbb{CP}^\infty$ is the qoutient of $\mathbb{C}^\infty$ by the component-wise $\mathbb{C}^*$-action, and γ_∞ is the complex line bundle whose fiber over $[x]$ is the line $\mathbb{C} \cdot x$. By Example 3.6 and Poincare duality (of singular homology with singular cohomology), for each $n \geq 0$, we have

$$H^i(\mathbb{CP}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 2k,\ 0 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, the ring $H^*(\mathbb{CP}^n, \mathbb{Z})$ is generated by $h = c_1(\gamma_n^*) = -c_1(\gamma_n)$ satisfying $h^{n+1} = 0$; i.e.

$$H^*(\mathbb{CP}^n, \mathbb{Z}) = \frac{\mathbb{Z}[h]}{(h^{n+1} = 0)}. \tag{4.1}$$

Here $\mathbb{Z}[h]$ is the integral polynomial ring with variable h. It is easy to see that γ_n^* admits a transversal section that vanishes along $\mathbb{CP}^{n-1} \subset \mathbb{CP}^{n-1}$. Therefore, h is the poincare dual of the hyperplane $\mathbb{CP}^{n-1}$. As $n \longrightarrow \infty$, the relation $h^{n+1} = 0$ disappears; i.e. in a suitable infinite dimensional sense we have

$$H^*(\mathbb{CP}^\infty, \mathbb{Z}) = \mathbb{Z}[h].$$

Theorem 4.1. *For every complex line bundle $E \longrightarrow M$, there is a map $f\colon M \longrightarrow \mathbb{CP}^\infty$ such that $f^*\gamma_\infty^* = E$. If f_1 and f_2 are two such maps, then f_1 is homotopic to f_2.*

The first statement in Theorem 4.1 allows us to define $c_1(E)$ to be

$$c_1(E) = f^*h. \tag{4.2}$$

By the second statement, (4.2) is well-defined.

Next, we will extend the construction above to the higher rank and prove the generalization of Theorem 4.1. For arbitrary $r \in \mathbb{Z}^{>0}$, the generalization of $\mathbb{CP}^n$ is the complex Grassmannian $\mathrm{Gr}_r(n,\mathbb{C})$ mentioned in HW 2.59. The complex manifold $Gr_r(n) = Gr_r(\mathbb{C}^n)$ is the space of k-dimensional subspaces of $\mathbb{C}^n$. Every r-dimensional subspace $W \subset \mathbb{C}^n$ is generated by r linearly independent vectors $v_1, \ldots, v_r \in \mathbb{C}^n$. These vectors form the the rows of an $r \times n$ matrix Y. The matrix Y' corresponding to a different basis is equal to Y for some $A \in GL(r,\mathbb{C})$. Therefore, $\mathrm{Gr}_r(n)$ is the quotient manifold

$$\mathrm{Gr}_r(n) = GL(r,\mathbb{C}) \backslash \{Y \in M_{r\times n}(\mathbb{C})\colon \mathrm{rank}(Y) = r\}.$$

From a slightly different perspective, the unitary group $U(n) \subset GL(n,\mathbb{C})$ acts transitively on the set of r-dimensional subspaces of $\mathbb{C}^n$, and the stabilizer of $\mathbb{C}^r \subset \mathbb{C}^n$ is $U(r) \times U(n-r)$. Therefore, $\mathrm{Gr}_r(n)$ is equal to the homogenous space (A Lie group quotient[10])

$$\frac{U(n)}{U(r)\times U(n-r)} = \frac{GL(n,\mathbb{C})}{GL(r,\mathbb{C})\times GL(n-r,\mathbb{C})}.$$

From the matrix point of view above, the space of $(r \times n)$ matrices

$$V = \big\{Y = [\mathrm{I}\ Z] \colon Z \in M_{r\times(n-r)}(\mathbb{C})\big\} \cong \mathbb{C}^{r\times(n-r)}$$

defines a (holomorphic) chart around $\mathbb{C}^r \subset \mathbb{C}^n$. The transitive action of $U(n)$ on V gives a chart around every other point in $\mathrm{Gr}_r(n)$. We conclude that $\mathrm{Gr}_r(n)$ has complex dimension $r\times(n-r)$. A Hermitian pairing

$$\mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$$

gives an isomorphism $\mathrm{Gr}(n) \longrightarrow \mathrm{Gr}_{n-r}(n)$ that sends an r-dimensional subspace W to its orthogonal complement $W^\perp$. Functorially, this is the isomorphism

$$\mathrm{Gr}_r(\mathbb{C}^n) \longrightarrow \mathrm{Gr}_{n-r}\big((\mathbb{C}^n)^*\big)$$

that sends W to the dual subspace $W^\perp \subset (\mathbb{C}^n)^*$ of those linear maps on $\mathbb{C}^n$ that vanish on W. A Hermitian metric identifies $\mathbb{C}^n$ with its dual space $(\mathbb{C}^n)^*$ and realizes $W^\perp$ as the orthogonal complement.

Generalizing Example 2.47, every $Gr(k,n)$ admits a rank r tautological vector bundle

$$\gamma_{r,n} \longrightarrow \mathrm{Gr}_r(n)$$

whose fiber over the k-dimensional subspace $W \subset \mathbb{C}^n$ is W it self. In other words,

$$\gamma_{r,n} = \{(W,z) \in \mathrm{Gr}_r(n)\times\mathbb{C}^n \colon z \in W\} \subset \mathrm{Gr}_r(n)\times\mathbb{C}^n.$$

The inclusion $\gamma_{r,n} \subset \mathrm{Gr}_r(n)\times\mathbb{C}^n$ gives rise to a short exact sequence of vector bundles

$$0 \longrightarrow \gamma_{r,n} \longrightarrow \mathrm{Gr}_r(n)\times\mathbb{C}^n \longrightarrow Q_{r,n} \longrightarrow 0 \tag{4.3}$$

[10] The quotient of a Lie group by closed subgroup is manifold by [**?**, p.120].

such that $Q_{r,n}$ has rank $n-r$. Dualizing this sequence we get

$$0 \longrightarrow Q^*_{r,n} \longrightarrow \mathrm{Gr}_r(n) \times (\mathbb{C}^n)^* \longrightarrow \gamma^*_{r,n} \longrightarrow 0.$$

Therefore, under the isomorphism $\mathrm{Gr}_r(\mathbb{C}^n) \cong \mathrm{Gr}_{n-r}\big((\mathbb{C}^*)^n\big)$, we have

$$Q^*_{r,n} = \gamma_{n-r,n} \qquad \text{and} \qquad \gamma^*_{r,n} \cong Q_{n-r,n}.$$

If we write

$$c(\gamma_{r,n}) = 1 - c_1 + c_2 + \cdots + (-1)^r c_r \quad \text{and} \quad c(Q_{r,n}) = 1 + c^*_1 + c^*_2 + \cdots + c^*_{n-r}$$

then

$$c(\gamma_{r,n})c(Q_{r,n}) = 1 + (c^*_1 - c_1) + (c^*_2 + c_2 - c^*_1 c_1) + \cdots = 1$$

and

$$c(\gamma_{n-r,n}) = 1 - c^*_1 + c^*_2 + \cdots + (-1)^r c^*_r \quad \text{and} \quad c(Q_{n-r,n}) = 1 + c_1 + c_2 + \cdots + c_r.$$

Remark 4.2. By HW 2.48, putting $r=1$, we get

- $Gr(1,n) = \mathbb{CP}^{n-1}$;
- $\mathcal{E}(1,n) = \gamma_n$;
- $Q(1,n) = T\mathbb{P}^{n-1} \otimes \gamma_n$.

Since $c(\gamma_n) = (1-h)$, by (4.3), we have

$$c(Q_{1,n}) = \frac{1}{1-h} = 1 + h + \ldots + h^{n-2}.$$

The following theorem generalizes (4.1).

Theorem 4.3. (1) *The cohomology ring of* $\mathrm{Gr}_r(n)$ *is given by*

$$H^*(\mathrm{Gr}_r(n)) = \frac{\mathbb{Z}[c_1,\ldots,c_r,c^*_1,\ldots,c^*_{n-r}]}{\big(c(\gamma_{r,n})c(Q_{r,n}) = 1\big)}\ ;$$

(2) *The chern classes* $c_1,\ldots,c_r$ *generate* $H^*(\mathrm{Gr}_r(n))$*;*

(3) *For a fixed* r *and every* $i \geq 0$*, there is sufficiently large* $n_0 = n_0(r,i)$ *such that for every* $n \geq n_0$ *there are no polynomial relations of degree* i *(or lower) among* $c_1,\ldots,c_k$*.*

We will comeback to this later in this section. As before, for every $m \geq n$, the natural inclusion $\mathbb{C}^n = \mathbb{C}^n \times \{0\}^{m-n} \subset \mathbb{C}^m$ results in a natural embedding $\mathrm{Gr}_r(n) \subset \mathrm{Gr}_r(m)$ such that $\gamma_{r,m}|_{\mathrm{Gr}_r(n)} = \gamma_r(n)$. Therefore, increasing n from r to ∞, we obtained a sequence of embeddings

$$\begin{array}{ccccccc}
\gamma_{r,r} & \hookrightarrow & \gamma_{r,r+1} & \hookrightarrow & \gamma_{r,r+2} & \hookrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Gr}_r(r) & \hookrightarrow & \mathrm{Gr}_r(r+1) & \hookrightarrow & \mathrm{Gr}_r(r+2) & \hookrightarrow & \cdots
\end{array}$$

We define $\gamma_{r,\infty} \longrightarrow \mathrm{Gr}_r(\infty)$ to be the limiting rank r complex vector bundle. The following is a corollary of Theorem 4.3.(3).

Corollary 4.4. $H^*(\mathrm{Gr}_r(\infty)) = \mathbb{Z}[c_1, \ldots, c_r]$.

Theorem 4.5. *For every rank r complex vector bundle $E \longrightarrow M$, there is a map $f\colon M \longrightarrow Gr(r, \infty)$ such that $f^*\mathcal{E}^*(r,\infty) = E$. If f and f' are two such maps, then f is homotopic to f'.*

Proof. For simplicity, we assume M is compact; otherwise, in the following argument, one must consider a countable cover and directly work with $\mathbb{C}^\infty$. Therefore, there is a sufficiently large n and n sections $\{s_1, \ldots, s_n\}$ of E that span (every fiber of) E. These sections result in a surjective complex linear bundle homomorphism

$$\varrho\colon M \times \mathbb{C}^n \longrightarrow E, \quad (p, (x_1, \ldots, x_n)) \longrightarrow \sum_{i=1}^{n} x_i s_i(p) \in E_p.$$

Taking the dual of ϱ we obtain an embedding of E^* into the trivial bundle

$$\varrho^*\colon E^* \longrightarrow (M \times \mathbb{C}^n)^* \cong M \times \mathbb{C}^n.$$

Define

$$f\colon M \longrightarrow \mathrm{Gr}_r(n), \qquad f(p) = [\varrho^*(E^*_p)]. \tag{4.4}$$

We have

$$\big(f^*\gamma^*_{r,n}\big)|_p = \gamma^*_{r,n}|_{f(p)} = \big(\varrho^*(E^*_p)\big)^* = E_p.$$

Therefore, $f^*\gamma^*_{r,\infty} = E$.

The second statement says: if we have two maps $f, f'\colon M \longrightarrow \mathrm{Gr}_r(\infty)$ such that $E = f^*\gamma^*_{r,\infty} = (f')^*\gamma^*_r(\infty)$ then f and f' are homotopic. Again, for simplicity we assume M is compact and f, f' have image in some $\mathrm{Gr}_r(n)$. A general proof of this based on obstruction theory may be found in Steenord [**?**, Sec 19] and Husemoller [**?**, Sec 7.6]. This proof has two steps. First, we show that every f is of the form (4.4), i.e. there are n-sections $s_1, \ldots, s_n$ such that such that f is the map corresponding to these sections in (4.4).

The surjective bundle map

$$\mathrm{Gr}_r(n) \times (\mathbb{C}^n)^* \longrightarrow \gamma^*_{r,n}$$

gives us n sections $e^*_1, \ldots, e^*_n$ of $\gamma^*_{r,n}$ that generate $\gamma^*_{r,n}$ at every point. For every $f\colon M \longrightarrow \mathrm{Gr}_r(n)$ let $s_i = f^*e^*_i$, for $i = 1, \ldots, n$, denote the corresponding pullback sections of $E = f^*\gamma^*_{r,n}$. If $f'\colon M \longrightarrow \mathrm{Gr}_r(n)$ is the map corresponding to these sections as in (4.4); show that $f = f'$.

Now suppose f and f' are two maps as in (4.4) given by sections $s_1, \ldots, s_n$ and $s'_1, \ldots, s'_n$. Consider the product $M \times [0,1]$. For some $N \geq n$, we can find N sections $S_1, \ldots, S_N$ such that for some $\varepsilon > 0$

- on $M \times [0, \varepsilon]$ only n of $S_1, \ldots, S_n$ are non-zero and they coincide with the trivial extensions of $s_1, \ldots, s_n$,
- on $M \times [1-\varepsilon, 1]$ only n of $S_1, \ldots, S_n$ are non-zero and they coincide with the trivial extensions of $s'_1, \ldots, s'_n$,
- $S_1, \ldots, S_N$ generate each fiber of E over $M \times [0,1]$.

Therefore, the map $F\colon M \times [0,1] \longrightarrow \mathrm{Gr}_r(N)$ gives a homotopy between

$$M \xrightarrow{f} \mathrm{Gr}_r(n) \longrightarrow \mathrm{Gr}_r(N) \quad \text{and} \quad M \xrightarrow{f'} \mathrm{Gr}_r(n) \longrightarrow \mathrm{Gr}_r(N).$$

$\square$

If a complex vector bundle E or rank r splits as a sum of complex line bundles

$$E = \bigoplus_{i=1}^{r} L_i \tag{4.5}$$

and $c_1(L_i) = x_i$, then we have

$$c(E) = \prod_{i=1}^{n}(1 + x_i) = 1 + \big(\sum_i x_i\big) + \big(\sum_{i<j} x_i x_j\big) + \cdots + \big(x_1 \ldots x_r\big).$$

Every term on the righthand side is a symmetric polynomial in $x_1, \ldots, x_r$. Moreover, the decomposition (4.5) allows us to calculate the chern classes of other vector bundles constructed functorially from E. For example,

- Since $E^* = \bigoplus_{i=1}^r L_i^*$ and $c_1(L_i^*) = -x_i$, we conclude that

$$c(E^*) = \prod_{i=1}^{n}(1 - x_i) = 1 - c_1(E) + c_2(E) \pm \ldots + (-1)^r c_r(E);$$

- Since $E \otimes E = \bigoplus_{i=1}^r \bigoplus_{j=1}^r L_i \otimes L_j$ and $c_1(L_i \otimes L_j) = x_i + x_j$, we conclude that

$$c(E \otimes E) = \prod_{i=1}^{r}\prod_{j=1}^{r}(1 + x_i + x_j) = 1 + (r+1)\big(\sum_i x_i\big) + \cdots = 1 + 2rc_1 + \cdots;.$$

In general, for every vector bundle F functorially constructed from E, $c(F)$ will be a symmetric polynomial in x_i; therefore, it can be written in terms of the chern classes of E. The problem is, not every arbitrary complex vector bundle admits a decomposition as in (4.5). Nevertheless, the following result, known as the splitting principle allows us to assume such a splitting exists to find $c(F)$.

Theorem 4.6. *To prove a polynomial identity in the chern classes of a complex vector bundle, it suffices to prove it under the assumption that the vector bundles are direct sum of line bundles.*

For the proofs of Theorems 4.6 and 4.3, we need to define and study Flag manifolds (varieties). Flag varieties can be defined in two ways.

Given $n \geq 1$, a flag in $\mathbb{C}^n$ is a sequence of subspaces

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_n = \mathbb{C}^n$$

such that $\dim_{\mathbb{C}} W_i = i$. Let $Fl(n)$ denote the set of all flags in $\mathbb{C}^n$. Similarly, one may use an abstract vector space V instead of $\mathbb{C}^n$ to define $Fl(V)$. The group $\mathrm{GL}(n,\mathbb{C})$ acts transitively on $Fl(n)$. The stabilizer of the standard flag

$$0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n$$

is the subgroup of upper-triangular matrices $H(n,\mathbb{C})$. Therefore,

$$Fl(n) = \mathrm{GL}(n,\mathbb{C})/H(n,\mathbb{C}) \tag{4.6}$$

is naturally a complex manifold.

More generally, for every rank n complex vector bundle $V \longrightarrow M$, the flag manifold $Fl(V)$ can be defined inductively in the following way. First, for each $x \in M$, we consider the of set of lines $W_1 \subset V_x$ which gives us the fiber bundle $\pi_1\colon F_1 = \mathbb{P}(V) \longrightarrow F_0 := M$ with fibers $\mathbb{CP}^{n-1}$. We have an exact sequence

$$0 \longrightarrow \gamma_1 \longrightarrow \pi_1^* V \longrightarrow Q_1 \longrightarrow 0$$

where γ_1 is the tautological line bundle on F_1 and Q_1 is the rank $(n-1)$ quotient bundle. Next, we consider the fiber bundle $\pi_2\colon F_2 = \mathbb{P}(Q_1) \longrightarrow F_1$ whose fibers are isomorphic $\mathbb{CP}^{n-2}$. Again, we have an exact sequence

$$0 \longrightarrow \gamma_2 \longrightarrow \pi_2^* Q_1 \longrightarrow Q_2 \longrightarrow 0$$

where γ_2 is the tautological line bundle on F_2 and Q_2 is the rank $(n-2)$ quotient bundle. Continuing inductively, we obtain a sequence of fiber bundles

$$F_n \xrightarrow{\pi_n} F_{n-1} \xrightarrow{\pi_{n-1}} \cdots \longrightarrow F_1 \xrightarrow{\pi_1} F_0 = M$$

such that the fiber of $F_k \xrightarrow{\pi_k} F_{k-1}$ is isomorphic to $\mathbb{CP}^{n-k}$ for all $k = 1, \ldots, n$. Define

$$Fl(V) = F_n \xrightarrow{\pi = \pi_1 \circ \pi_{n-1} \circ \pi_n} M.$$

Lemma 4.7. *Each fiber of $Fl(V)$ is the flag variety $Fl(V)$; i.e. if $M =$ point and V is a vector bundle of rank n then $Fl(V)$ is the flag variety defined in (4.6).*

Proof. For every $k = 1$, a point in F_1 corresponds to a line W_1 in V. A point in F_2 over W_1 in F_1 is a line in Q_1; that is equivalent to the quotient W_2/W_1 of a 2-dimensional subspace W_2 of V including W_1. Then, inductively, a line in Q_k is equivalent to the quotient of a k-dimensional subspace of V and the $k-1$-dimensional subspace of that constructed in the previous steep. □

Let

$$F_\ell \xrightarrow{\pi_{\ell,k} := \pi_{k+1} \circ \ldots \circ \pi_\ell} F_k \qquad \forall\, \ell > k.$$

It also follows from the inductive construction that

$$\pi^* V = \pi_{n,1}^* \gamma_1 \oplus \pi_{n,2}^* \gamma_2 \oplus \cdots \pi_{n,n-1}^* \gamma_{n-1} \oplus \gamma_n;$$

i.e. the pullback of V to F_n splits! The line bundles $\pi_{n,1}^*\gamma_1, \pi_{n,2}^*\gamma_2, \ldots \gamma_n$ can alternatively be described in the following way. For every flag

$$x = \Big(0 \subset W_1 \subset W_2 \subset \cdots \subset W_n = V\Big) \in Fl(V),$$

the fiber of $\pi_{n,1}^*\gamma_1$ over x is the line W_1, the fiber of $\pi_{n,2}^*\gamma_2$ over x is the line W_2/W_1, and in general, the fiber of $\pi_{n,k}^*\gamma_k$ over x is the line W_k/W_{k-1}.

Remark 4.8. Note that $Q_n = 0$. Also, $F_n = F_{n-1}$ and $\gamma_n = Q_{n-1}$. To keep the notation symmetric, we have included F_n as the last step (even though, the last step is trivial).

For every $1 \leq k \leq n$, by Leray-Hirsh (see (3.23)), we know that $H^*(F_k, \mathbb{Z})$ is a free module over $H^*(F_{k-1}, \mathbb{Z})$ generated by $c_1(\gamma_k^*)$ subject to one relation

$$c_1(\gamma_k^*)^{n-k+1} + c_{k-1,1}\; c_1(\gamma_k^*)^{n-k} + \ldots + c_{k-1,n-k+1} = 0$$

where $c_{k-1,1}, \ldots, c_{k-1,n-k+1}$ are the chern classes of Q_{k-1}. Inductively, we conclude that $H^*(F_n, \mathbb{Z})$ is a free module over $H^*(M, \mathbb{Z})$ generated by

$$h_1 = c_1(\pi_{n,1}^*\gamma_1^*), \quad h_2 = c_1(\pi_{n,2}^*\gamma_2^*) \ldots \quad h_n = c_1(\gamma_n^*)$$

subject to the only relation (known as Whitney Product Formula)

$$\prod_{i=1}^{n}(1 - h_i) = c(V)$$

which corresponds to the isomorphism

$$\pi^* V \cong \pi_{n,1}^*\gamma_1 \oplus \pi_{n,2}^*\gamma_2 \oplus \cdots \oplus \gamma_n.$$

From Leray-Hirsch, we conclude that

$$H^*(Fl(V), \mathbb{Z}) = \frac{H^*(M, \mathbb{Z})[h_1, h_2, \ldots, h_n]}{\prod_{i=1}^{n}(1 - h_i) = c(V)}. \tag{4.7}$$

For M = point, since $c(V) = 1$, we get the following corollary. Moreover, in each step of the induction

Corollary 4.9. *For every $n \geq 1$, we have*

$$H^*(Fl(n), \mathbb{Z}) = \frac{\mathbb{Z}[h_1, \ldots, h_n]}{\prod_{i=1}^{n}(1 - h_i) = 1}.$$

Proof of Theorem 4.6. Suppose E and F are complex vector bundles on M and we want to prove a polynomial relation

$$P(c(E), c(F)) = 0.$$

By the discussion above, after passing to the fiber product of $\pi\colon Fl(E) \times_M Fl(F) \longrightarrow M$, E and F decompose to complex line bundles. So if we know that

$$P(c(\pi^* E), c(\pi^* F)) = \pi^* P(c(E), c(F)) = 0,$$

by the infectivity of

$$\pi^*\colon H^*(M, \mathbb{Z}) \longrightarrow H^*(Fl(E) \times_M Fl(F), \mathbb{Z}),$$

we conclude that $P(c(E), c(F)) = 0$. □

Proof of Theorem 4.3. Part (1). Consider the exact sequence

$$0 \longrightarrow \gamma_{r,n} \longrightarrow \mathrm{Gr}_r(n) \times (\mathbb{C}^n) \longrightarrow Q_{r,n} \longrightarrow 0.$$

First, we apply the flag manifold construction to $\gamma_{r,n}$ to get $\pi_1\colon F_1 = Fl(\gamma_{r,n}) \longrightarrow \mathrm{Gr}_r(n)$. Let $Q' = \pi_1^* Q_{r,n}$. Next, we apply the flag manifold construction to Q' to get $\pi_2\colon F_2 = Fl(Q') \longrightarrow F_1$. Let $\pi = \pi_1 \circ \pi_2\colon F_2 \longrightarrow \mathrm{Gr}_r(n)$. It is easy to see that $F_2 = Fl(n)$. By (4.7), we have

$$H^*(F_2, \mathbb{Z}) = \frac{H^*(\mathrm{Gr}_r(n))[x_1, \ldots, x_k, y_1, \ldots, y_{n-k}]}{\prod_{i=1}^{k}(1 - x_i) = c(\gamma_{r,n}), \prod_{i=1}^{n-k}(1 - y_i) = c(Q_{r,n})}$$

On the other hand

$$H^*(Fl(n),\mathbb{Z}) = \frac{\mathbb{Z}[h_1,\ldots,h_n]}{\prod_{i=1}^n (1-h_i)=1}$$

where $(x_1,\ldots,x_k,y_1,\ldots,y_{n-k}) = (h_1,\ldots,h_n)$. Therefore, in $H^*(\mathrm{Gr}_r(n))$, the cohomology classes $c(\gamma_{r,n})$ and $c(Q_{r,n})$ satisfy no relation other than $c(\gamma_{r,n})c(Q_{r,n}) = 1$, because any other relation among them will give additional relations among x_i and y_j and thus among h_i. We conclude that there is an injection of rings

$$\frac{\mathbb{Z}[c(\gamma_{r,n}),c(Q_{r,n})]}{c(\gamma_{r,n})c(Q_{r,n})=1} \hookrightarrow H^*(\mathrm{Gr}_r(n)).$$

We will show later that every cohomology class in $\mathrm{Gr}_r(n)$ can be written in terms of the classes in $c(\gamma_{r,n})$ and $c(Q_{r,n})$, concluding that the injection above is an isomorphism. For every manifold M, its poincare polynomial is defined by

$$P(t) = \sum_{i=0}^{\dim(M)} \dim\ H^i(M,\mathbb{R})\ t^i.$$

One can also use the Poincare polynomial to make the same conclusion; see [**?**, p. 294].

Parts (2) and (3) of Theorem 4.3 follows from expanding the equation $c(\gamma_{r,n})c(Q_{r,n}) = 1$. □

4.2 Equivariant cohomology and localization

The infinite projective space $\mathbb{CP}^\infty$ and the identity

$$H^*(\mathbb{CP}^n,\mathbb{Z}) = \mathbb{Z}[h]$$

have another meaning and application that we will discuss in this section.

Suppose a group G acts on a manifold M. This action induces an action of G on the vector space $H^*(M,\mathbb{R})$. By Theorem 2.57 (or 2.54), if G acts freely and properly on M, then M/G is a manifold. The projection $\pi\colon M \longrightarrow M/G$ gives a pull back map $\pi^*\colon H^*(M/G,\mathbb{R}) \longrightarrow H^*(M,\mathbb{R})$ whose image is the subset of $H^*(M,\mathbb{R})$ preserved by the action of G; i.e. the trivial component of the representation $G \longrightarrow \mathrm{Aut}(H^*(M,\mathbb{R}))$. We

HW 4.10. Explain the map $\pi^*\colon H^*(M/G,\mathbb{R}) \longrightarrow H^*(M,\mathbb{R})$ for the example of $M = S^{2n+1}$, $G = S^1$, and $M/G = \mathbb{CP}^n$.

The question is, if the action is not free and proper, what can we say about the G-equivariant cohomology classes on M, i.e. the subset of $H^*(M,\mathbb{R})$ that is preserved by G.

Example 4.11. Consider the action of $\mathbb{T}^n = (S^1)^n$ on $\mathbb{CP}^n$ given by

$$(\mathrm{e}^{\mathrm{i}\theta_1},\ldots,\mathrm{e}^{\mathrm{i}\theta_n})\cdot[x_0,x_1,\ldots,x_n] \longrightarrow [x_0,\mathrm{e}^{\mathrm{i}\theta_1}x_1,\ldots,\mathrm{e}^{\mathrm{i}\theta_n}x_n].$$

This torus action has $n+1$ fixed points

$$p_0 = [1,0,\ldots,0],\ p_0 = [0,1,\ldots,0],\ p_n = [0,0,\ldots,n].$$

There is an idea to turn every action into a free action without changing the homotopy type of M as follows. For every G, the idea is to find a contractible space EG on which the group G acts freely and properly. Then $EG \times M$ will have the same homotopy type as M and the diagonal action of G on $EG \times M$ will always be free and proper. The homotopy quotient M_G of M by G, also called the Borel construction, is defined to be the quotient of $EG \times M$ by the diagonal action of G. We can also see M_G as the fiber product of M and the principal G-bundle

$$EG \longrightarrow BG = EG/G.$$

The quotient space BG is called the classifying space of G. The equivariant cohomology $H^*_G(M, -)$ of M is defined to be the cohomology $H^*(M_G, -)$ of the homotopy quotient M_G. Here H^* denotes singular cohomology with any coefficient ring. We will also develop a de Rham version of this cohomology (with coefficients in $\mathbb{R}$).

Remark 4.12. A principal G-bundle is a fiber bundle $P \longrightarrow M$ such that (i) fibers of P are isomorphic to G, (ii) G acts on P and preserves each fiber, (iii) the action on each fiber is by multiplication in G. The theory of principal bundles is closely related to theory of vector bundles. Associated to every representation $\varrho\colon G \longrightarrow \mathrm{Aut}(V)$ of G, we obtain a vector bundle

$$E = \big(P \times V\big)/G$$

where $g \in G$ acts on the product by $(p, v) \longrightarrow (p \cdot g, g^{-1}v)$. Conversely, if E is a vector bundle of real rank r, the space of frames on E is a principal $GL(r)$-bundle. If we fix a metric and consider orthonormal frames, we obtain a principal $O(r)$-bundle. As another example, complex line bundles (with a Hermitian metric) are equivalent to principal $U(1) = S^1$-bundles.

The following is the analogue/generalization of Theorem 4.13.

Theorem 4.13. *For every principal G-bundle $P \longrightarrow M$, there is a map $f\colon M \longrightarrow BG$ such that $f^*EG = P$. If f and f' are two such maps, then f is homotopic to f'.*

Example 4.14. These classifying spaces are often infinite dimensional. Here are some examples.

- For $G = S^1$, by the relation between S^1-bundles and complex line bundles, we have $BG = \mathbb{CP}^\infty$ and $EG = S^\infty = \lim_{n\longrightarrow\infty} S^{2n+1}$. Show that S^∞ is contractible!
- For $G = \mathbb{Z}$, we have $BG = S^1$ and $EG = \mathbb{R}$.
- For $G = \mathbb{Z}_2$, we have $BG = \mathbb{RP}^\infty$ and $EG = S^\infty$.
- For $G = U(n)$, by the relation between $U(r)$-bundles and complex vector bundles of rank r, we have $BG = \mathrm{Gr}_r(\infty)$ and EG is the frame bundle of $\gamma_{r,\infty}$.

If we apply the construction above to $M = \mathrm{pt}$ with the trivial action of G, we get

$$H^*_G(\mathrm{pt}) = H^*(BG).$$

For every other manifold M with an action of G, the obvious G-equivariant map $M \longrightarrow \mathrm{pt}$ gives $H^*_G(M)$ the structure of a module over $H^*(BG)$. Therefore, this process will result in working with the often larger coefficient ring $H^*(BG)$ instead of the original coefficient ring ($\mathbb{R}$ or $\mathbb{Z}$). Also, the projection $M \times EG \longrightarrow EG$ descends to a fiberation $M_G \longrightarrow BG$ whose fibers are isomorphic to M. Therefore, M_G is a fiber bundle over BG with fibers M_G. Restriction to a fiber defines a map

$$H^*_G(M) \longrightarrow H^*(M) \tag{4.8}$$

that does not need not be surjective.

Example 4.15. Here are the $H^*(BG)$s corresponding to the first two examples in 4.14.

- For $G = S^1$, we get $H^*(BG, \mathbb{R}) = H^*(\mathbb{CP}^\infty, \mathbb{R})$ is the polynomial ring $\mathbb{R}[h]$.
- For $G = \mathbb{Z}$, we get $H^*(BG, \mathbb{R}) = \mathbb{R}[x]/x^2 = 0$.

Remark 4.16. If the action of G on M is free and proper so that M/G is a manifold, the projection $M \times EG \longrightarrow M$ descends to a projection

$$\pi \colon M_G \longrightarrow M/G$$

whose fibers are EG. Since EG is contractible, we obtain

$$H^*_G(M) = H^*(M/G)$$

with the trivial module structure.

If G acts on M and N is a G-invariant submanifold of M then we get an inclusion $\iota_N \colon N_G \longrightarrow M_G$; therefore, at the level of cohomology, we get a restriction map $\iota_N^* \colon H^*_G(M) \longrightarrow H^*_G(N)$. For example, if $N = p$ is a fixed point of G, then $H^*_G(p) = H^*(BG)$ and the map above is an evaluation

$$\iota_p^* \colon H^*_G(M) \longrightarrow H^*(BG) \tag{4.9}$$

corresponding to the fixed point p. These evaluations will appear in localization formula.

In what follows, I will follow the paper of Atiyah and Bott in a reverse order. First, we discuss a de Rham (Cartan) model of $H^*_G(M, \mathbb{R})$ when $G = S^1$, then, we study localization and do some interesting computations.

Associated to any action of S^1 on a manifold M we obtained a vector field ζ on M given by

$$\zeta(x) = \frac{\mathrm{d}}{\mathrm{d}\theta}(\mathrm{e}^{\mathrm{i}\theta} \cdot x)|_{\theta=0} \qquad \forall\, x \in M$$

The S^1-action is the flow of ζ.

Fix a formal variable u of degree 2. Let

$$\Omega^*(M, \mathbb{R}[u]) = \Omega^*(M, \mathbb{R}[u]) \otimes \mathbb{R}[u] = \Big\{ \sum_\alpha \eta_\alpha u^{n_\alpha} \colon \eta_\alpha \in \Omega^*(M, \mathbb{R}) \Big\}$$

denote the space of differential forms with coefficients in $\mathbb{R}[u]$. Therefore, for each $d \geq 0$, the degree piece $\Omega^d(M, \mathbb{R}[u])$ consists of finite sums of the form

$$\widetilde{\eta} = \eta_d + \eta_{d-2} u + \cdots + \eta_{d-2k} u^k + \cdots$$

such that $\eta_i \in \Omega^i(M, \mathbb{R})$. The map

$$\Omega^d(M, \mathbb{R}[u]) \xrightarrow{D = \mathrm{d} + u\,\iota_\zeta} \Omega^{d+1}(M, \mathbb{R}[u])$$

is $\mathbb{R}[u]$-linear and

$$D \circ D = (\mathrm{d} + u\,\iota_\zeta) \circ (\mathrm{d} + u\,\iota_\zeta) = \mathrm{d} \circ \mathrm{d} + uL_\zeta + u^2 \iota_\zeta \circ \iota_\zeta = uL_\zeta.$$

Let $\Omega^*_{S^1}(M, \mathbb{R})$ denote the space of differential forms η where $L_\zeta \eta = 0$. By (2.20), $\Omega^*_{S^1}(M, \mathbb{R})$ is the space of S^1-invariant differential forms on M. By the identity above, restricted to $\Omega^*_{S^1}(M, \mathbb{R}[u])$ we have $D \circ D$.

Theorem 4.17. *There is an isomorphism between the cohomology of $(\Omega^*_{S^1}(M, \mathbb{R}[u]), D)$ the equivariant cohomology $H^*_{S^1}(M, \mathbb{R})$ that maps u to h.*

Remark 4.18. The restriction map (4.8) sends the cohomology class of a D-closed form

$$\widetilde{\eta} = \eta_d + \eta_{d-2}u + \cdots + \eta_{d-2k}u^k + \cdots$$

to $[\eta_d]$ (Check that if $\widetilde{\eta}$ is D-closed then η_k is d-closed); i.e. (4.8) is given by putting $u = 0$. This map need not to be surjective. Starting from a closed and S^1-invariant d-form η_d, in order to build a D-closed $\widetilde{\eta}$ that starts with η_d, we need a sequence of S^1-invariant differential forms $\eta_{d-2}, \eta_{d-4}, \ldots$ such that

$$\iota_\zeta \eta_{d-2k} + d\eta_{d-2(k+1)} = 0 \qquad \forall\, k \geq 0. \tag{4.10}$$

In particular, η_d is itself D-closed if and only if $\iota_\zeta \eta_d = 0$ (which is stronger that $L_\zeta \eta_d = d\iota_\zeta \eta_d = 0$). Given a closed S^1-invariant η_d, an extension $\widetilde{\eta}$, if it exists, is called an equivariant extension of η_d.

For

$$\widetilde{\eta} = \sum \eta_\alpha P_\alpha(u) \in \Omega^*(M, \mathbb{R}[u]),$$

if M is oriented, we define

$$\int_M \widetilde{\eta} = \sum_\alpha P_\alpha(u) \int_M \eta_\alpha.$$

Of course, $\int_M \eta_\alpha$ can only be non-trivial if $\deg(\eta_\alpha) = \dim M$. One peculiarity of equivariant integration is the possibility of obtaining a nonzero answer while integrating a form over a manifold whose dimension is not equal to the degree of the form.

Example 4.19. With notation as above $\Omega^0_\zeta(M, \mathbb{R}[u]) = \Omega^0_\zeta(M, \mathbb{R})$ is the space of S^1-invariant smooth functions $f\colon M \longrightarrow \mathbb{R}$. Therefore, $H^0_{S_1}(M)$ calculates the number of connected components of M/S^1 which is equal to $H^0(M)$ because S^1 is connected. In degree 1, $\Omega^1_\zeta(M, \mathbb{R}[u]) = \Omega^1_\zeta(M, \mathbb{R})$ is the space of S^1-invariant 1-forms η. Furthermore,

$$D\eta = \mathrm{d}\eta + u\,\iota_\zeta \eta$$

is zero if and only if η is closed and $\iota_\zeta \eta = 0$.

Example 4.20. Consider $M = \mathbb{C}$ and let S^1 act by $z \longrightarrow \mathrm{e}^{\mathrm{i}m\theta} z$. This is called a linear action of weight m. We denote $\mathbb{C}$ with this action by $\mathbb{C}_m$. Since $\mathbb{C}_{S_1}$ is a complex line bundle over $BS^1 = \mathbb{CP}^\infty$, we conclude that

$$H^*_{S^1}(\mathbb{C}_m, \mathbb{R}) = H^*(\mathbb{CP}^\infty, \mathbb{R}) = \mathbb{R}[u].$$

Example 4.21. Consider $M = S^2$ and the standard action of S^1 by rotation (one may also consider a weighted action). Write $S^2 = U_0 \cup U_\infty$ where $U_0 \cong \mathbb{C}$ and $U_\infty \cong \mathbb{C}$ are neighborhoods of the north and south poles, respectively. The action on $U_0 \cap U_\infty$ is free. Therefore, by the previous example

$$H^*_{S^1}(U_0, \mathbb{R}) = H^*_{S^1}(U_\infty, \mathbb{R}) = \mathbb{R}[u]$$

and

$$H^*_{S^1}(U_0 \cap U_\infty, \mathbb{R}) = H(S^1/S^1, \mathbb{R}) = \mathbb{R}$$

is concentrated in degree zero. From Mayer-Vietoris, we can conclude that

$$H^*_{S^1}(S^2,\mathbb{R}) = \{(f_0, f_\infty) \in \mathbb{R}[u] \oplus \mathbb{R}[u]\colon f_0(0) = f_\infty(0)\}.$$

This is a a free $\mathbb{R}[u]$-module where a basis is given by $(1, 1)$ and $(u, -u)$. For the fixed points p_0 and p_∞, the evaluation maps in (4.9) map (f_0, f_∞) to f_0 and f_∞ respectively. So every equivariant differential form on S^2 is uniquely specified by the information of its restriction to the fixed points! With respect to the polar coordinates (r, θ) on U_0, the Fubini-Study volume form of S^2 is the closed S^1-invarinat 2-form

$$\omega = \frac{2r\mathrm{d}r \wedge \mathrm{d}\theta}{(1 + r^2)^2}.$$

The equivariant extension $\widetilde{\omega}$ of ω exists and is given by a sum of the form $\widetilde{\omega} = \omega + uH$ where H is a function satisfying

$$-\iota_{\partial\theta}\omega = \frac{2r\mathrm{d}r}{(1 + r^2)^2} = \mathrm{d}H.$$

see (4.10). Solving this equation we get

$$H(r) = H_0 + \int_0^r \frac{\mathrm{d}s^2}{(1 + s^2)^2} = H_0 + \frac{r^2}{(1 + r^2)}$$

The restrictions of $\widetilde{\omega} = \omega + uH$ to p_0 and p_∞ are $f_0(u) = H_0 u$ and $f_\infty(u) = H_\infty u = (1 + H_0)u$, respectively. Note that

$$1 = \int_{S^2} \omega = \int_{S^2} \widetilde{\omega} = -\Big(\frac{f_0(u)}{u} + \frac{f_1(u)}{-u}\Big).$$

This is a special case of the localization formula below. In this example, if we replace the standard S^1-action with $z \longrightarrow e^{\mathrm{i}m\theta}z$, over $\mathbb{Z}$, we get

$$H^*_{S^1}(S^2,\mathbb{Z}) = \{(f_0, f_\infty) \in \mathbb{R}[u] \oplus \mathbb{R}[u]\colon f_0 - f_\infty = muh(u)\}.$$

However, over $\mathbb{R}$, there is no torsion and we get the same answer as above. Consequently, H, f_0, and f_∞ will be replaced with mH, mH_0u, and $mH_\infty u$ and we get

$$1 = \int_{S^2} \omega = -\Big(\frac{f_0(u)}{mu} + \frac{f_1(u)}{-mu}\Big).$$

Remark 4.22. For $G = \mathbb{T}^k = (S^1)^k$, we have $BG = \big(\mathbb{CP}^\infty\big)^k$. Therefore, $H^*(B\mathbb{T}^k,\mathbb{Z}) = \mathbb{Z}[h_1, \ldots, h_k]$. In this case, the torus action is given by k vector fields $\zeta_1, \ldots, \zeta_k$ such that $[\zeta_i, \zeta_j] = 0$ for all $1 \le i, j \le k$. Similarly to the case of S^1 above, a de Rham model is obtained by considering $\mathbb{T}^k$-invariant differential forms with values in $\mathbb{R}[u_1, \ldots, u_d]$ and the deformed derivative map

$$\Omega^*(M,\mathbb{R}[u_1, \ldots, u_k]) \xrightarrow{D=\mathrm{d}+u_1\iota_{\zeta_1}+\cdots+u_k\iota_{\zeta_k}} \Omega^*(M,\mathbb{R}[u_1, \ldots, u_k]).$$

What kind of information can be mined from the fixed points of an action? This is the question we want to answer next.

If a Lie group G acts on a manifold by diffeomorphisms,

$$\varphi\colon G\times M\longrightarrow M,\quad (g,x)\longrightarrow \varphi_g(x),$$

at a fixed point $p\in M$, the differential $\mathrm{d}\varphi_g\colon T_pM\longrightarrow T_pM$ is a linear automorphism of the tangent space, giving rise to a representation of the group G on the tangent space T_pM. Invariants of the representation are then invariants of the action at the fixed point. If $\dim_{\mathbb{C}}M=2n$, for a circle action, at an isolated fixed point p, the tangent space T_pM decomposes into a direct sum $\mathbb{C}_{m_1}\oplus\cdots\oplus\mathbb{C}_{m_n}$, where $\mathbb{C}_m$ is the representation of weight m in Example 4.20. The integers $m_1,\ldots,m_n$ are defined only up to sign, but if M is oriented, the sign of the product $m_1\cdots m_n$ is well defined by the orientation of M. The localization formula then states that for every equivariant form $\widetilde{\omega}$ we have

$$\int_M\widetilde{\omega}=\sum_{p\in\mathrm{Fix}(S^1)}\frac{\iota_p^*\widetilde{\omega}}{\mathrm{e}_p}$$

where e_p is the equivariant Thom-form/Euler-form of p. In our notation, e_p is equal to

$$\mathrm{e}_p=m_1\cdots m_n(-u)^n\in H^*_{S^1}(p,\mathbb{R})=\mathbb{R}[u].$$

In general, the fixed locus of a $\mathbb{T}^k$-action may have higher dimensional components, but each connected component $F\subset M$ will be a smooth manifold. Similarly to above, we obtain an action of $\mathbb{T}^k$ on the normal bundle $\mathcal{N}_MF$. The formula above will continue to hold with e_F instead of e_p, where

$$\mathrm{e}_F\in H^*_{\mathbb{T}^k}(F,\mathbb{R})$$

is the equivariant Thom-form/Euler-form of $\mathcal{N}_MF$.

Below, we only outline the proof of localization formula. For each $\mathbb{T}^k$-invariant embedding $\iota\colon N\longrightarrow M$, we have the pullback map $\iota^*\colon H^*_{\mathbb{T}^k}(M)\longrightarrow H^*_{T^k}(N)$. There is also a push-forward map

$$\iota_*\colon H^*_{\mathbb{T}^k}(N)\longrightarrow H^{*+\dim(M)-\dim(N)}_{T^k}(M)$$

that is multiplication by Thom-form of N; i.e.

$$\iota_*\iota^*\widetilde{\eta}=\widetilde{\eta}\wedge\mathrm{e}_N$$

where e_N is the degree $\dim(M)-\dim(N)$ Thom-form of $N\subset M$. Let $\mathcal{R}=\mathbb{R}[u_1,\ldots,u_k]$. Both $H^*_{\mathbb{T}^k}(N)$ and $H^*_{\mathbb{T}^k}(N)$ are $\mathcal{R}$-modules and the maps above are maps of $\mathcal{R}$-modules. For any $\mathcal{R}$-module H, the set

$$\mathrm{Ann}(H)=\{f\in\mathcal{R}\colon f\alpha=0\quad\forall\ \alpha\in H\}$$

is an ideal. For $\mathcal{R}=\mathbb{R}[u_1,\ldots,u_k]$, the support of H is the affine variety (possibly singular complex manifold)

$$\mathrm{Supp}(H)=\{(x_1,\ldots,x_k)\in\mathbb{C}^k\colon f(x_1,\ldots,x_k)=0\quad\forall\ f\in\mathrm{Ann}(H)\}.$$

Let $\mathbb{F}$ denote the field of fractions of $\mathbb{R}[u_1,\ldots,u_k]$. For every $\mathbb{R}[u_1,\ldots,u_k]$-module H, by killing the torsion elements, i.e. elements that are annihilated by some nontrivial element in $\mathrm{Ann}(H)$, we obtain an $\mathbb{F}$-vector space called the localization $H_{\mathcal{R}}$ of H. Suppose $\mathrm{Fix}(\mathbb{T}^k)=\bigcup_\alpha F_\alpha$ and define

$$Q=\sum_\alpha\frac{\iota_{F_\alpha *}\iota^*_{F_\alpha}}{\mathrm{e}_{F_\alpha}}\colon H^*_{\mathbb{T}^k}(M)_{\mathcal{R}}\longrightarrow H^*_{\mathbb{T}^k}(M)_{\mathcal{R}}.$$

The localization formula follows from showing that this is an isomorphism of $\mathbb{F}$-vector spaces.

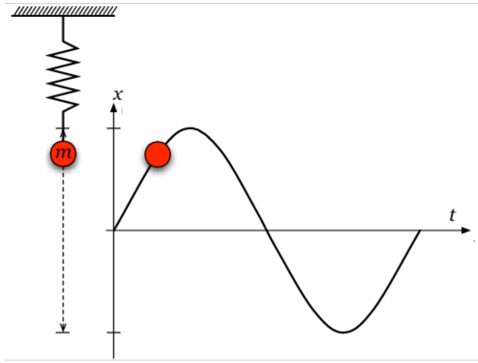


Figure 7: Left. Harmonic oscillator. Right. Graph of location vs. time.

5 An introduction to symplectic topology

Symplectic topology has origin in the study of Hamiltonian dynamical systems. Let us start with a classical and simple example in physics. The differential equation governing the motion of a harmonic oscillator (Figure 7) with no damping is the second order ODE

$$m\ddot{x} = -\kappa x, \tag{5.1}$$

where m is the mass of the object attached to the spring and κ is a constant that depends on the spring. Let $p = m\dot{x}$ denote the momentum. We can rewrite (5.1) as

$$\dot{x} = \frac{p}{m} \qquad \dot{p} = -\kappa x \tag{5.2}$$

to turn in into a pair of first order ODEs in the "phase space" $\mathbb{R}^2$ with coordinates (x,p) =(location, momentum). The phase space of an object moving in $\mathbb{R}^n$ is $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ with coordinates $(x,p) = (x_1,\ldots,x_n,p_1,\ldots,p_n)$ such that (x_a,p_a) is the (location, momentum) in the a-th direction. More generally, the phase of an object moving in a manifold M is the cotangent bundle T^*M. Every solution of (5.2) has the form

$$(x(t),p(t)) = A\big(\cos(wt+\theta_0), -mw\sin(wt+\theta_0)\big)$$

where $w = \sqrt{\kappa/m}$. The curve $(x(t),p(t))$ traces an ellipse in the phase space $\mathbb{R}^2$ with that initial (location, momentum)

$$(x_0,p_0) = A\big(\cos(\theta_0), -mw\sin(\theta_0)\big);$$

see Figure 8. For each time t, the map

$$\varphi_t\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad (x_0,p_0) \longrightarrow (x(t),p(t))$$

is a diffeomorphism of $\mathbb{R}^2$ such that $\varphi_0 = \mathrm{id}$ and $\varphi_t \circ \varphi_s = \varphi_{t+s}$. In other words, the flow of a (time independent) ODE defines an action of $\mathbb{R}$ on the underlying manifold. One of the major goals of studying such systems is to understand the dynamics of φ_t. Define

$$H\colon \mathbb{R}^2 \longrightarrow \mathbb{R}, \qquad H(x,p) = \frac{1}{2}\Big(\frac{p^2}{m} + \kappa x^2\Big).$$

The first term $\frac{1}{2}\frac{p^2}{m} = \frac{1}{2}m\dot{x}^2$ is the kinetic energy of the moving object and the second term $\frac{1}{2}\kappa x^2$ is the potential energy; thus, H is the total energy of the system. Observe that the equation (5.2) has the form

$$\dot{x} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial x}.$$

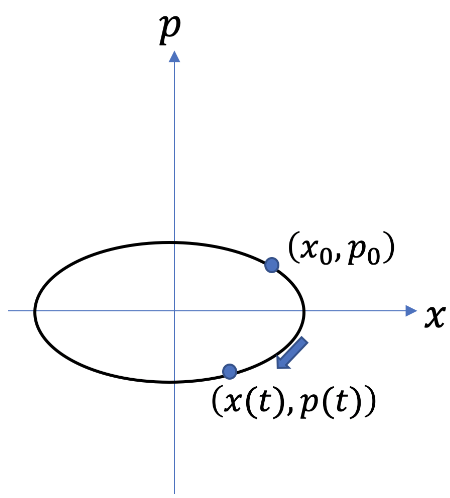


Figure 8: Orbit of a harmonic oscillator in phase space

Definition 5.1. A Hamiltonian ODE in $\mathbb{R}^{2n}$ with coordinates $(x,p) = (x_1, \ldots, x_n, p_1, \ldots, p_n)$ is an equation of the form

$$\dot{x}_a = \frac{\partial H}{\partial p_a} \qquad \dot{p}_a = -\frac{\partial H}{\partial x_a} \qquad \forall\ a = 1, \ldots, n, \tag{5.3}$$

for some "energy" function $H\colon \mathbb{R}^{2n} \longrightarrow \mathbb{R}$.

It is easy to show that the value of H does not change along any orbit $(x(t), p(t))$; i.e. H is a conserved quantity and the orbits of (5.3) live on the level sets of the function H. The ODE (5.3) can be re-written in the compact form

$$\frac{\mathrm{d}}{\mathrm{d}t}(x,p) = \zeta_H(x,p)$$

where

$$\zeta_H = \sum_{a=1}^{n} \frac{\partial H}{\partial p_a}\partial_{x_a} - \frac{\partial H}{\partial x_a}\partial_{p_a}$$

is a vector field associated to H on $\mathbb{R}^{2n}$. The vector field ζ_H and the 1-form $\mathrm{d}H$ are related by

$$\iota_{\zeta_H}\omega_0 = \mathrm{d}H \tag{5.4}$$

where ω_0 is the 2-form

$$\omega_0 = \sum_{a=1}^{n} \mathrm{d}x_a \wedge \mathrm{d}p_a.$$

The 2-form ω_0 is called the standard symplectic form on $\mathbb{R}^{2n}$. Let ω be a 2-form on a manifold M; the following properties are equivalent:

- $\dim(M) = 2n$ and ω^n is a volume form on M;
- for every $x \in M$, the map
$$T_xM \longrightarrow T_x^*M, \qquad \zeta \longrightarrow \iota_\zeta\omega$$
is an isomorphism.

If these properties hold, we say ω is a non-degenerate 2-form on M.

Definition 5.2. A symplectic manifold is a smooth manifold M equipped with a closed non-degenerate 2-form ω.

Example 5.3. The standard 2-form ω_0 defines a symplectic structure on the phase space $\mathbb{R}^{2n}$.

Example 5.4. An area form on every oriented surface Σ defines a symplectic structure on Σ.

The following is a fundamental theorem in symplectic topology which shows symplectic manifolds are all locally trivial.

Theorem 5.5. *Suppose (M,ω) is a symplectic manifold. For every $q \in M$ there is a chart $\psi\colon U \longrightarrow V \subset \mathbb{R}^{2n}$ around x (known as Darboux chart) such that $\psi^*\omega_0 = \omega$.*

Definition 5.6. Let (M,ω) be a symplectic manifold. A Hamiltonian ODE on M is an ODE of the form

$$\dot{x} = \zeta_H(x) \qquad \forall\, x \in M$$

where $H\colon M \longrightarrow \mathbb{R}$ is a smooth function and ζ_H is uniquely determined by the equation

$$\mathrm{d}H = -\iota_{\zeta_H}\omega. \tag{5.5}$$

Remark 5.7. Note that the identity (5.5) differs by a minus sign from (5.4). This is the common convention in symplectic topology to make certain actions counter clock-wise.

Lemma 5.8. *Let $\{\varphi_t\colon M \longrightarrow M\}_{t\in\mathbb{R}}$ denote the flow of the Hamiltonian ODE corresponding to the function H on the symplectic manifold M. Then*

$$\varphi_t^*\omega = \omega \qquad \forall\, t \in \mathbb{R}.$$

Proof. By (2.20),

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^*\omega = \varphi_t^* L_{\zeta_H}\omega = \varphi_t^*(\iota_{\zeta_H}\mathrm{d}\omega + \mathrm{d}\iota_{\zeta_H}\omega) = \varphi_t^*(0 - \mathrm{d}\mathrm{d}H) = 0.$$

The claim follows. □

Definition 5.9. Let (M,ω) be a symplectic manifold, we say a diffeomorphism $\psi\colon M \longrightarrow M$ is a symplectomorphism if $\psi^*\omega = \omega$. We say $\varphi\colon M \longrightarrow M$ is a Hamiltonian diffeomorphism if there exists H such that $\varphi = \varphi_t$ for some $t \in \mathbb{R}$ (we can always change H so that $t = 1$).

By Lemma 5.8, every Hamiltonian diffeomorphism is a symplectomorphism but the converse is not true for arbitrary manifold M.

Let M be a smooth manifold and $(x_1, \dots, x_n)$ be local coordinates on an open set $U \subset M$. For $x \in U$, every cotangent vector $\eta \in T_x^*M$ can be written as

$$\eta = \sum_{a=1}^{n} \eta_a \mathrm{d}x_a.$$

The assignment $\eta \longrightarrow (x_1, \dots, x_n, p_1, \dots, p_n)$ defines a local trivialization of $T^*M|_U$. Let

$$\omega_U = \sum_{a=1}^{n} \mathrm{d}x_a \wedge \mathrm{d}p_a \in \Omega^2(T^*M|_U, \mathbb{R}).$$

The following lemma follows from Chain Rule.

Lemma 5.10. *The two forms ω_U are independent of the choice of local coordinates on U and patch together to define a canonical symplectic structure on T^*M*

Next, we study Hamiltonian torus actions on symplectic manifolds and describe a way of constructing more complicated symplectic manifolds as quotient spaces.

Let (M,ω) be a symplectic manifold and

$$\mathbb{T}^k \times M \longrightarrow M, \qquad \big((\mathrm{e}^{\mathrm{i}\theta_1},\dots,\mathrm{e}^{\mathrm{i}\theta_k}),x\big) \longrightarrow \mathrm{e}^{\mathrm{i}\theta_1}\cdots\mathrm{e}^{\mathrm{i}\theta_k}\cdot x$$

be a torus action given by the vector fields

$$\zeta_i(x) = \frac{\mathrm{d}}{\mathrm{d}\theta}(\mathrm{e}^{\mathrm{i}\theta_i}\cdot x)|_{\theta_i=0} \qquad \forall\, x\in M.$$

We say this action is Hamiltonian if there are functions $H_i\colon M\longrightarrow\mathbb{R}$ such that $\zeta_i=\zeta_{H_i}$; i.e. $\iota_{\zeta_i}\omega=-\mathrm{d}H_i$. Then we say

$$H=(H_1,\dots,H_k)\colon M\longrightarrow\mathbb{R}^k$$

is the moment map of the action. The example of harmonic oscillator is a Hamiltonian S^1-action on $\mathbb{R}^2$. We will implicitly assume that the ζ_i are linearly independent, i.e. the action is generically free.

Theorem 5.11. *(1) If $a=(a_1,\dots,a_k)$ is a regular value of H, then*

$$M_a = H^{-1}(a)/\mathbb{T}^k$$

is a $2(n-k)$-dimensional manifold and admits a canonical symplectic structure ω_a such that $\pi^*\omega_a=\omega|_{H^{-1}(a)}$. Here $\pi\colon H^{-1}(a)\longrightarrow M_a$ is the quotient map. (2) If M is closed, then the image of H is a rational polyhedral polytope Δ in $\mathbb{R}^k$. (3) If $k=n$, then M can be uniquely reconstructed from Δ.

The symplectic manifold (M_a,ω_a) is known as a symplectic reduction of M.

Example 5.12. Consider $(\mathbb{C}^n=\mathbb{R}^{2n},\omega_0)$ and the function

$$H(x,p)=\frac{1}{2}\sum_a(x_a^2+p_a^2)=\frac{1}{2}|z|^2, \tag{5.6}$$

where

$$z=(z_1=x_1+\mathrm{i}p_1,\dots,z_n=x_n+\mathrm{i}p_n).$$

Then

$$\zeta_H=\sum_a -p_a\partial x_a + x_a\partial_{p_a}=\sum_a\partial_{\theta_a}$$

where θ_a is the angle coordinate in the (x_a,p_a)-plane. The flow of ζ_H is the diagonal S^1-action

$$\big(\mathrm{e}^{\mathrm{i}\theta},z\big)\longrightarrow\mathrm{e}^{\mathrm{i}\theta}z=(\mathrm{e}^{\mathrm{i}\theta}z_1,\dots,\mathrm{e}^{\mathrm{i}\theta}z_n).$$

The level set $H^{-1}(R^2)$ is the sphere S^{2n+1}_R of radius R. For each R, the quotient space

$$M_{R^2}=H^{-1}(R^2)/S^1$$

is the complex projective space $\mathbb{CP}^{n-1}$. The symplectic form $\omega_{R^2}=R^2\omega_1$ is a multiple of the well-known Fubini-Study Kähler form on $\mathbb{CP}^{n-1}$.

Symplectic reduction can also be used to define symplectic blowup. For a holomorphic manifold M of complex dimension n, the blowup $\widetilde{M}$ of M at a point $q \in M$ is another holomorphic manifold that has a copy of $E = \mathbb{CP}^{n-1}$ in place of q. It admits a projection map $\pi\colon \widetilde{M} \longrightarrow M$ that collapses E to q and is an isomorphism outside E. Intuitively, $\widetilde{M}$ is obtained by replacing q with all the (complex) tangent directions at q. Explicitly, it is constructed in the following way. Fix local holomorphic coordinates $(z_1, \ldots, z_n) \in \mathbb{C}^n$ on an open neighborhood $U \ni q$ such that q corresponds to the origin. Then the pre-image of U in $\widetilde{M}$ is the open set

$$\big\{\big((z_1, \ldots, z_n), [x_1, \ldots, x_n]\big) \in U \times \mathbb{CP}^{n-1} \colon z_i x_j = z_j x_i\big\}$$

with

$$E = 0 \times \mathbb{CP}^{n-1}.$$

Note that if $z_i \neq 0$ for some $i = 1, \ldots, n$, then $[x_1, \ldots, x_n]$ is uniquely determined by

$$x_j = \frac{z_j}{z_i} x_i \;\Rightarrow [x_1, \ldots, x_n] = [z_1/z_i, z_2/z_i, \ldots, z_n/z_i].$$

Topologically, blowup corresponds to the connect sum of M and $\overline{\mathbb{CP}^n}$ ($\mathbb{CP}^n$ with the reversed orientation) at a point. In the symplectic world, blowup is constructed in the following way. First, by Theorem 5.5, there are local coordinates (x, p) around q with respect to which ω coincides with ω_0. Let

$$H\colon U \longrightarrow \mathbb{R}, \qquad H(z) = \frac{1}{2}|z|^2$$

denote the restriction of the function H in (5.6) to U. For sufficiently small $\varepsilon > 0$, the level set $H^{-1}(\varepsilon)$ divides M into two components $M_+ \cup M_-$ such that

$$M_- = H^{-1}([0, \epsilon]) \quad \text{and} \quad M_+ = M - H^{-1}([0, \epsilon)).$$

Then M_+ is a manifold with boundary $\partial M_+ = H^{-1}(\varepsilon) \cong S^1$. Roughly speaking, the blowup manifold $\widetilde{M}$ is obtained by collapsing the boundary of M_+ using the S^1-action into manifold $E \cong \mathbb{CP}^{n-1}$ of real dimension $2(n-2)$. More precisely, $\widetilde{M}$ is a union of two charts

$$\widetilde{M} = W \cup (M - H^{-1}([0, \epsilon])$$

where W is constructed in the following way. Extend H to function $\widetilde{H}$ on $U \times \mathbb{C}$ by

$$\widetilde{H}(z, t) = H(z) - \frac{1}{2}|t|^2.$$

Then,

$$\widetilde{H}^{-1}(\varepsilon) = \big\{(z, t) \in U \times \mathbb{C} \colon H(z) \geq \varepsilon,\ \frac{1}{2}|t|^2 = H(z) - \varepsilon\big\}.$$

Let $(W = \widetilde{H}^{-1}(0)/S^1, \omega_W)$ denote the symplectic reduction of $U \times \mathbb{C}$ with respect to $\widetilde{H}$ at the level set ε. Then,

- The subset
$$E = \Big(\widetilde{H}^{-1}(\varepsilon) \cap U \times \{0\}\Big)/S^1 \cong H^{-1}(\varepsilon)/S^1 \cong \mathbb{CP}^{n-1}$$
is a symplectic submanifold of W;

- The complement of E is naturally identified with the open set $U - H^{-1}(\varepsilon) \subset (M - H^{-1}([0,\epsilon])$ by
$$W - E = \{(z,t) \in U \times \mathbb{C} \colon H(z) > \varepsilon,\ \frac{1}{2}|t|^2 = H(z) - \varepsilon\} \cong U - H^{-1}(\varepsilon).$$
- The symplectic forms ω_W and ω agree on this overlap region.

Therefore, ω_W and ω patch together to define a symplectic form on $\widetilde{M} = W \cup (M - H^{-1}([0,\epsilon])$.

Remark 5.13. Unlike in holomorphic blowup, in the case of symplectic blowup, there is no canonical projection map $\pi \colon \widetilde{M} \longrightarrow M$.

Mohammad Farajzadeh Tehrani,
mohammad-tehrani@uiowa.edu

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