Deformation Theory of Pseudoholomorphic Curves Relative to an SNC Divisor

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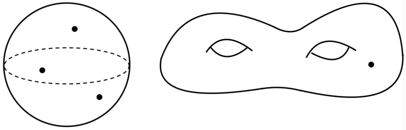
Notation

- X: smooth complex proj variety OR symplectic manifold with an ω-tame almost complex structure J
- Genus g curves (Σ, j) with k marked points $\vec{z} = (z_1, \ldots, z_k)$



- $A \in H_2(X, \mathbb{Z})$
- $\mathcal{M}_{g,k}(X, A) =$

$$\Big\{ \big(u, (\Sigma, \mathrm{j}, \vec{z})\big) : \Sigma \xrightarrow{u} X,\ \bar{\partial} u = 0,\ [u(\Sigma)] = A \Big\} / \sim$$

Relative case

- SNC divisor in X: $D = \bigcup_{i=1}^{N} D_i$
- **Remark.** In Symp case, definition of SNC divisor is due to McLean, Zinger, and I (2014). We prove existence of J that is compatible with D and several other strcutures
- $\mathcal{M}_{g,k}(X,A) \supset \mathcal{M}_{g,\mathfrak{s}}(X,D,A) =$

$$\Big\{[u,(\Sigma,\mathfrak{j},\vec{z})]\colon u^{-1}(D) \subset \{z_1,\ldots,z_k\},\ \mathsf{ord}_{z_a}(u,D_i) = s_{ai}\Big\}$$

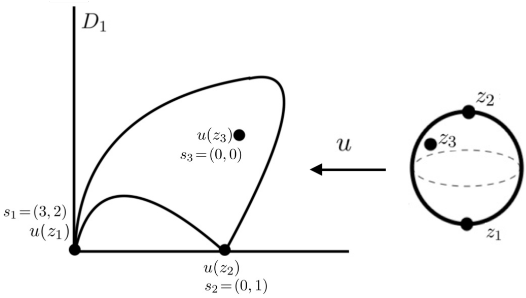
- Tangency orders: $\mathfrak{s} = (s_1,\ldots,s_k)$

$$s_a = (s_{ai})_{i=1}^{N} \in \mathbb{N}^N, \qquad A\cdot D_i = \sum_{a=1}^{k} s_{ai}$$

- $\text{Exp-dim}_{\mathbb{C}}\, \mathcal{M}_{g,\mathfrak{s}}(X,D,A) = c_1^{TX}(A) + (n-3)(1-g) + k - A\cdot D$

Example 1

- $X = \mathbb{P}^2$, $\quad D = D_1 \cup D_2$ union of two hyperplanes (lines)
- $A = [3] \in H_2(\mathbb{P}^2, \mathbb{Z}) \cong \mathbb{Z}$, $\; g = 0$, $\; k = 3$
- $\mathfrak{s} = \big(s_1 = (3,2), s_2 = (0,1), s_3 = (0,0)\big)$
- $u\big([z,w]\big) = [z^3, z^2 w, p(z,w)]$



Example 2

- (X, D) arbitrary, $A = 0$ (constant maps), $\mathfrak{s} = \vec{0} = \big(0, \ldots, 0\big)$
- $\mathcal{M}_{g,\vec{0}}(X, D, 0) \cong \mathcal{M}_{g,k} \times (X - D)$ has dimension $n + 3(g-1) + k$
- The expected dimension is $n(1-g) + 3(g-1) + k$, which is $\underline{ng}$ less than the actual dimension
- **Question**: What is the obstruction bundle?
- In the classical case,

$$\overline{\mathcal{M}}_{g,k}(X, 0) \cong \overline{\mathcal{M}}_{g,k} \times X$$

and the obstruction bundle is

$$\pi_1^* \mathcal{E}_g^* \otimes \pi_2^* TX$$

GOAL (analytical approach)

- Compactify $\mathcal{M}_{g,\mathfrak{s}}(X,D,A) \rightsquigarrow \overline{\mathcal{M}}_{g,\mathfrak{s}}(X,D,A)$

$$\begin{array}{ccc} \mathcal{M}_{g,\mathfrak{s}}(X,D,A) & \hookrightarrow & \mathcal{M}_{g,k}(X,A) \\ \big\downarrow & & \big\downarrow \text{stable compactification} \\ \overline{\mathcal{M}}_{g,\mathfrak{s}}(X,D,A) & \overset{?}{\hookrightarrow} & \overline{\mathcal{M}}_{g,k}(X,A) \end{array}$$

- Set up the deformation theory
- Construct VFC

- **Compactness Theorem (–, 2017)** For suitable choice of J (including holomorphic case), there exists a metrizable compactification $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$ such the natural forgetful map

$$\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A) \longrightarrow \overline{\mathcal{M}}_{g,k}(X,A)$$

  is locally an embedding. It is an embedding if $g=0$.

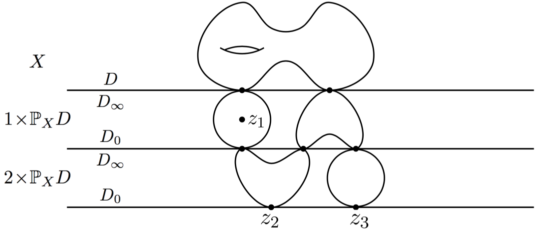
- **Proposition (–, 2017)** If D is smooth, there is a surjective map

$$\overline{\mathcal{M}}^{\mathsf{rel}}_{g,\mathfrak{s}}(X,D,A) \longrightarrow \overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$$

Previous works (smooth D, early 2000)

Relative compactification of Jun Li, Ionel-Parker, and Li-Ruan

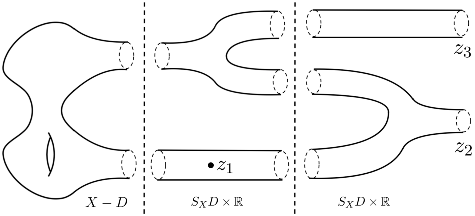
- **Jun Li and Ionel-Parker setup:**



- Ionel-Parker's work is limited to the semi-positive setting

- It does not contain a dedicated deformation-obstruction theory or a gluing analysis

- **Li-Ruan setup** is a Morse-Bott version of the SFT setup



- SFT setup can be modified to address the analytical problems
- Working with $X - D$ is hard if D is not smooth

Previous works (SNC D and more, mid 2000-current)

- Gross-Siebert, Abramovich-Chen, ... (2010-2012, Working with log varieties)
  - **idea:** They consider pairs of holomorphic maps and maps between certain **sheaves of monoids** on domains and a fixed sheaf of monoids on the target
  - Also, recent results by Ruddat, Ranganathan, Wise, ...
- Brett Parker (analytical, 2007-Current, working with his Exploded category)
  - **idea:** Similarly, pairs of holomorphic maps and maps of certain analytical sheaves
- Ionel (analytical, 2015, assuming certain J exists)
  - Claimed a construction of GW invariants relative to SNC divisors using expanded degenerations

Deformation theory and transversality

- When is $\mathcal{M}_{g,\mathfrak{s}}(X, D, A)$ an orbifold of the expected dimension?
- What are the deformation/obstruction spaces?
- How to achieve transversality?
- and, the analogue of these questions for nodal maps in the compactification

Classical setup

- For fixed $(\Sigma, \mathfrak{j})$ and A, we consider the ∞-dimensional bundle

$$\mathcal{E} \longrightarrow \mathcal{B} = \mathsf{Map}_A(\Sigma, X), \qquad \mathcal{E}|_u = \Gamma(\Sigma, \Omega^{0,1}_{\Sigma,\mathfrak{j}} \otimes u^*TX)$$

- $\bar{\partial}\colon \mathcal{B} \longrightarrow \mathcal{E}$ is a smooth section; where $\bar{\partial}u = \frac{1}{2}\big(\mathsf{d}u + J\mathsf{d}u \circ \mathfrak{j}\big)$
- $\bar{\partial}^{-1}(0)$ is the set of $(J, \mathfrak{j})$-holomorphic maps from Σ into X
- $\mathsf{D}_u\bar{\partial}\colon \Gamma(\Sigma, u^*TX) \longrightarrow \Gamma(\Sigma, \Omega^{0,1}_{\Sigma,\mathfrak{j}} \otimes u^*TX)$
- $\mathsf{D}_u\bar{\partial} = \bar{\partial}_{\mathsf{std}} +$ compact perturbation, and

$$\mathsf{Def}(u) = \ker(\mathsf{D}_u\bar{\partial})$$
$$\mathsf{Obs}(u) = \mathsf{coker}(\mathsf{D}_u\bar{\partial})$$

are finite dimensional

- By Riemann-Roch

$$\dim_{\mathbb{R}}\mathrm{Def}(u) - \dim_{\mathbb{R}}\mathrm{Obs}(u) = 2(c_1^{TX}(A) + n(1-g))$$

- If $D_u\bar{\partial}$ is surjective ($\mathrm{Obs}(u) = 0$) $\Rightarrow$ <u>around u</u>, $\mathcal{M}_{g,k}(X,A)$ is a smooth oriented orbifold of real dimension

$$2\big(c_1^{TX}(A) + n(1-g) + 3(g-1) + k\big) = \\ 2\big(c_1^{TX}(A) + (n-3)(1-g) + k\big)$$

- How to achieve transversality? Consider
global (Ruan-Tian) or
local (Li-Tian, Fukaya-Ono, etc) deformations of $\bar{\partial}$

- For $f = \big(u, C = (\Sigma, \mathfrak{j}, \vec{z})\big)$

$$0 \longrightarrow \mathsf{Def}(u) \longrightarrow \mathsf{Def}(f) \longrightarrow \mathsf{Def}(C) \longrightarrow \mathsf{Obs}(u) \longrightarrow \mathsf{Obs}(f) \longrightarrow 0$$

- $\mathsf{Def}(C) = H^1(T\Sigma(-\log \vec{z})) = \mathbb{E}\mathsf{xt}^1(\Omega_\Sigma(\vec{z}))$
- We just need $\mathsf{Obs}(f) = 0$ for $\mathcal{M}_{g,k}(X, A)$ to be smooth orbifold near f

Ruan-Tian perturbations

- Consider "regular" covering $\overline{\mathfrak{M}}_{g,k} \longrightarrow \overline{\mathcal{M}}_{g,k}$ admitting a universal family $\overline{\mathfrak{U}}_{g,k} \longrightarrow \overline{\mathfrak{M}}_{g,k}$
- If $g=0$, we can simply take $\overline{\mathfrak{U}}_{0,k} = \overline{\mathcal{M}}_{0,k+1} \longrightarrow \overline{\mathcal{M}}_{0,k}$
- $\Omega^{0,1}_{g,k} \longrightarrow \overline{\mathfrak{U}}_{g,k}$ whose restriction to each curve C is the sheaf of smooth $(0,1)$-forms on C supported away from the nodes
- Perturbation: $\nu \in \Gamma(\overline{\mathfrak{U}}_{g,k} \times X,\ \Omega^{0,1}_{g,k} \otimes_{\mathbb{C}} TX)$
- $\overline{\mathcal{M}}_{g,k}(X,A,\nu) = \Big\{ \big(\phi, u, C=(\Sigma, \mathfrak{j}, \vec{z})\big) :$
  $\Sigma \xrightarrow{u} X,\ C \xrightarrow{\phi} \overline{\mathfrak{U}}_{g,k},\ \bar\partial u = (\phi,u)^*\nu,\ [u(\Sigma)] = A \Big\} / \sim$

GW invariants

- Theorem (Ruan-Tian, 97): If X is semi-positive, for generic (J,ν),

$$\text{st} \times \text{ev}\colon \overline{\mathcal{M}}_{g,k}(X,A,\nu) \longrightarrow \overline{\mathcal{M}}_{g,k} \times X^k$$

  is a pseudo-cycle of the expected dimension

- Theorem (McDuff-Salamon, 94): Same result for generic J (no ν) if $g=0$ and X is positive

- Intersecting homology classes in $\overline{\mathcal{M}}_{g,k} \times X^k$ with the homology class of st $\times$ ev gives GW invariants

Log tangnet bundle

- Observation: Exp-dim$_{\mathbb{C}}$ $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)$

$$\begin{aligned} &= c_1^{TX}(A) + (n-3)(1-g) + k - A\cdot D \\ &= c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k \end{aligned}$$

- Log tangent bundle can also be defined for SNC symplectic divisors
- Construction of $TX(-\log D)$ uses the notion of regularization defined by McLean, Zinger, and I
- Deformation equivalence class of the complex vector bundle $TX(-\log D)$ only depends on ω
- There is a $\mathbb{C}$-linear homomorphism $\iota\colon TX(-\log D) \longrightarrow TX$ that is an isomorphism away from D

Setup

- For a fixed $(\Sigma, \mathfrak{j}, \vec{z})$, A, and $\mathfrak{s}$, we can construct a configuration space

$$\mathsf{Map}_{A,\mathfrak{s}}((\Sigma, \vec{z}), (X, D)) \subset \mathsf{Map}_A(\Sigma, X)$$

such that for each map u in this space

$$T_u\mathsf{Map}_{A,\mathfrak{s}}((\Sigma, \vec{z}), (X, D)) \cong \Gamma(\Sigma, u^*TX(-\log D))$$

and $\bar{\partial}u$ lifts to a section $\bar{\partial}_{\log}u \in \Gamma(\Sigma, \Omega^{0,1}_{\Sigma,\mathfrak{j}} \otimes u^*TX(-\log D))$:

$$\begin{array}{ccc} & & u^*TX(-\log D) \\ & \overset{\bar{\partial}_{\log}u}{\nearrow} & \downarrow \iota \\ T^{0,1}\Sigma & \xrightarrow{\bar{\partial}u} & u^*TX\ . \end{array}$$

Log linearization of CR operator

- Therefore, $\bar{\partial}_{\log}$ defines a section of ∞-dimensional bundle

$$\mathcal{E}_{\log} \longrightarrow \mathsf{Map}_{A,\mathfrak{s}}((\Sigma, \vec{z}), (X, D))$$
$$\mathcal{E}_{\log}|_u = \Gamma(\Sigma, \Omega^{0,1}_{\Sigma,\mathrm{j}} \otimes u^*TX(-\log D))$$

- $\mathcal{M}_{g,\mathfrak{s}}(X, D, A) = \bar{\partial}^{-1}_{\log}(0)/\sim$

-

$$\begin{array}{ccc} \Gamma(\Sigma, u^*TX(-\log D)) & \xrightarrow{\mathsf{D}_u \bar{\partial}_{\log}} & \Gamma(\Sigma, \Omega^{0,1}_{\Sigma,\mathrm{j}} \otimes u^*TX(-\log D)) \\ \downarrow \iota & & \downarrow \iota \\ \Gamma(\Sigma, u^*TX) & \xrightarrow{\mathsf{D}_u \bar{\partial}} & \Gamma(\Sigma, \Omega^{0,1}_{\Sigma,\mathrm{j}} \otimes u^*TX) \end{array}$$

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$$\mathsf{Def}_{\log}(u) = \mathsf{ker}(\mathsf{D}_u \bar{\partial}_{\log})$$
$$\mathsf{Obs}_{\log(u)} = \mathsf{coker}(\mathsf{D}_u \bar{\partial}_{\log})$$

Transversality (Main stratum)

- By Riemann-Roch

$$\mathsf{dim}_{\mathbb{R}}\mathsf{Def}_{\log}(u) - \mathsf{dim}_{\mathbb{R}}\mathsf{Obs}_{\log}(u) = 2\big(c_1^{TX(-\log D)}(A) + n(1-g)\big)$$

- If $\mathsf{Obs}_{\log}(u) = 0 \;\Rightarrow$ Around u, $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)$ is a smooth oriented orbifold of real dimension

$$2\big(c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k\big)$$

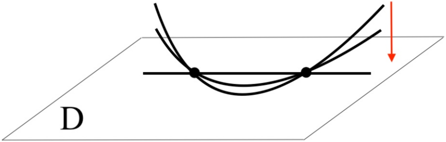
- For $f = \big(u, C = (\Sigma, \mathfrak{j}, \vec{z})\big)$

$$0 \longrightarrow \mathsf{Def}_{\log}(u) \longrightarrow \mathsf{Def}_{\log}(f) \longrightarrow \mathsf{Def}(C)$$
$$\longrightarrow \mathsf{Obs}_{\log}(u) \longrightarrow \mathsf{Obs}_{\log}(f) \longrightarrow 0$$

- We just need $\mathsf{Obs}_{\log}(f) = 0$ for $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)$ to be a smooth orbifold near f

Log maps with smooth domain and image in D

- A sequence of curves sinking into D_I gives us a J-holomoprhic map
$$u\colon \Sigma \longrightarrow D_I = \bigcap_{i\in I} D_i,$$
and holomorphic sections ζ_i of $u^*\mathcal{N}_X D_i$, for all $i\in I$



- ζ_i is only well-defined up to the action of $\mathbb{C}^*$
- The pair $(u, \zeta = (\zeta_i)_{i\in I})$ allows us to define a tangency order vector in $\mathbb{Z}^N$ for each $x \in \Sigma$:
$$\mathrm{ord}_x(u,\zeta) = (\mathrm{ord}^i_x(u,\zeta))_{i=1}^N, \quad \mathrm{ord}^i_x(u,\zeta) = \begin{cases} \mathrm{ord}_x(u, D_i) & \text{if } \; i \notin I \\ \mathrm{ord}_x(\zeta_i) & \text{if } \; i \in I \end{cases}$$

Log maps with smooth domain and image in D

- The equivalence class of a tuple $f=(u,\zeta,(\Sigma,\mathfrak{j},\vec{z}))$ defines a log curve in $\overline{\mathcal{M}}_{g,\mathfrak{s}}(X,D,A)$ if

$$\mathrm{ord}_{z_a}(u,\zeta)=s_a \qquad \text{and} \qquad \mathrm{ord}_x(u,\zeta)=0 \;\; \forall\, x\in\Sigma-\{z_1,\ldots,z_k\}$$

- Denote the stratum of such maps by $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)_I$
- There is a forgetful embedding map

$$\begin{aligned}\mathcal{M}_{g,\mathfrak{s}}(X,D,A)_I &\longrightarrow \mathcal{M}_{g,\bar{\mathfrak{s}}}(D_I,\partial D_I,A)\\ (u,\zeta,(\Sigma,\mathfrak{j},\vec{z})) &\longrightarrow (u,(\Sigma,\mathfrak{j},\vec{z}))\end{aligned}$$

where

$$\partial D_I=\bigcup_{j\in[N]-I} D_{I\cup j}$$

and

$$\bar{\mathfrak{s}}=(\bar{s}_1,\ldots,\bar{s}_k), \qquad \bar{s}_a=(s_{aj})_{j\in[N]-I}.$$

Log maps with smooth domain and image in D

- **Lemma**: There exists a map

$$P\colon \mathcal{M}_{g,\bar{\mathfrak{s}}}(D_I,\partial D_I,A) \longrightarrow (\mathsf{Pic}^0_g)^I$$

such that

$$\mathcal{M}_{g,\mathfrak{s}}(X,D,A)_I = P^{-1}(\mathcal{O}^I)$$

- **Conclusion**: If $\mathcal{M}_{g,\bar{\mathfrak{s}}}(D_I,\partial D_I,A)$ is cut transversely at u and $\mathcal{O}^I$ is a regular value of P at u then

$$\mathcal{M}_{g,\mathfrak{s}}(X,D,A)_I$$

is a smooth oriented orbifold of $\mathbb{C}$-dimension

$$c_1^{TD_I(-\log\partial D_I)}(A)+(n-|I|-3)(1-g)+k-|I|g =$$
$$c_1^{TX(-\log D)}(A)+(n-3)(1-g)+k-|I|$$

around f

Nodal log maps

An element of $\overline{\mathcal{M}}_{g,\mathfrak{s}}(X, D, A)$ is the equivalence class of a tuple

$$f = \big(u_v, \zeta_v = (\zeta_{v,i})_{i \in I_v}, (\Sigma_v, \mathfrak{j}_v, \vec{z}_v \cup q_v)\big)_{v \in \mathbb{V}}$$

such that

1. each tuple $\big(u_v, \zeta_v, (\Sigma_v, \mathfrak{j}_v, \vec{z}_v \cup q_v)\big)$ is as above except that $\zeta_{v,i}$ can have poles

2. forgetting ζ_v we get a stable map

$$\overline{f} = \big(u_v, (\Sigma_v, \mathfrak{j}_v, \vec{z}_v \cup q_v)\big)_{v \in \mathbb{V}}$$

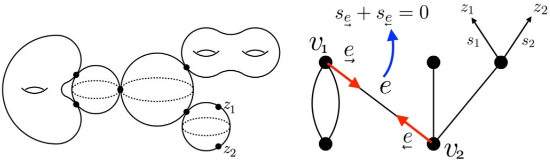
   in $\overline{\mathcal{M}}_{g,k}(X, A)$

3. tangency order vector at the marked point z_a is s_a
4. tangency orders at nodal points are dual of each other
5. tangency order vector at any other point is trivial

6. plus two more conditions!

Dual Graph and associated structures

Nodal Curve and its Dual Graph:



- $\varrho\colon \mathbb{Z}^{\mathbb{E}} \oplus \bigoplus_{v\in\mathbb{V}} \mathbb{Z}^{I_v} \longrightarrow \bigoplus_{e\in\mathbb{E}} \mathbb{Z}^{I_e}$
- $\mathrm{Ker}(\varrho) = \Big\{\big((\lambda_e)_{e\in\mathbb{E}}, (s_v)_{v\in\mathbb{V}}\big)\colon \quad s_{v_2} - s_{v_1} = \lambda_e s_{\vec{e}} \quad \forall v_1 \xrightarrow{\vec{e}} v_2\Big\}$
- $\exp(\varrho)\colon (\mathbb{C}^*)^{\mathbb{E}} \times \prod_{v\in\mathbb{V}} (\mathbb{C}^*)^{I_v} \longrightarrow \prod_{e\in\mathbb{E}} (\mathbb{C}^*)^{I_e}$
- $\mathcal{G}$ = cokernel of $\exp(\varrho)$
- In the classical case ϱ is the trivial map $\mathbb{Z}^{\mathbb{E}} \longrightarrow 0$ and $\mathcal{G}$ is trivial

Log maps

- **Lemma:** There is a map $f \longrightarrow \mathrm{ob}(f) \in \mathcal{G}$
- A log map is a tuple f as above satisfying the additional conditions

  1. **Condition 1 (combinatorial)**: $\ker(\varrho)$ has an element in the positive quadrant
  2. **Condition 2 (non-combinatorial)**: $\mathrm{ob}(f) = 1 \in \mathcal{G}$

- Condition 1 is equivalent to the existence of certain tropical curves in $\mathbb{R}^N$ modeled on the dual graph Γ (used in works of AC-GS)
- Condition 2 has no explicit analogue in the literature (it is needed for the construction of gluing map)

Expected dimension

- $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$ is coarsely stratified by $\bigcup_{\Gamma}\mathcal{M}_{g,\mathfrak{s}}(X,D,A)_{\Gamma}$
- **Lemma**: If
  1. $\mathcal{M}_{g_v,\bar{\mathfrak{s}}_v}(D_{I_v},\partial D_{I_v},A_v)$ is cut transversely at u_v and $\mathcal{O}^{I_v}$ is a regular value of P at u_v, for each $v\in\mathbb{V}$,
  2. the evaluation map at the nodal points are transverse to the diagonal,
  3. 1 is a regular value of the map ob

  then $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)_{\Gamma}$ is an oriented smooth orbifold of real dimension

  $$2\big(c_1^{TX(-\log D)}(A)+(n-3)(1-g)+k-\text{dim Ker}(\varrho)\big)$$

  around f
- If $N>1$, there are configurations with arbitrary large number of nodes but dim Ker$(\varrho)=1$

How to achieve transversality?

- Genus 0 simple maps $\longrightarrow$ generic J
- Higher genus simple maps $\longrightarrow$ generic (J,ν)

What is the right class of perturbations?

- Classical case (Ruan-Tian): $\nu \in \Gamma(\overline{\mathfrak{U}}_{g,k} \times X,\ \Omega^{0,1}_{g,k} \otimes_{\mathbb{C}} TX)$
- Relative case (Ionel-Parker): ν with conditions along D

$$\nu|_D \in \Gamma(\overline{\mathfrak{U}}_{g,k} \times D,\ \Omega^{0,1}_{g,k} \otimes_{\mathbb{C}} TD),$$
$$\frac{1}{2}\big(J\nabla_\nu J + \nabla_{J\nu} J\big)w - \big(\widetilde{\nabla}_w \nu + J\widetilde{\nabla}_{Jw}\nu\big) \in \Omega^{0,1}_{g,k} \otimes_{\mathbb{C}} T_x D;$$

for all $x \in D$, $w \in T_x X$

Logarithmic Ruan-Tian perturbations

- Recall: $\bar{\partial}_{\log} u \in \Gamma(\Sigma, \Omega^{0,1}_{\Sigma,\mathfrak{j}} \otimes u^* TX(-\log D))$
- **Definition**: Logarithmic Ruan-Tian perturbation

$$\nu_{\log} \in \Gamma(\overline{\mathfrak{U}}_{g,k} \times X,\ \Omega^{0,1}_{g,k} \otimes_{\mathbb{C}} TX(-\log D))$$

- $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X, D, A, \nu_{\log})$ can be defined for such $\nu_{\log}$
- **Lemma:** via $\iota\colon TX(-\log D) \longrightarrow TX$, from each $\nu_{\log}$ we obtain a classical ν satisfying IP conditions
- The forgetful map

$$\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X, D, A, \nu_{\log}) \longrightarrow \overline{\mathcal{M}}_{g,k}(X, A, \nu)$$

is still a local embedding. It is an embedding if $g = 0$.

Transversality theorem

- A (J, ν)-holomorphic map (ϕ, u, C) is called **simple** if no bubble (a non-trivial contracted component of ϕ) is a multiple cover, and images of every two bubbles are different
- A log map is called simple if the underlying stable map is simple
- **Theorem (–, 2019)** For generic (J, ν) the subspace of simple maps

$$\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\log,\star}(X, D, A, \nu_{\log}) \subset \overline{\mathcal{M}}_{g,\mathfrak{s}}^{\log}(X, D, A, \nu_{\log})_\Gamma$$

is an oriented smooth manifold of real dimension

$$2\left(c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k - \mathsf{dim}\ \mathsf{Ker}(\varrho)\right)$$

Semi-positive pairs

- We say $(X, D=\bigcup_{i=1}^N D_i, \omega)$ is **semi-positive** if

  $A\cdot D_i \geq 0$, and $c_1^{TX(-\log D)}(A) \geq 1-n \;\Rightarrow\; c_1^{TX(-\log D)}(A) \geq 0$

  for all $A \in \pi_2(M)$ such that $\omega(A) > 0$.

- Other notions of semi-positivity (and positivity) can be defined

- For each $a=1,\ldots,k$, let

$$I_a = \{i\colon s_{ai} \neq 0\} \subset \{1,\ldots,N\}$$

- $X^{\mathfrak{s}} = \prod_{a=1}^k D_{I_a}$

- Evaluation map at marked points has image in $X^{\mathfrak{s}}$

Claim. If (X, D, ω) is semi-positive, for generic (J, ν),

1. the map

$$\mathsf{st} \times \mathsf{ev}\colon \mathcal{M}_{g,\mathfrak{s}}(X, D, A, \nu_{\log}) \longrightarrow \overline{\mathcal{M}}_{g,k} \times X^{\mathfrak{s}}$$

   defines a pseudo-cycle of $\mathbb{C}$-dimension

$$c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k;$$

2. the integral homology class $\widetilde{\mathsf{GW}}^{X,D}_{g,\mathfrak{s},A}$ in $\overline{\mathcal{M}}_{g,k} \times X^{\mathfrak{s}}$ determined by this pseudo-cycle is independent of the choice of (J, ν);
3. furthermore, the rational class

$$\mathsf{GW}^{X,D}_{g,\mathfrak{s},A} \equiv \frac{1}{\deg\, p} \widetilde{\mathsf{GW}}^{X,D}_{g,\mathfrak{s},A} \in H_*(\overline{\mathcal{M}}_{g,k} \times X^{\mathfrak{s}}, \mathbb{Q}),$$

   where $\deg\, p$ is the degree of the regular covering used to define ν, is an invariant of the deformation equivalence class of ω.