Problem Solving 2

Lecture 13   Apr 04, 2021

- Q1 (warm up). Find the number of solutions of

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = d$$

where $a < b < c < d$ are natural numbers.

- Solution.

- $a < b < c \ \rightarrow \ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 1 + \frac{1}{2} + \frac{1}{3} < 2 \ \rightarrow d = 1$ which is impossible!

- Q2. Suppose a, b, c are three positive integers so that $a < 2b$ and
- the remainder of a divided by b is even (say $2r$)
- the remainder of a divided by c is r
- the remainder of b divided by c is r

Show that $\frac{a+b}{2}$ is an integer and c divides $\frac{a+b}{2}$?

- Hint. Write a and b as something x c + remainder

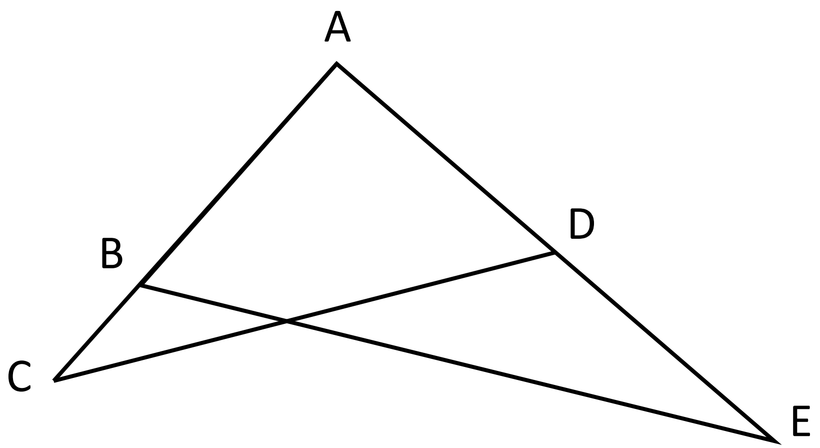
- **Solution.** Write
- $a = xc + r$
- $b = yc + r$
- $a = z\, b + 2r$

- $a < 2b \;\rightarrow\; z = 0 \;\; or \;\; 1$ Show that $z = 1$

Then, $a + b = a + a - 2r = 2(a - r) \rightarrow \frac{a+b}{2} = a - r = xc$

- Q3. Suppose $\angle CDE = CBE$.   Which of the following relations hold:

- [ ] $BC \cdot AC = DE \cdot AE$
- [ ] $AB \cdot AC = AD \cdot AE$
- [ ] $AD \cdot AC = AB \cdot AE$
- [ ] $AB \cdot BC = AD \cdot DE$
- [ ] $AB \cdot DE = AD \cdot BC$

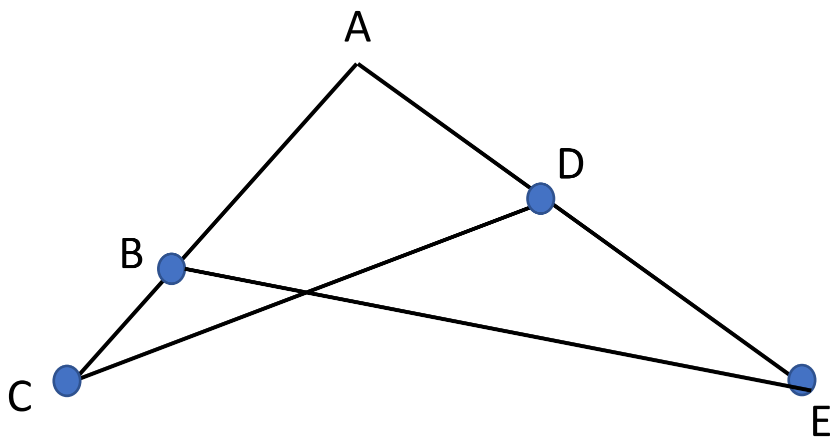


- Solution. Suppose $\angle CDE = CBE$.

Show that the triangles ACD and ABE are similar

SO $\frac{AD}{AB} = \frac{AC}{AE} = \frac{CD}{BE}$

SO $\underline{AD \cdot AE = AB \cdot AC}$



- Q4. For all $n \in \mathbb{N}$, show that the equation

$$\frac{1}{x} - \frac{1}{y} = \frac{1}{n}$$

has a unique integer solution (x, y) if and only if n is a prime

- Hint. If n is not prime, find explicit solutions

- Solution (part 1).

(1) Suppose n is a prime p $\rightarrow$ we can rewrite $\frac{1}{x} - \frac{1}{y} = \frac{1}{p}$ as $\frac{y-x}{xy} = \frac{1}{p}$ or

$$p(y-x) = xy$$

(2) Since p divides xy and is prime, it must divide at least one of them

(3) If $x = pk \rightarrow \; y - pk = ky \rightarrow y(k-1) = -pk \quad which\ is\ impossible$

(3) If $y = pk \rightarrow \; pk - x = kx \rightarrow x(k+1) = \; pk \; \rightarrow x = \frac{pk}{k+1}$. Since $k+1$ and k are relatively prime, $k+1$ must divide p. Therefore, it must be p. Therefore,

$$x = k = p - 1, \qquad and\ y = p(p-1)$$

- Solution (part 2)

(1) Suppose n is not a prime $\rightarrow n = ab \;\; with\; a, b > 1$

(2) Here are two solutions

$$x = n - 1, \qquad y = n(n-1) \qquad (just\ like\ prime\ case)$$

$$x = a(b-1), \qquad y = ab(b-1)$$

The key is that if $\frac{1}{x} - \frac{1}{y} = \frac{1}{b}$ then $\frac{1}{ax} - \frac{1}{ay} = \frac{1}{ab}$

So if you have a solution (x, y) for $n = b$, then you get a solution (ax, ay) for $n = ab$

- Q5. Consider a 25x25 table. In first row we write 1 to 25 from left to right. In the second row we write 26 to 50 from left to write, and we continue this way to fill the table. If we choose 25 squares from the table so that no two of them are in the same row and same column, which of the followings can be sum of these 25 numbers?

- Always $\binom{25}{2}$
- A number between $\binom{25}{2}$ and $\binom{25}{3}$
- Always $\frac{25+25^3}{2}$
- A number between $\binom{25}{2}$ and $\frac{25+25^3}{2}$

- Solution.

The numbers in first row are of the form $0\times 25+j \quad where\ j=1,2,3,\dots,25$

The numbers in second row are of the form $1\times 25+j \quad where\ j=1,2,3,\dots,25$

…

The numbers in the $i-th$ column are of the form $(i-1)\times 25+j \quad where\ j=1,2,3,\dots,25$

…

The numbers in 25th row are of the form $24\times 25+j \quad where\ j=1,2,3,\dots,25$

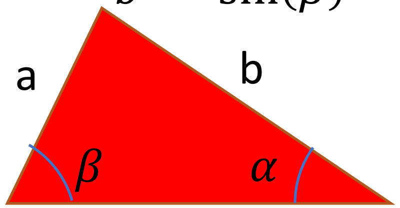
Each $0\leq i\leq 24$ and $each\ \ 1\leq j\leq 25$ appear once so:

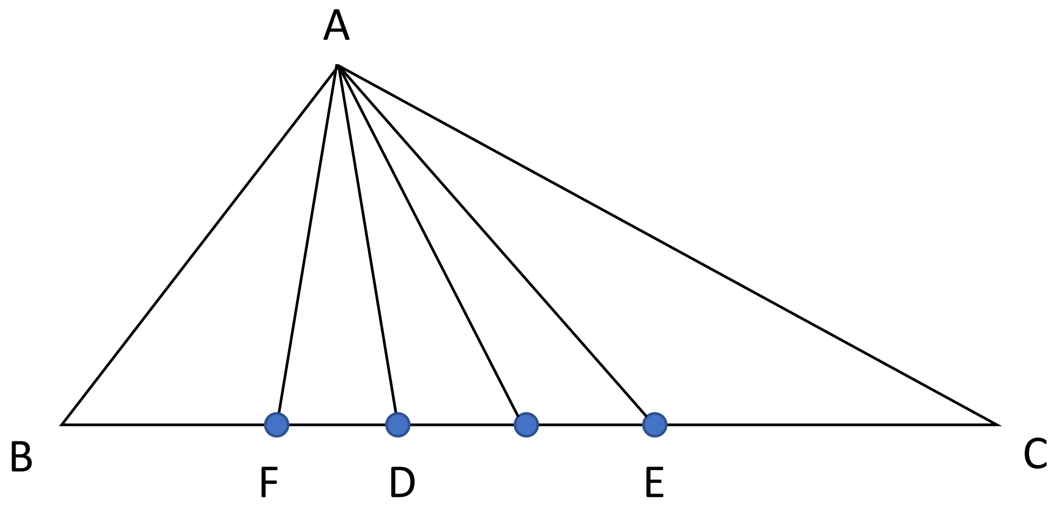
Answer is always = $25\,(0+1+\cdots+24)+(1+2+\cdots+25)$ =

- Q6. Consider a triangle ABC. Let D denote the intersection point of the angle bisector of A and BC. Let E be the mirror of D with respect to the middle of BC. Choose the point F on BC such that $\angle BAF = \angle EAC$.

Show that $\frac{BF}{FC} = \left(\frac{AB}{AC}\right)^3$

Hint. $\frac{a}{b} = \frac{\sin(\alpha)}{\sin(\beta)}$





- **Answer.**

$$(1)\ \frac{\sin(B)}{\sin(\angle BAF)} = \frac{AF}{BF} \quad \Rightarrow \quad \text{So} \quad \frac{BF}{CE} = \frac{AF}{AE}\frac{\sin(C)}{\sin(B)} = \frac{AF}{AE}\frac{AB}{AC} \quad \text{(i)}$$

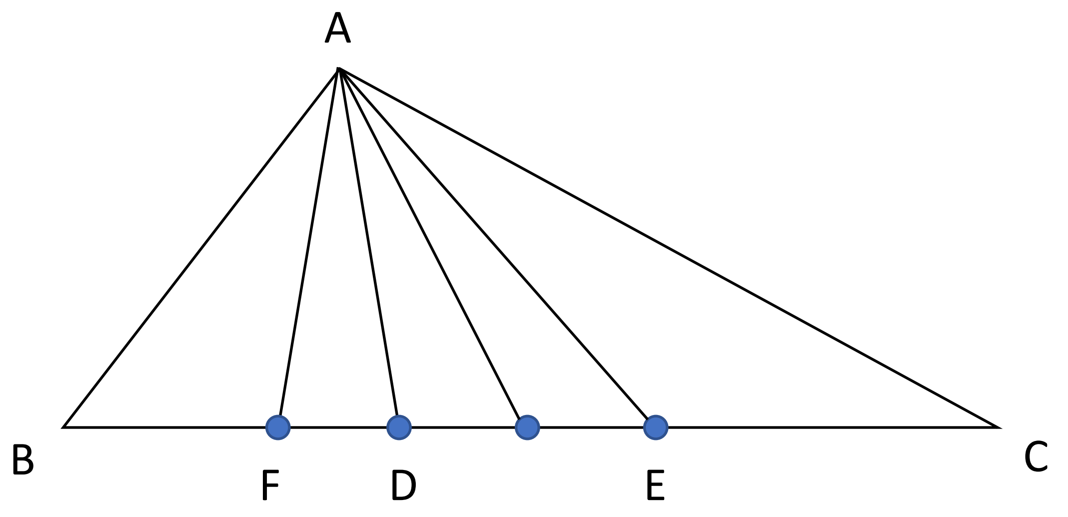
$$(2)\ \frac{\sin(C)}{\sin(\angle CAE)} = \frac{AE}{CE}$$

$$(3)\ \frac{\sin(B)}{\sin(\angle BAE)} = \frac{AE}{BE} \quad \Rightarrow \quad \text{So} \quad \frac{BE}{CF} = \frac{AE}{AF}\frac{\sin(C)}{\sin(B)} = \frac{AE}{AF}\frac{AB}{AC} \quad \text{(ii)}$$

$$(4)\ \frac{\sin(C)}{\sin(\angle CAF)} = \frac{AF}{CF}$$

$$\text{(i)+(ii)} \quad \Rightarrow \quad \frac{BF}{CE}\frac{BE}{CF} = \left(\frac{AB}{AC}\right)^2 \quad \Rightarrow$$

$$\frac{BF}{CF} = \left(\frac{AB}{AC}\right)^2 \left(\frac{CE}{BE}\right) = \left(\frac{AB}{AC}\right)^2 \left(\frac{BD}{CD}\right) = \left(\frac{AB}{AC}\right)^3$$



- Q7. What is the size of largest subset S of $\{1,2,\dots,99\}$ with the property that no number in S is twice another number in S.

- Solution. For each odd number $n \in \{1,3,\ldots,99\}$, consider the subset

- $\{n, 2n, 4n, \ldots, 2^k n\} \subset \{1,\ldots,99\}$
- At most $\left\lceil \frac{k}{2} \right\rceil$ of these numbers can be in S
- So we may assume they are $\{n, 4n, 16n, 64\,n, \ldots\}$

- Answer: # Odds + # 4 Odds + … = 50 + 13 + 2 + 1 = 66

- Q8. Show that there is only one sequence of positive integers $n_1, n_2, \ldots$ such that

$$n_{k+1} > n_{n_k}$$

Hint. Show the sequence is increasing!

- Answer.

(1) We show n_1 is the smallest: if the smallest is n_k with $k > 1$, then by ★ $n_k > n_{n_{k-1}}$

So $n_{n_{k-1}}$ is smaller than n_k which is a contradiction

(2) Now consider a new sequence $n'_1 = n_2,\ n'_2 = n_3, \ldots$

Show that this new sequence has the property ★

(3) So n_2 is the smallest of $n_2, n_3, \ldots$

(4) Repeating this idea (by induction) we conclude that $n_1 < n_2 < \cdots$

(5) Therefore, $n_k \geq k$

(6) On the other hand, by ★ we must have $n_{k-1} < k$

So $n_k = k$

- Q-end. Consider a triangle ABC. Let H denote the intersection point of the three heights. If AH=BC, what are the possible values of

Hint. You need to expand this picture!