Problem Solving 1

Lecture 12   Mar 28, 2021

- Q1. We keep multiplying even numbers $2, 4, 6, \ldots$ until we get a number N that is divisible by 1375. What is the last number used in the multiplication?

- Hint. Decompose 1375 into a product of prime numbers:

$$\mathbf{1375 = 5^3 \times 11}$$

- Solution.

(1) In the sequence

$$2, 4, 6, 8, 10, \ldots$$

the first number that is divisible by 11 is 22

(2) The numbers divisible by 5 are in the sequence of even numbers are

$$10, 20, 30, 40, \ldots$$

We need 5^3. So we need 10, 20, and 30.

Conclusion: The last number in the product is **30**

- Q2. Suppose $9x + 5y$ is divisible by 11. Which of the following numbers is also divisible by 11:

$a)$ $10x + 2y$

$b)$ $10x + 4y$

$c)$ $10x + 6y$

$d)$ $10x + 8y$

$e)$ $10x + 10y$

- Hint. Work mod 11 (Recall the discussion on congruence)

- Solution.

(1) To change 9 to 10, we find a number $\boldsymbol{a}$ modulo 11 such that

$$9\,a \equiv 10 \equiv -1 \quad modulo\ 11$$

(2) Note that $9 \equiv -2 \ \ mod\ 11,$ so. $a \equiv -5 \equiv 6\ \ mod\ 11$ works

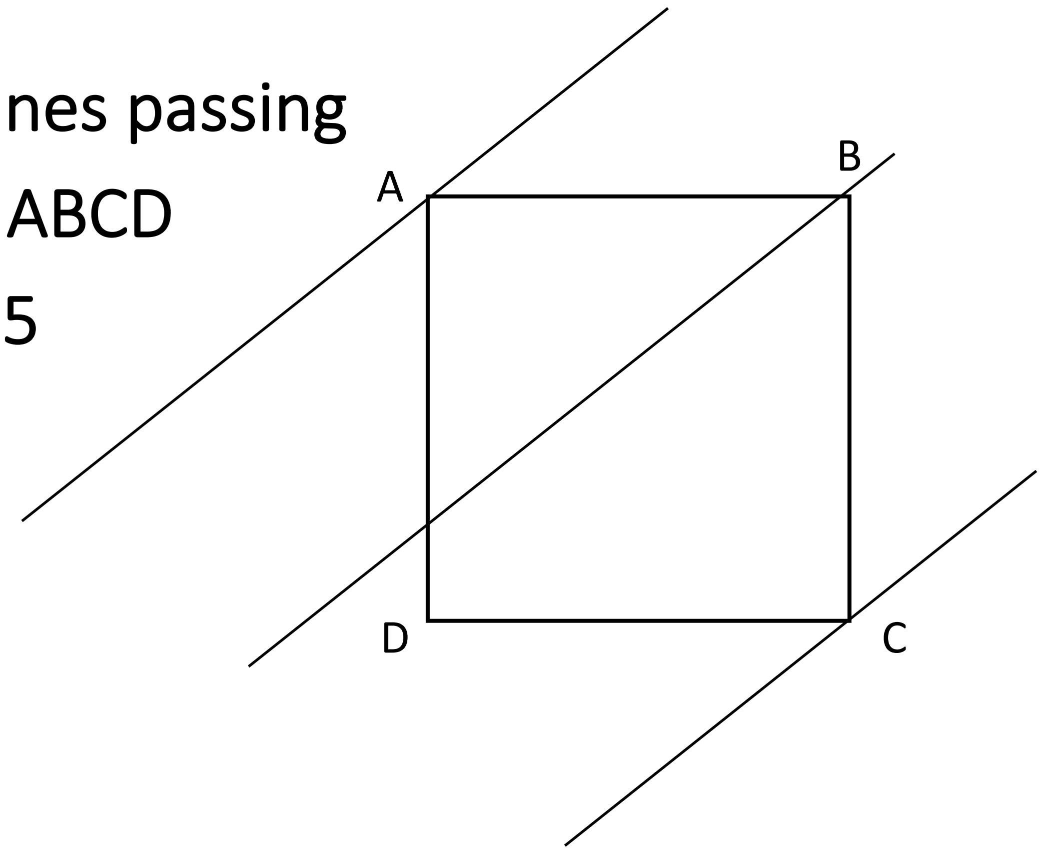
(3) We know $0 \equiv 9x + 5y\ mod\ 11$

(4) Multiply by 6, you get

$$0 \equiv 6(9x + 5y) \equiv 10x + 30y \equiv 10x + 8y \quad mod\ 11$$

- Q3. Suppose L_1, L_2, L_3 are 3 parallel lines passing Through the vertices A, B, C of a square ABCD Suppose the distance between L_1, L_2 is 5 and the distance between L_2, L_3 is 7.

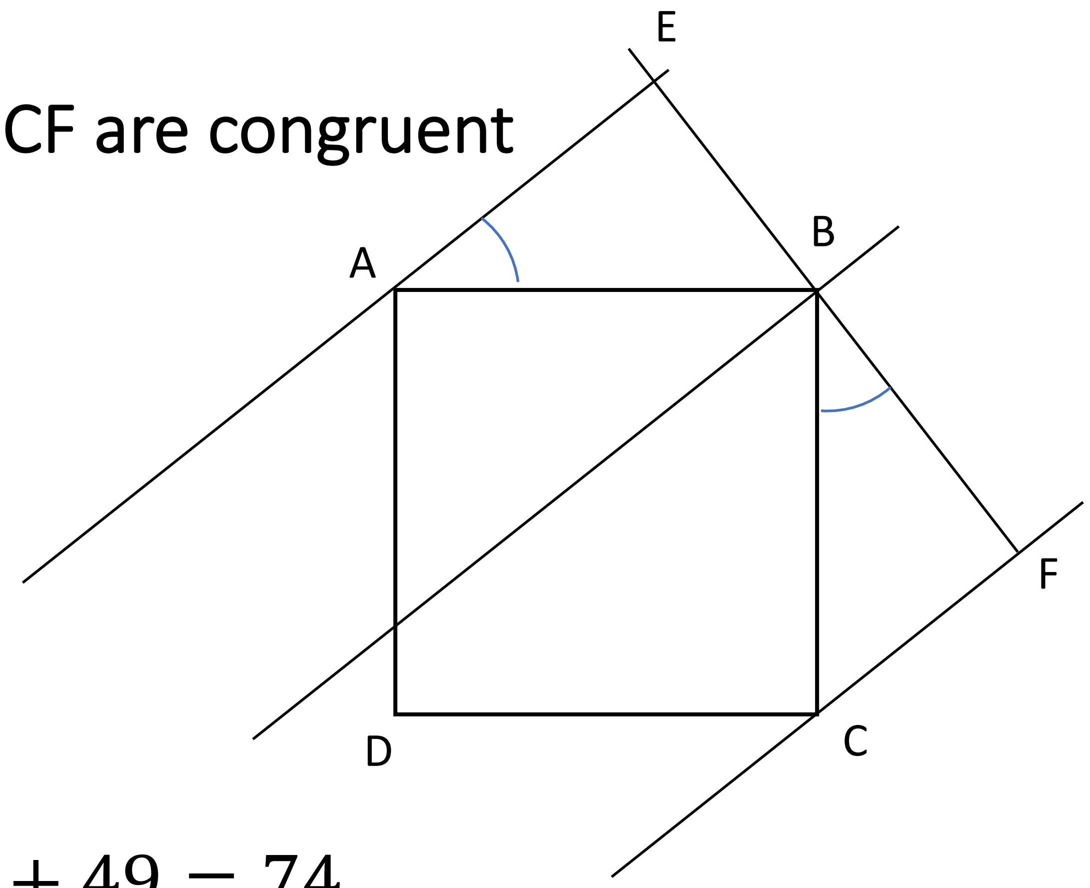
What is the area of square?



- Hint. Use Pythagorean theorem and similar triangles

- Solution.
- 1) The right triangles ABE and BCF are congruent
- 2) So
- AE=BF=7
- BE=CF=5
- 3) So

AREA= $AB^2 = AE^2 + BE^2 = 25 + 49 = 74$



- **Q4.** Find integer solutions (x, y, z) of the equations

$$x^2 + y - z = 100 \quad and \;\; x + y^2 - z = 124$$

- **Hint.** Try to reduce the number of variables

- Solution.

(1) Second equation – First equation is: $x + y^2 - x^2 - y = 24$

(2) The left-hand side can be written as:

$$(y^2 - x^2) - (y - x) = (y - x)(y + x) - (y - x) = (y - x)(y + x - 1)$$

(3) $(y - x)$ **and** $(y + x - 1)$ have different parities

So one must be 3 and the other 8

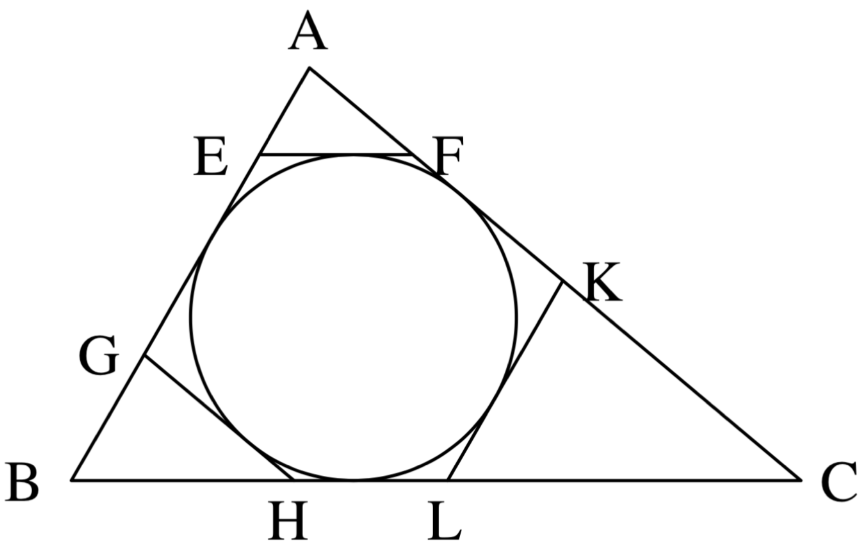
Or one must be 1 and the other 24

Or one must be -3 and the other -8

Or one must be -1 and the other -24

Write all the possibilities and solve for x, y

- Q5. The following picture shows a triangle ABC, its inscribed circle of radius r, and three tangent lines to it EF, GH, KL, which are parallel to BC, AC, AB.
- If r_a, r_b, r_c are radii of the inscribed circles in AEF, BGH, CKL, then which of the following relations holds
- (1) $r < r_a + r_b + r_c$
- (2) $r = r_a + r_b + r_c$
- (3) $r > r_a + r_b + r_c$
- (4) None of them



- Hint. Look for similar triangles again

- Solution.

(1) The triangles AEF, BGH, and CLK are all similar to ABC (check the angles)

(2) Therefore:

AEF vs. ABC: $\frac{r_a}{r} = \frac{AE}{AB} = \frac{AF}{AC} = \frac{EF}{BC} = \frac{AE+AF+EF}{AB+AC+BC} = \frac{p_{AEF}\ (perimeter\ of\ AEF)}{p_{ABC}\ (perimeter\ of\ ABC)}$

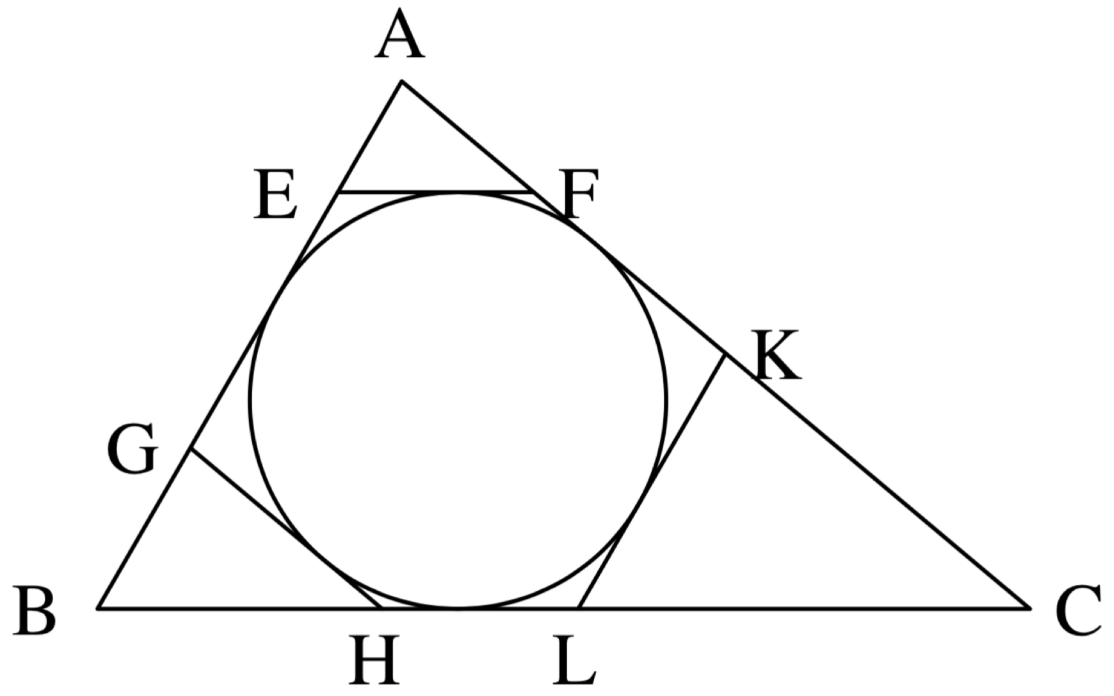
Similarly

$\frac{r_b}{r} = \frac{BG+BH+GH}{AB+AC+BC}$ and $\frac{r_c}{r} = \frac{CK+CL+KL}{AB+AC+BC}$

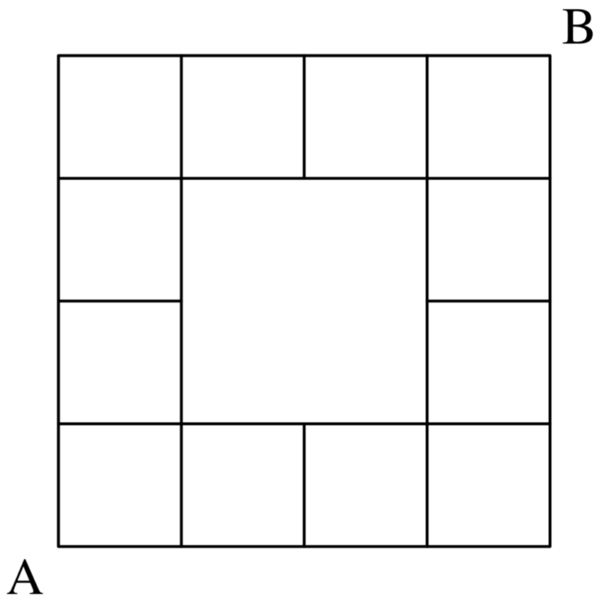
(3) Lastly, we show that

$$\frac{r_a + r_b + r_c}{r} = \frac{AE + AF + EF + BG + BH + GH + CK + CL + KL}{AB + AC + BC} = 1$$

We want to show that: $\frac{r_a+r_b+r_c}{r} = \frac{AE+AF+EF+BG+BH+GH+CK+CL+KL}{AB+AC+BC} = 1$



- Q6. In the following pictures, in how many ways we can go from A to B so that we have followed the shortest path (8 steps = 4 forward + 4 up).



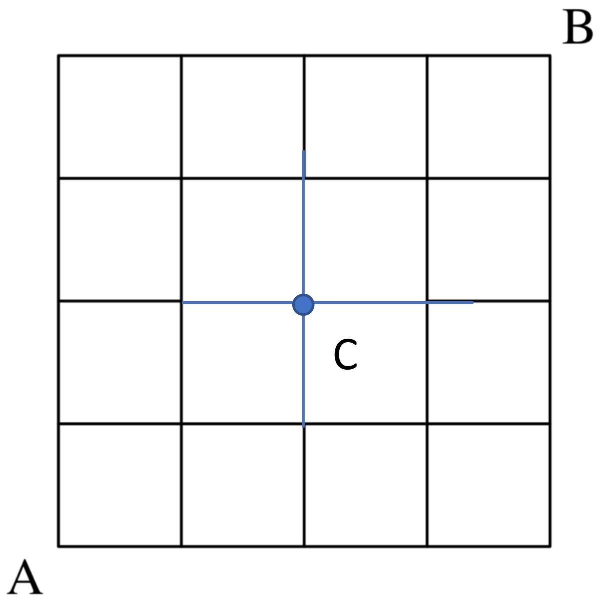
- Hint. Inclusion exclusion counting

- Solution.

(1) We can first complete the grid by adding the node in the middle

(2) Paths avoiding C = All paths from A to B – Path passing through C

(3) All paths = $\binom{8}{4}$. (Why?)



(4) Paths passing through C = $\binom{4}{2}\binom{4}{2}$

Answer: 70 – 36 = 34

- Q7. Let N be a natural number with digits $a_k a_{k-1} \cdots a_1$.

We say N is symmetric if reversing the location of the digits of N yields the same number N; i.e. $a_k a_{k-1} \cdots a_1 = a_1 a_2 \cdots a_k$

- How many good numbers **between**

1 to 100,000 are there ?

- Hint. There are 5 possibilities, $k = 1, 2, 3, 4, 5$

- Solution.

(1) # symmetric numbers with 1 digit = 9

(2) # symmetric numbers with 2 digits = 9

(3) # symmetric numbers with 3 digits = 9 x 10

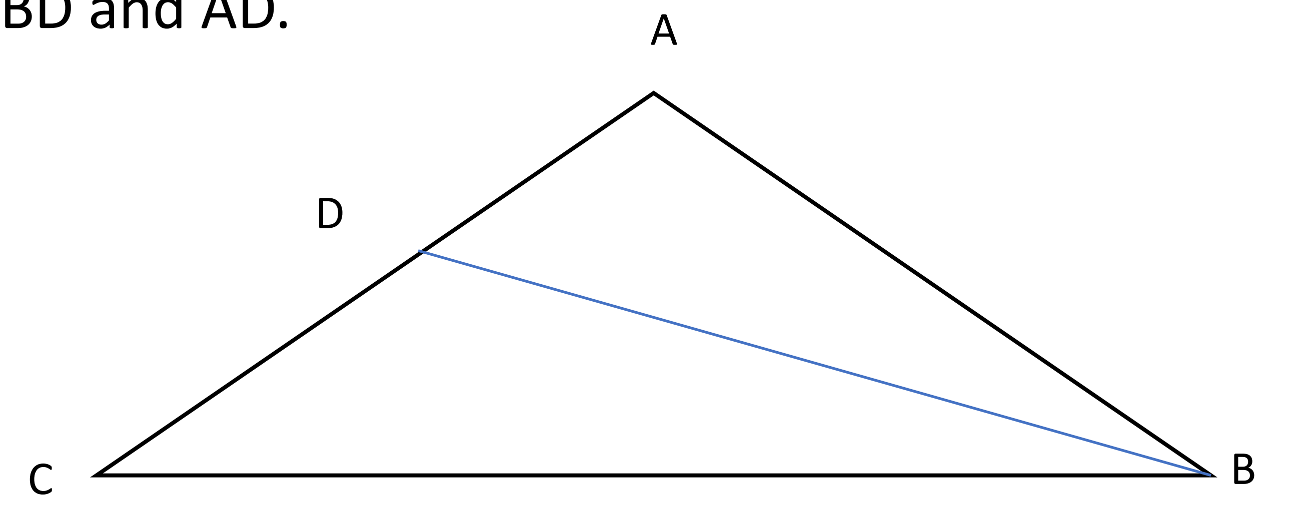
(4) # symmetric numbers with 4 digits = 9 x 10

(5) # symmetric numbers with 5 digits = 9 x 100

Answer= 1098

- **Q8.** Consider an isosceles triangle ABC (AB=AC). Suppose the angle bisector of angle B intersects AC at the point D, and BC=BD+AD. What is the angle A?

- **Hint.** Divide BC into pieces of size BD and AD.



- Solution. Consider an isosceles triangle ABC (AB=AC). Suppose the angle bisector of angle B intersects AC at the point D, and BC=BD+AD. What is the angle A?
- (1) Choose E so that BD=BE
- (2) Therefore, CE=AD
- (3) Since BD is an angle bisector, we have $\frac{BC}{CD} = \frac{BA}{AD}$
- (4) (2)+(3) → $\frac{BC}{CD} = \frac{BA}{CE}$
- (5) Angle C is common between CAB and CED
- (6) So the triangles ABC and CED are similar
- (7) Deduce that angle A = 100 degrees