Coefficients of polynomial expansions

Polynomial expansion.

• \((x + y)^2 = (x + y)(x + y) = x^2 + xy + yx + y^2 = 1 \ x^2 + 2 \ xy + 1 \ y^2\)
• \((x + y)^3 = (x + y)^2(x + y) = (x^2 + 2xy + y^2)(x + y) = 1 \ x^3 + 3 \ x^2y + 3 \ xy^2 + 1 \ y^3\)

What is the expansion formula for arbitrary \((x + y)^n\)?
Expansion formula:

\[(x + y)^n = (x + y) \cdots (x + y) = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \cdots + \binom{n}{k} x^{n-k} y^k + \cdots + \binom{n}{n} y^n\]

- **What is \(\binom{n}{k}\):** The number of ways you can choose \(k\) things from a set of size \(n\)

- **Why the coefficients are \(\binom{n}{k}\):**

  To get \(x^{n-k}y^k\) you need to choose \(y\) from \(k\) of the \(n\) terms in the product
Examples

• \((x + y)^2 = 1 \cdot x^2 + 2 \cdot xy + 1 \cdot y^2\) 😁

So \(\binom{2}{0} = 1\) \(\binom{2}{1} = 2\) \(\binom{2}{2} = 1\)

• \((x + y)^3 = 1 \cdot x^3 + 3 \cdot x^2y + 3 \cdot xy^2 + 1 \cdot y^3\)

So \(\binom{3}{0} = 1\) \(\binom{3}{1} = 3\) \(\binom{3}{2} = 3\) \(\binom{3}{3} = 1\)
Properties

\[(x + y)^n = (x + y) \cdots (x + y) = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \cdots + \binom{n}{k} x^{n-k}y^k + \cdots + \binom{n}{n} y^n\]

(1) Coefficients of \(x^{n-k}y^k\) and \(x^k y^{n-k}\) must be the same \(\Rightarrow\) \(\binom{n}{k} = \binom{n}{n-k}\)

(2) \((x + y)^{n+1} = (x + y)^n (x + y) \Rightarrow\)

\[\binom{n+1}{0} x^{n+1} + \binom{n+1}{1} x^n y + \cdots + \binom{n+1}{k} x^{n+1-k} y^k + \cdots + \binom{n+1}{n+1} y^{n+1} =\]

\[\left(\binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \cdots + \binom{n}{k} x^{n-k}y^k + \cdots + \binom{n}{n} y^n\right) (x + y) =\]

\[\left(\binom{n}{0} x^n + \left(\binom{n}{1} + \binom{n}{0}\right) x^n y + \cdots + \left(\binom{n}{k} + \binom{n}{k-1}\right) x^{n+1-k} y^k + \cdots + \binom{n}{n} y^n\right)\]
By comparing the coefficients, we conclude that:

\[
\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}
\]

Here is another way of proving ♠:

Suppose you want to choose a subset S of size \( k \) from the set \{ \( a_1, ..., a_{n+1} \) \}

- Total number of choices: \( \binom{n+1}{k} \)
- Number of those S that do not include \( a_{n+1} \) : \( \binom{n}{k} \)
- Number of those S that do include \( a_{n+1} \) : \( \binom{n}{k-1} \)
Pascal's triangle

\[
\begin{array}{ccccccccccc}
& & & & & & 0 & & & & \\
& & & & & 1 & & & & \\
& & & & 2 & & & & \\
& & & 3 & & & & \\
& & 4 & & & & \\
& 5 & & & & \\
& 6 & & & \\
& 7 & & \\
& \vdots & \\
\end{array}
\]

\[
\begin{array}{ccccccccccc}
& & & & & & 1 & & & & \\
& & & & & 1 & & & & \\
& & & & 1 & & & & \\
& & & 1 & & & & \\
& & 1 & & & & \\
& 1 & & & & \\
& 1 & & & \\
& 1 & & \\
& \vdots & \\
\end{array}
\]
Exercises:

- Prove the followings:

\[
\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n} = 2^n
\]

\[
\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n} = 0
\]

\[
k \binom{n}{k} = n \binom{n-1}{k-1}
\]
What is the formula for \( \binom{n}{k} \)

- **Lemma.** \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \)

- **Comment:** \( n! = 1 \times 2 \times \cdots \times n \). and \( 0! = 1 \)

- **Proof of Lemma:**

To choose \( k \) elements from a set of size \( n \):

You have (1) \( n \) choice for the first pick, (2) \( n - 1 \) choice for the second pick ... (k) \( n - k + 1 \) choice for the last pick

- Totally: that is \( n(n-1) \cdots (n-k+1) \) choices

- However, the order in which the \( k \) items are chosen is not important
Proof continued

- That is

\[ k! = \text{the number of ways to order } k \text{ things} \]

redundancy.

Therefore:

\[
\binom{n}{k} = \frac{n \times (n-1) \times \cdots \times (n-k+1)}{k!} = \frac{n \times (n-1) \times \cdots \times (n-k+1) \times (n-k)!}{k! \times (n-k)!} = \frac{n!}{k!(n-k)!}
\]
More variables

What is the expansion formula for more than two variables?

\[(x_1 + x_2 + \cdots + x_k)^n\]

- Example:

\[(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz\]

\[(x + y + z)^3 =
1 \ (x^3 + y^3 + z^3) +
3 \ (x^2y + y^2x + x^2z + z^2x + y^2z + z^2y) +
6 \ xyz\]
Sigma notation

- In order to write big sums with compact notation, people use the notation

\[ \sum_{\text{lower limit of index}}^{\text{upper limit of index}} \text{summand} \]

- Example

Instead of writing \( \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \cdots + \binom{n}{k} x^{n-k}y^k + \cdots + \binom{n}{n} y^n \)

We write \( \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \) or \( \sum_{0 \leq k \leq n} \binom{n}{k} x^{n-k} y^k \)
General expansion formula

\[(x_1 + x_2 + \cdots + x_k)^n = \sum_{m_1 + \cdots + m_k = n} \binom{n}{m_1 \, m_2 \cdots m_k} x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}\]

\[\binom{n}{m_1 \, m_2 \cdots m_k} = \text{Number of ways to divide a set of } n \text{ objects into } k \text{ ordered groups of sizes } m_1, m_2, \ldots, m_k\]

Note that for \(k = 2\), \(\binom{n}{m_1 \, m_2} = \binom{n}{m_1}\) in our previous notation: Once you pick the first group the rest automatically go the next group.

\[\binom{n}{m_1 \, m_2 \cdots m_k} = \frac{n!}{m_1! \, m_2! \cdots m_k!}\]

Example: \(\binom{6}{2 \, 2} = \frac{6!}{2!2!2!} = \frac{720}{2 \times 2 \times 2} = 90\)
Stirling number \( S(n, k) \) = the number of ways to partition \( n \) different objects into \( k \) non-empty groups, where there is no order on the sets

- There is no easy formula for \( S(n, k) \) but the can be calculated recursively

Examples:

\[
S(n, n) = 1
\]

\[
S(n, n - 1) = \binom{n}{2}
\]
Recursive formula

\[ S(n, k) = S(n - 1, k - 1) + k \cdot S(n - 1, k) \]

- Proof: Set = \{1,2,3,\ldots,n\}
- There are two types of partitions of this set into \( k \) non-empty un-ordered subsets
- (1) Those for which \{n\} a group by itself
- (2) Those for which \( n \) belongs to a larger subset
- \#(1) = S(n - 1, k - 1) : you need to divide \{1,2,\ldots n - 1\} into \( k - 1 \) subsets
- \#(2) = k \cdot S(n - 1, k) : you divide \{1,2,\ldots n - 1\} into \( k \) subsets and put \( n \) into one of them
Problems*

- You are creating a 4-digit pin code. How many choices are there in the following cases?
  
  (a) With no restriction.
  
  (b) No digit is repeated.
  
  (c) No digit is repeated, digit number 3 is a 0.
  
  (d) No digit is repeated, and they must appear in increasing order.
  
  (e) No digit is repeated, 2 and 5 must be present.

*: Questions taken from Combinatorics and counting, by Per Alexandersson
How many shuffles are there of a deck of cards, such that A♥ is not directly on top of K♥, and A♠ is not directly on top of K♠?
How many different words can be created by rearranging the letters in SELFIE STICK?

*: Questions taken from Combinatorics and counting, by Per Alexandersson
Problems*

- In how many ways can 8 people form couples of two?

*: Questions taken from Combinatorics and counting, by Per Alexandersson
Problems*

- How many integer solutions does $x_1 + x_2 + \cdots + x_k = n$ have, with $x_i \geq 0$?

*: Questions taken from Combinatorics and counting, by Per Alexandersson
Problems

- Show that

\[ x(x - 1)(x - 2) \cdots (x - n + 1) = \sum_{0 \leq k \leq n} S(n, k) x^k \]

What do you get if you plug in \( x = n \)?