Description of the idea

- **Idea.** We will study basic **moves** that change shapes in plane.
- We learn how these moves change basic shapes such as a line or circle, or quantities such as length and angle.
- Finally, we combine these moves to make more complicated moves.

We call these moves geometric transformations.
(1) **Translation**: We move every point in certain direction, with certain distance.

- A translation $T_v$ is described by a vector $v$:
  \[ x \rightarrow T_v(x) = x + v \]
- Translations **preserve** length and angle
- It does not fixes any point unless $v = 0$
- If $v = 0$, then $T$ does not make any change
Doing multiple moves

- **Description:** If you first move points with a vector $u$ and then with a vector $v$, it is as if you have moved points with sum of two vectors $u + v$

- **Mathematical notation:** $T_v \circ T_u = T_{u+v}$

WE SAY: Composition of a series of translations by vectors $v_1, \ldots, v_k$ is a translation by the vector $v = v_1 + \cdots + v_k$
(2) **Homothecy (or Homothety)**: A transformation of plane which **dilates** distances with respect to a fixed point (center of Homothecy)

- A Homothecy $H_{o,r}$ is described by its center point $o$ and the dilation factor $r$:

  \[ x \rightarrow H_{o,r}(x) = o + r (x - o) \]

- Homothecy **preserve** angle but changes length by a factor of $r$

- It **only fixes** the center point $o$ (unless $r = 1$)

- If $r = 1$, then $H$ does not make any change
- $r > 1$ : things get bigger.  
- $r < 1$ : things get smaller

- Effect of Homothecy on Lines:
  
  (1) every line $L$ will be mapped to another line parallel to $L$
  
  (2) If $L$ passes through the center it will be preserved

- Circles:
  every Circle $C$ will be mapped to another circle
Composition of two Homothecies

- If we first do a Homothecy $H_1 = H_{o_1, r_1}$ with center $o_1$ and dilation factor $r_1$, and then do another a Homothecy $H_2 = H_{o_2, r_2}$ with center $o_2$ and dilation factor $r_2$, what is the overall transformation?

- **Mathematical notation**: What can we say about $H = H_2 \circ H_1$?

- **Observations**:
  1. $H$ does not change angles.
  2. $H$ changes distances by a total factor of $r = r_1 r_2$

(Q) Could $H$ possibly be a Homothecy as well with a dilation factor of $r$?

(Q) If yes, where will be its center?
Where is the center of $H$?

- **Lemma**: If $r_1 r_2 = 1$, then the composition of $H_1$ and $H_2$ is a translation by the vector $v = r_2 \overrightarrow{o_1 o_2}$

- **Lemma**: If $r_1 r_2 \neq 1$, then the composition of $H_1$ and $H_2$ is a homothety with ratio $r = r_1 r_2$ and the center $o$ will be somewhere on the line $o_1 o_2$

- Four cases:
  
  (i) $r, r_2 > 1$,  
  (ii) $r, r_2 < 1$,  
  (iii) $r < 1 < r_2$,  
  (iV) $r_2 < 1 < r$
4. Given three circles with centers $A$, $B$, $C$ and distinct radii, show that the exsimilicenters of the three pairs of circles are collinear.
Strategy of the proof

1. We perform 3 homotheties with centers $x, y, z$ and ratios $\frac{r_B}{r_C}, \frac{r_A}{r_B}, \text{ and } \frac{r_C}{r_A}$ respectively.

2. The circle $C$ is fixed in this process.

3. The overall ratio is 1.

4. So the composition of these 3 homotheties does not move any point.

Now

5. Track the line passing through $xy$ in this process to conclude that $z$ must be on it as well.
Proof of Menelaus (version 2) with homothety

**Theorem.** Suppose \( \frac{B'A}{B'C} \times \frac{A'C}{A'B} \times \frac{C'B}{C'A} = 1 \) then

\( A', B', C' \) are on one line \( L \) (we say \( A', B', C' \) are **collinear**)

Repeat the same proof as the previous one.
Problems to solve with Homothecy

- https://www.cut-the-knot.org/pythagoras/Transforms/ProblemsByHomothety.shtml
Homothety with negative scaling factor

We can also consider Homotheties $H_{(o,r)}$ where the dilation factor $r < 0$:

$$x \to H_{o,r}(x) = o + r(x - o)$$

The same rule as before applies:

Composition of Homotheties is a homothety

(sometimes a translation)
Proof of Menelaus (version 1) with Homothety

**Theorem.** Suppose \( \frac{B'A}{B'C} \times \frac{A'C}{A'B} \times \frac{C'B}{C'A} = 1 \) then

A', B', C' are on one line L (we say A', B', C' are **collinear**)

![Diagram of a triangle with points A, B, C, A', B', C']
More geometric transformations

- (3) **Reflection** across a line $L$:
  
  - Properties:
    - Applying $R_L$ twice, we get identity
    - It preserves distance and angle
(3) If we reflect across a line $L$ and then $L'$, what is the composition of two reflections?

\[ \begin{align*}
  x &\rightarrow y = R_L(x) \\
  y &\rightarrow z = R_{L'}(y)
\end{align*} \]
More geometric transformations

- (4) Rotation around a point $o$ with angle $\theta$

- Properties:
  - It preserves distance and angle
  - The only point fixed is the center $o$
More geometric transformations

- (5) **Inversion** with respect to a point $o$ and product distance $r^2$:

  It takes any point $x$ besides the center $o$ to another point $y = I(x)$ colinear with $x$ and $o$ such that $ox \times oy = r^2$.

  It changes any line not passing through the center $o$ to a circle (as in the picture). In this regard, it is different from transformations we had before.