Counting 2

Lecture 10 Mar 14, 2021

Recall:

$\binom{\boldsymbol{n}}{\boldsymbol{k}}$: The number of ways you can choose k things from a set of size n

$$(x+y)^n = (x+y)\cdots(x+y) = \binom{n}{0}\ x^n + \binom{n}{1} x^{n-1}y + \cdots + \binom{n}{k} x^{n-k}y^k + \cdots + \binom{n}{n} y^{\mathrm{n}}$$

Properties:

- $\binom{n}{k} = \binom{n}{n-k}$
- $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$
- $\binom{n}{0} = 1$
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Warmup:

- Show that. $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n} = 0$

- Show that $k \binom{n}{k} = n \binom{n-1}{k-1}$

- * How many integer solutions does $x_1 + x_2 + \cdots + x_k = n$ have, with $x_i \geq 0$?

*: Questions taken from Combinatorics and counting, by Per Alexandersson

Inclusion-Exclusion (simple case)

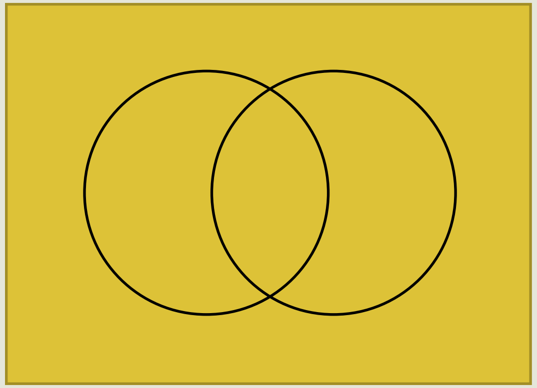
- Last time we saw one example of an inclusion-exclusion counting problem:

The number of objects that do not have property A or B is

+ # All objects

- # objects that have property A - # objects that have property B

+ # objects that have both properties A and B



Inclusion-Exclusion (general case)

The number of objects that do not have property A_1, A_2, A_3 , … is

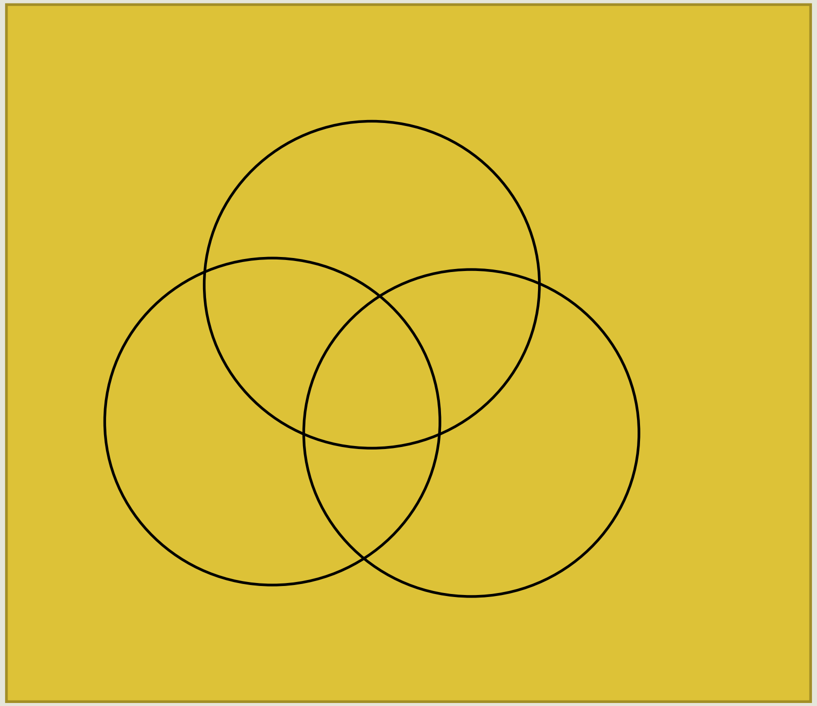
All objects

- # objects that have property A_i

+ # objects that have both properties A_i and A_j

- # objects that have 3 of the properties $A_i, A_{j,} A_k$

+ …



Examples*

- There are five people of different height. In how many ways can they stand in a line, so there is no 3 consecutive people with increasing height?

*: Questions taken from Combinatorics and counting, by Per Alexandersson

Examples*

- Find the number of positive integers between 1 to 100 which are not prime but they are not divisible by 2,3, and 5.

*: https://artofproblemsolving.com/wiki/index.php/2005_AMC_12A_Problems/Problem_18

Random questions

In the rest of our time today we go over questions from the book:

102 combinatorial problems

- Q2. The student lockers at Olympic High are numbered consecutively beginning with locker number 1. The plastic digits used to number the lockers cost two cents a piece. If it costs $137.94 to label all the lockers, how many lockers are there at the school?

- Q3. Let n be an odd integer greater than 1. Prove the sequence $\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{\frac{n-1}{2}}$ contains an odd number of odd numbers!
- Hint: the number of odd numbers in a set $a_1, a_2, \dots, a_k$ is odd $\Leftrightarrow$ the sum $a_1 + a_2 + \cdots + a_k$ is odd

- Q4. How many positive integers not exceeding 2001 are multiples of 3 or 4 but not 5.

- Twenty five boys and twenty five girls sit around a table. Show that it is always possible to find a person both of whose neighbors are girls.
- Hint: Assuming that there is an arrangement not satisfying the property, estimate the number of blocks of consecutive girls and boys to get a contradiction.

- Q8. A spider has one sock and one shoe for each of its eight legs. In how many different orders can the spider put on its socks and shoes, assuming that, on each leg, the sock must be put on before the shoe!

- Q11. Determine the number of ways to choose five numbers from $1,2,\ldots,18$ such that any two chosen numbers differ by at least 2.

- Q17. Prove that among any 16 distinct integers between 1 to 100, there are four different ones a, b, c, d such that $a + b = c + d$

- Q20. Two of the squares of 7x7 checkboard are painted yellow, and the rest are painted green. Two color schemes are the same if one can be obtained from the other by a rotation. How many different color schemes are possible?